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Local Cohomology and Čech Complexes

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致　　謝

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Contents

1 Introduction

Through out this thesis, $(R, \mathbf{m}, k)$ is a Noetherian local ring with maximal ideal $\mathbf{m}$. Let A be an R-module and define

$$\Gamma_{\mathbf{m}}(A) = \{y \in A \mid \mathbf{m}^k y = 0 \text{ for some } k \geq 0\}.$$

It is not difficult to check that $\Gamma_{\mathbf{m}}(-)$ is a left exact additive functor on the category of R-modules. The right derived functor associate to $\Gamma_{\mathbf{m}}(-)$, denoted by $H^n_{\mathbf{m}}(-)$, is called the local cohomology functor. In other words, take an injective resolution $\mathcal{I}$ of A and delete A, then we get a cochain complex $\Gamma_{\mathbf{m}}(\mathcal{I})$ by applying the functor $\Gamma_{\mathbf{m}}$ to every term in $\mathcal{I}$. Then $H^n_{\mathbf{m}}(A)$ is the nth cohomology associate to $\Gamma_{\mathbf{m}}(\mathcal{I})$, i.e.,$H^n_{\mathbf{m}}(A) = H^n(\Gamma_{\mathbf{m}}(\mathcal{I}))$.

Let $x_1, x_2, \ldots, x_n$ be a sequence of elements of R. The Čech complex with respect to the sequence $x_1, x_2, \ldots, x_n$ is a cochain complex

$$\mathcal{C} : 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n \longrightarrow 0$$

where $C^t = \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}}$ and $C^0 = R$, and the differentiation $d^t : C^t \to C^{t+1}$ is given on the component $R_{x_{i_1} \cdots x_{i_t}} \to R_{x_{j_1} x_{j_2} \cdots x_{j_{t+1}}}$ to be

$$\begin{cases} (-1)^{s-1} \cdot nat : R_{x_{i_1} \cdots x_{i_t}} \to (R_{x_{i_1} \cdots x_{i_t}})_{x_{j_s}} & \text{if } \{i_1, \ldots, i_t\} = \{j_1, \ldots, \hat{j_s}, \ldots, j_{t+1}\}, \\ 0 & \text{otherwise} . \end{cases}$$

Note that $nat : R_{x_{i_1} \cdots x_{i_t}} \to (R_{x_{i_1} \cdots x_{i_t}})_{x_{j_s}}$ is the natural R-module homomorphism defined by $\frac{r}{(x_{i_1} \cdots x_{i_t})^l} \mapsto \frac{x^l_{j_s} r}{(x_{i_1} \cdots x_{i_t} x_{j_s})^l}$.

It is well-known that if the ideal $(x_1, x_2, \ldots, x_n)$ is $\mathbf{m}$-primary, then we have

$$H^t_{\mathbf{m}}(A) \cong H^t(A \otimes \mathcal{C})$$

for all R-modules A and $t \geq 0$. In this thesis, we will give a proof with complete details for this isomorphism. In order to prove this isomorphism, we consider additive functors on the category of R-modules. In Chapter 2, we introduce *connected pairs*, *connected sequences*, and the definition of being universal for a connected sequence. We will also show that if two universal connected sequences $\{T^n, E^n\}_{n \geq 0}$ and $\{T^{'n}, E^{'n}\}_{n \geq 0}$ have the same initial $T^0 = T^{'0}$, then $T^n(A) \cong T^{'n}(A)$ for all R-module A and $n \geq 0$.

