1 Introduction and preliminaries

Recently, by following the method of Joo [1], Horvath [2] gave a general result in the theory of minimax theorems, which requires neither Hahn-Banach's theorem nor Brouwer's fixed point theorem. Horvath's minimax theorem has been generalized to non-compact case by Kim [3]. Lin and Yu [4] further gave a two functions version of Horvath's minimax theorem. On the minimax theorem of Lin and Yu's paper, two functions are defined on convex regions. In the first part of this paper, we shall replace the convexity of the regions by *η-connectedness*, defined as follows.

Let Y be a subset of a vector space E, and η be a function from $Y \times Y \times [0,1]$ to E so that for each y_0, y_1 in Y, $\eta(y_0, y_1, \cdot)$ is continuous. We say Y is *η-connected* if

$$\forall\ y_0, y_1 \in Y \quad \forall\ t \in [0,1],\ \eta(y_0, y_1, t) \in Y \ \textit{and} \left\{ \begin{array}{l} \eta(y_0, y_1, 0) = y_0 \\ \eta(y_0, y_1, 1) = y_1 \end{array} \right.$$

On the other hand, in 1972, Terkelsen [7] gave a generalization of von Neumann's minimax theorem by mixing topological and algebraic conditions. As shown in [7], Terkelsen's minimax theorem is different from Sion's minimax theorem [8] and Fan's minimax theorem [9].

Let X and Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$. For $t \in (0,1)$, f is said to be *t-convex* on Y if for any y_1, y_2 in Y, there exists y_0 in Y such that for all x in X,

$$f(x, y_0) \leq t \max\{f(x, y_1), f(x, y_2)\} + (1-t) \min\{f(x, y_1), f(x, y_2)\}.$$

Replacing the midpoint convexity in Terkelsen's minimax theorem [7] by *t-convexity*, a generalization of Terkelsen's minimax theorem is given by Geraghty and Lin [10].

A function $f : X \longrightarrow Y$ is said to be *upward* on Y if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that for all x in X,

$$f(x, y_3) \leq \max\{f(x, y_1), f(x, y_2)\},$$

and

$$|f(x, y_1) - f(x, y_2)| \geq \epsilon \Longrightarrow f(x, y_3) \leq \max\{f(x, y_1), f(x, y_2)\} - \delta.$$

Simons [11] introduced the upwardness concept which generalizes *t-convexity* and obtained a minimax theorem that includes the result of Geraghty-Lin [6].

We say that f and g are *jointly upward* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that for all $x \in X$,

$$\max\{f(x, y_3), g(x, y_3)\} \leq \max\{g(x, y_1), f(x, y_2)\},$$

and

$$|g(x,y_1)-f(x,y_2)|\geq \epsilon \Longrightarrow \max\{f(x,y_3),g(x,y_3)\}\leq \max\{g(x,y_1),f(x,y_2)\}-\delta.$$

It is easy to see that when $f=g$ and the map $y\longmapsto f(x,y)$ is convex, f and g are *jointly upward.*

Lin and Yu [5] give a two functions version of Simon's minimax theorem with two jointly upward functions. In fact, they require the usual changeless proportions between the two functions on the defining region:

$$f(x,y)\leq g(x,y),\ \forall\ (x,y)\in X\times Y.$$

In the second part of this paper, we shall present a two functions version of minimax theorems without the above restriction and convexity, but need merely some kind of connectedness.

Finally, we shall restrict the feasible region, and define a class of X-*quasiconcave* sets as follows. Let $T:X\to Y$ be a multifunction. The set $H_T=\{g:X\times Y\to\mathbb{R}\}$ is said to be a X-*quasiconcave* set of T if for all $g\in H_T$, and $x,x_1,x_2\in X$, there exists $x_3\in X$, $h\in H_T$ such that

$$h(x_3,y)\geq \max\{g(x_1,y),g(x_2,y)\}\quad\ \forall\ y\in T(x).$$

Clearly, if T is sigle-valued, then H_T is X-*qusiconcave*. In the sequel, we extend the two functions minimax theorem to the graph of the multifunction under such a X-*quasiconcave* property.