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Simple Solutions of the Harmonic Oscillator Integral Equation

Jeng-Yih Su
Department of Physics, National Taiwan Normal University

Abstract

We employ a simple method to solve the integral equation for the energy eigenstates of the linear harmonic oscillator. The energy eigenvalues and eigenstates are all obtained in our approach. The integral equation approach of solving the oscillation problem is very helpful to the study of quantum mechanical scattering theory and complements the conventional methods of solving the differential equation or employing the operator method.

Key words: energy eigenvalue equation integral equation quantum harmonic oscillator.

In a recent publication, Muñoz finds the energy eigenstates for the simple harmonic oscillator (SHO) through the solution of an integral equation (Muñoz, 1998). In fact, he first derives the integral equation using Fourier transforms and then employs properties of Fourier transforms and a method parallel to the abstract algebraic method of ladder operators to solve the integral equation. The purpose of this paper is to write down the integral equation immediately from the relation between the wave functions in momentum space and in coordinate space, which is due to the same form of differential equations in both spaces, and to solve it in a fairly simple way by observing that the Fourier transform of a Gaussian function is another Gaussian. To make the integral equation approach an independent method of solving the energy eigenvalue problem, we also obtain the eigenfunctions as well as eigenvalues (which are not obtained in Muñoz's paper) without reference to those obtained from well-known methods (see, eg., Schiff, 1968; Merzbacher, 1970; Cohen-Tannoudji, Diu, and Laloë, 1977; Shankar, 1994; Griffiths, 1994). In our approach, we have obtained four classes of solutions for the integral equation, which is an interesting property of the integral equation approach.

The Schrödinger equation for the energy eigenstates of the SHO is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x). \tag{1}$$

Introducing the dimensionless independent variable $\xi=\alpha x=(m\omega/\hbar)^{1/2}x$ leads to the simpler form (Schiff 1968)

$$\frac{d^2\overline{\psi}(\xi)}{d\xi^2}+(\lambda-\xi^2)\overline{\psi}(\xi)=0, \tag{2}$$

where $\overline{\psi}(\xi)=\overline{\psi}(\alpha x)=\psi(x)$, $\lambda = 2E/\hbar\omega$.

To derive the integral equation for the SHO wave function, one simply uses the relation between the wave functions $\varphi(p)$ in momentum space and $\psi(x)$ in coordinate space (Merzbacher, 1970, p.144)

$$\varphi(p) = d\,\psi(p/m\omega) = d\,\overline{\psi}(\alpha p/m\omega)\,, \tag{3}$$

where d is a constant, and

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)\,e^{-ipx/\hbar}dx,$$

$$= \frac{1}{\sqrt{2\pi\, m\omega}}\int_{-\infty}^{\infty}\overline{\psi}(\xi)\,e^{-ip\xi/\alpha\hbar}d\,\xi\,. \tag{4}$$

By using Eq. (3) and letting $K = p/\alpha\hbar$, one has the integral equation

$$c\,\overline{\psi}(K) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\psi}(\xi)\,e^{-iK\xi}\,d\,\xi\,, \tag{5}$$

where $c=(m\omega)^{1/2}d$. This integral equation is equivalent to Eq. (6) in Muñoz (1998). To determine the constant c, one can iterate Eq. (5) and obtain,

$$c^2\,\overline{\psi}(K) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\,\xi\,e^{-iK\xi}\int_{-\infty}^{\infty}\overline{\psi}(K')\,e^{-iK'\xi}\,dK'$$

$$= \int_{-\infty}^{\infty}\delta(K'+K)\,\overline{\psi}(K')\,dK' = \overline{\psi}(-K)\,, \tag{6}$$

which is equivalent to Eq. (7) in Muñoz (1998).

From the above equation, one has $c^4 = 1$ or $c = (\mp i)^N$, $N = 0, 1, 2, \cdots$, and Eq. (3) becomes (Merzbacher, 1970, p.144)

$$\varphi(p) = \frac{(\mp i)^N}{\sqrt{m\omega}} \overline{\psi}(\alpha p / m\omega) . \qquad (3')$$

The integral equation (5) states that the Fourier transform of $\overline{\psi}(\xi)$ is itself $\overline{\psi}(K)$ except for the constant $c = (\mp i)^N$, which is due to the same of differential equations in the coordinate space and in the momentum space. One also observes that the Fourier transform of a Gaussian function is another Gaussian. This observation may be an important clue in solving the integral equation (5).