In Chapter 3, we first prove that $\{H^n_{\mathbf{m}}(-), E^n\}_{n\geq 0}$ is a universal connected sequence with initial $H^0_{\mathbf{m}}(-) = \Gamma_{\mathbf{m}}(-)$. We will also show that $\{H^n(- \otimes \mathcal{C}), E^n\}_{n\geq 0}$ is a universal connected sequence with initial $H^0(- \otimes \mathcal{C}) = \Gamma_{\mathbf{m}}(-)$. In order to show $H^n(- \otimes \mathcal{C})$ is universal, we make two important observations: any nonzero injective R-module is a direct sum of indecomposable injective R-modules, and an indecomposable injective R-module is isomorphism to an injective hull of $R/\mathbf{p}$ for some $\mathbf{p} \in \mathrm{Spec}(R)$. Finally we use what we prove in Chapter 2 to see that $H^n_{\mathbf{m}}(A) \cong H^n(A \otimes \mathcal{C})$ for all R-module A and $n \geq 0$.

2 Connected Pairs and Connected Sequences

2.1 Preliminaries

Before introducing *connected pairs* and *connected sequences*, we consider the category $\mathcal{E}$ whose objects are short exact sequences of R-modules $E : 0 \to A \to B \to C \to 0$ and whose morphisms are triple R-module homomorphisms $(\alpha, \beta, \gamma) : E \to E^{'}$ such that the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ E' : 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

is commutative, where $E : 0 \to A \to B \to C \to 0$ and $E^{'} : 0 \to A^{'} \to B^{'} \to C^{'} \to 0$ are two objects, i.e., two short exact sequences of R-modules.

Definition 2.1.1. *Let $E : 0 \to A \to B \to C \to 0$ and $E' : 0 \to A \to B' \to C \to 0$ be two short exact sequences. We say that E is* ***congruent*** *to $E^{'}$ if there is a morphism $(1_A, \delta, 1_C) : E \to E'$ in $\mathcal{E}$, i.e., there exists an R-module homomorphism $\delta : B \to B'$ such that the diagram*

$$\begin{tikzcd} E : 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & B \arrow[r] \arrow[d, "\delta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E' : 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative.

Remark 2.1.2.

(1) *In Definition* 2.1.1*, because $(1_A, \delta, 1_C) : E \to E'$ is a morphism in the category $\mathcal{E}$, δ is an isomorphism by the Five Lemma. Thus there exists an R-module isomorphism $\delta^{-1} : B' \to B$ such that $\delta\delta^{-1} = 1_{B'}$ and $\delta^{-1}\delta = 1_B$. Then $(1_A, \delta^{-1}, 1_C) : E' \to E$ is a morphism in $\mathcal{E}$, i.e.,the diagram*

$$\begin{tikzcd} E' : 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & B' \arrow[r] \arrow[d, "\delta^{-1}"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative. Hence if E is congruent to $E^{'}$, then E' is congruent to E.

(2) *Let* $E: 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *be a split short exact sequence. Then there exists an* R*-module homomorphism* $f: B \to A$ *such that* $f\sigma = 1_A$. *Suppose* $E': 0 \to A \xrightarrow{\sigma'} B' \xrightarrow{\tau'} C \to 0$ *is a short exact sequence that is congruent to* E. *Then there exists an* R*-module homomorphism* $\delta: B \to B'$ *such that the diagram*

$$\begin{tikzcd} E: 0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, "1_A"] & B \arrow[r, "\tau"] \arrow[d, "\delta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E': 0 \arrow[r] & A \arrow[r, "\sigma'"] & B' \arrow[r, "\tau'"] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative. By (1), *we have an* R*-module homomorphism* $\delta^{-1}: B' \to B$ *such that* $\delta\delta^{-1} = 1_{B'}$, $\delta^{-1}\delta = 1_B$ *and the diagram*

$$\begin{tikzcd} E': 0 \arrow[r] & A \arrow[r, "\sigma'"] \arrow[d, "1_A"] & B' \arrow[r, "\tau'"] \arrow[d, "\delta^{-1}"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E: 0 \arrow[r] & A \arrow[r, "\sigma"] & B \arrow[r, "\tau"] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative. We take $f': B' \to A$ *to be the composition of the homomorphisms* $B' \xrightarrow{\delta^{-1}} B \xrightarrow{f} A$, *i.e.,* $f' = f\delta^{-1}$. *Therefore,* $f'\sigma' = f\delta^{-1}\sigma' = f\sigma = 1_A$. *Hence if* E *is split and* E' *is congruent to* E, *then* E' *is also split.*

