1 Introduction

In this paper, we study the following initial value problem:

$$\begin{cases} g'' - \beta y g^{\frac{1}{m}-1} g' + \alpha m g^{\frac{1}{m}} = 0, \quad y > 0, \\ g'(0) = m g^{1+\frac{k}{m}}(0), \end{cases} \tag{1}$$

where $k < 0$, $m < 0$, and α, β are given by

$$\alpha := \frac{1}{1-m-2k}, \ \beta := \frac{-k}{1-m-2k} = -k\alpha.$$

This problem arises in the study of the quenching behavior of the problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x, & x \in (0,1), \ t > 0, \\ (u^{m-1}u_x)(0,t) = u^{m+k}(0,t), & t > 0, \\ (u^{m-1}u_x)(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in [0,1]. \end{cases} \tag{2}$$

We say that a solution *quenches*, if its minimum reaches zero in finite time. Quenching phenomena (*when, where, and how the solution quenches*) have been studied by many authors, since the first work of Kawarada [10] in 1975. For more references, we refer the reader to the survey papers by Levine [11, 12] and Chan [1]. See also the references [3, 4, 5, 6, 7, 9].

We assume that u_0 is a C^1 function in $[0,1]$ such that

$$u_0 \geq \delta > 0, \ u_0' \geq 0 \text{ in } [0,1], \ u_0'(0) = u_0^{1+k}(0), \ u_0'(1) = 0. \tag{3}$$

Note that from (3) and the maximum principle, it follows that $u_x > 0$ as long as $u > 0$. We shall prove in §2 that there is $T \in (0,\infty)$ such that $u > 0$ in $[0,1] \times [0,T)$ and $\liminf_{t \uparrow T^-} u(0,t) = 0$.

We are concerned with the existence of the self-similar positive solutions of (2) in the form

$$u(x,t) = (T-t)^{\alpha} \varphi(x/(T-t)^{\beta}). \tag{4}$$

Set $y = x/(T-t)^{\beta}$. It follows that u satisfies (2) if and only if φ satisfies

$$\begin{cases} (\varphi^{m-1}\varphi')' - \beta y \varphi' + \alpha\varphi = 0, \quad 0 < y < (T-t)^{-\beta}, \\ (\varphi^{m-1}\varphi')(0) = \varphi^{m+k}(0), \\ (\varphi^{m-1}\varphi')((T-t)^{-\beta}) = 0. \end{cases} \tag{5}$$

Let $g = \varphi^m$. Then φ satisfies (5) if and only if g satisfies

$$\begin{cases} g'' - \beta y g^{\frac{1}{m}-1} g' + \alpha m g^{\frac{1}{m}} = 0, \quad 0 < y < (T-t)^{-\beta}, \\ g'(0) = m g^{1+\frac{k}{m}}(0), \\ g'((T-t)^{-\beta}) = 0. \end{cases}$$

Since we are interested in the behavior of u as $t \uparrow T$, we end up with the problem (1) and we shall prove the existence of the globally monotone decreasing solution of (1).

The case $k = 0$ has been studied by Ferreira-de Pablo-Quirós-Rossi in [2]. In [2], they proved that the quenching always occurs for any positive solution. They also study the quenching set, quenching rate, and beyond quenching. In particular, they found the so-called super-fast quenching for certain cases. Indeed, this is due to the lack of positive self-similar solutions. Therefore, it is very important to study the self-similar solutions. We shall apply the shooting method to study the initial value problem (1).

The organization of this paper is as follows. In §2, we shall first prove that quenching always occurs and then give some preliminary results related to the initial value problem (1). Then we prove in §3 the existence of self-similar solutions by a shooting method. Finally, we study the asymptotic behavior of any globally monotone decreasing solution of (1) in §4.