In fact, let $f(\xi, b) = e^{-\xi^2/a} e^{2b\xi}$, where a>0, one has its Fourier transform

$$F\{f(\xi, b)\} = \sqrt{\frac{a}{2}}\, e^{-aK^2/4} e^{-iabK} e^{ab^2} , \qquad (7)$$

by using the well-known formula

$$\int_{-\infty}^{\infty} e^{-iK'\xi} e^{-\xi^2/a} d\xi = \sqrt{a\pi} e^{-aK'^2/4} . \qquad (8)$$

The requiremennt that $f(\xi, b)$ resembles its Fourier transform as much as possible forces one to choose a=2. Then one has the more symmetric form

$$F\{f(\xi, b)\} = f(K, -ib)\, e^{2b^2} . \qquad (9)$$

Using the expansion (Schiff, 1968; Merzbacher, 1970; Cohen-Tannoudji et al., 1977; Arfken and Weber, 1995), $H_n(\xi)$ being the Hermite polynomials of order n

$$e^{2b\xi} e^{-b^2} = \sum_{n=0}^{\infty} \frac{b^n}{n!} H_n(\xi) , \qquad (10)$$

and equating the coefficients of equal powers of b on both sides of Eq. (9), one obtains

$$(-i)^n e^{-K^2/2} H_n(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iK\xi} e^{-\xi^2/2} H_n(\xi)\, d\xi . \qquad (11)$$

By comparing the above equation with the integral equation Eq. (5), one can write more conveniently $c = (-i)^n$, where n is the quantum number charactering the SHO energy eigenstates, the integral equation then takes the form

$$(-i)^n \, \overline{\psi}_n(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iK\xi} \, \overline{\psi}_n(\xi) \, d\xi \, , \tag{5'}$$

and its solutions are the standard ones (Schiff,1968; Merzbacher, 1970; Cohen-Tannoudji et al., 1977; Shankar, 1994; Griffiths, 1994)

$$\psi_n(x) = \overline{\psi}_n(\xi) = N_n H_n(\xi) \, e^{-\xi^2/2}, \tag{12}$$

where N_n is the normalization constant.

To make the integral equation approach an independent effective method of solving the energy eigenvalue problem, i.e., Eq. (2), it would be imperative to obtain the eigenfunctions as well as eigenvalues without reference to those given in the textbooks mentioned above. To this end we first write the eigenvalues of the integral equation (5) as $c = (-i)^s = (-i)^{s+4k}$, where s=0,1,2,3 and k=0,1,2,3,⋯. By using Eq. (11), we obtain the general solutions to the integral equation (5):

$$\overline{\psi}^{(p)}(\xi) = e^{-\xi^2/2} \sum_{k=0}^{\infty} a_{4k+s} H_{4k+s}(\xi) \tag{13}$$

By substituting it in Eq. (2), we get

$$\sum_{k=0}^{\infty} a_{4k+s} \left[H_{4k+s}(\xi) - 2\xi H_{4k+s}(\xi) + (\lambda - 1) H_{4k+s}(\xi) \right] = 0,$$

or

$$\sum_{k=0}^{\infty} a_{4k+s} \left[-2(4k+s) - 1 + \lambda \right] H_{4k+s}(\xi) = 0, \tag{14}$$

if the equation (Arfken and Weber, 1995)

$$H_q''(\xi) - 2\xi H_q'(\xi) + 2q H_q(\xi) = 0, \quad q=0,1,2,\cdots \tag{15}$$

is used. (Eq.(15) can easily be obtained from Eq. (10)).

By multiplying both sides of Eq. (14) by $e^{-\xi^2} H_n(\xi)$ and integrating with respect to ξ from $-\infty$ to ∞, and by using the orthogonality property (Arfken and Weber, 1995)

$$\int_{-\infty}^{\infty} e^{-\xi^2} H_n(\xi) H_{4k'+s}(\xi) d\zeta = 0 \quad \text{if} \quad n \neq 4k' + s \tag{16}$$

one gets

$$a_{4K+s}\left[-2(4K+s)-1+\lambda\right]=0 \quad \text{if} \quad n=4K+s \tag{17}$$

From this equation if follows that $E_n = (n+\frac{1}{2})\hbar\omega$ and

$\psi_n^{(p)}(x) = \overline{\psi}_n^{(p)}(\xi) = a_n H_n(\xi)e^{-\xi^2/2}$, which is just Eq. (12). We can now classify the energy eigenstates into four classes with kp=0,1,2,3 according to the eigenvalues c=1, *-i*, -1, *i* of the integral equation (5) instead of two classes of parity eigenstates. It's significance or impact on the SHO dynamics remains to be explored, and warrants further studies.

In conclusion, we this special property have solved the integral equation, first derived in Muñoz (1998), by the simple method described above. In this method we use the generating function of the Hermite polynomials, i.e., Eq. (10), which often serves as the definition for them (Arfken and Weber, 1995). All the properties (Schiff, 1968; Merzbacher, 1970; Cohen-Tannoudji et al., 1977; Arfken and Weber 1995) of $H_n(\xi)$ follow from Eq. (10), and thus everyone studying quantum mechanics has to know it. We also obtain the energy eigenvalues from the general solutions of the integral equation. The familiarity with the integral equation approach in quantum mechanics will be very helpful to the study of quantum mechanical scattering theory.

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簡諧振盪積分方程的簡易解

蘇正義

國立臺灣師範大學物理學系

吾人採用一簡單方法，以解量子力學簡諧振盪器之能量固有態的積分方程式。在此積分方程式解法中，吾人可得到所有得能量固有值與能量固有態。積分方程式的解法有助於探討量力散射理論；與解微分方程式或用算符方法的傳統解法有互補的功能。

關鍵字：能量固有值方程式　積分方程式　量力簡諧振盪器。