Lemma 2.1.3. *Let* $E: 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *be a short exact sequence. Suppose* $\alpha: A \to A'$ *is an* R*-module homomorphism. Then there is a short exact sequence* $E': 0 \to A' \xrightarrow{\sigma'} B' \xrightarrow{\tau'} C \to 0$ *and an* R*-module homomorphism* $\beta: B \to B'$ *such that the diagram*

$$\begin{tikzcd} E: 0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, "\alpha"] & B \arrow[r, "\tau"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E': 0 \arrow[r] & A' \arrow[r, "\sigma'"] & B' \arrow[r, "\tau'"] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative, i.e., $\lambda(\alpha, \beta, 1_c): E \to E'$ *is a morphism in* $\mathcal{E}$. *Moreover, the pair* (λ, E') *is unique up to a congruence of* E'.

Proof. We want to find an R-module B' and R-module homomorphisms σ', τ', β such that $E': 0 \to A' \xrightarrow{\sigma'} B' \xrightarrow{\tau'} C \to 0$ is exact and the diagram

$$\begin{tikzcd} E: 0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, "\alpha"] & B \arrow[r, "\tau"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E': 0 \arrow[r] & A' \arrow[r, "\sigma'"] & B' \arrow[r, "\tau'"] & C \arrow[r] & 0 \end{tikzcd} \tag{1}$$

is commutative. Take $B' = (A' \oplus B)/N$, where $N = \{(-\alpha(a), \sigma(a)) \in A' \oplus B \mid a \in A\}$. Let σ' be the composition of the natural homomorphisms $A' \to A' \oplus B \to (A' \oplus B)/N = B'$, i.e., $\sigma' : A' \to B'$ is the homomorphism defined by $\sigma'(a') = (a', 0) + N$ for all $a' \in A'$; let β be the composition of the natural homomorphisms $B \to A' \oplus B \to A' \oplus B/N = B'$, i.e., $\beta : B \to B'$ is the R-module homomorphism defined by $\beta(b) = (0, b) + N$ for all $b \in B$. Also, because N is contained in the kernel of the composite $A' \oplus B \to B \xrightarrow{\tau} C$, we have an induced R-module homomorphism $\tau' : (A' \oplus B)/N = B' \to C$ with $\tau'((a', b) + N) = \tau(b)$ for all $(a', b) + N \in B'$.

We first show that E' is exact. Note that

- because σ is one-to-one, it is not difficult to see that σ' is also one-to-one;
- because τ is onto, it is not difficult to check that τ' is also onto;
- because $\tau'\sigma'(a') = \tau'((a', 0) + N) = \tau(0) = 0$ for all $a' \in A'$, we have $\operatorname{Im}\sigma' \subseteq \operatorname{Ker}\tau'$.

Therefore, it remains to check that $\operatorname{Ker}\tau' \subseteq \operatorname{Im}\sigma'$. Let $(a', b) + N \in \operatorname{Ker}\tau'$. Then $\tau(b) = \tau'((a', b) + N) = 0$, so $b \in \operatorname{Ker}\tau$. Because E is exact, $\operatorname{Ker}\tau = \operatorname{Im}\sigma$ and so $b = \sigma(a_1)$ for some $a_1 \in A$. Then $(a', b) + N = (a', \sigma(a_1)) + N$. Moreover, because $(a', \sigma(a_1)) - (a' + \alpha(a_1), 0) = (-\alpha(a_1), \sigma(a_1)) \in N$, we have $(a', b) + N = (a', \sigma(a_1)) + N = (a' + \alpha(a_1), 0) + N = \sigma'(a' + \alpha(a_1)) \in \operatorname{Im}\sigma'$. Hence $\operatorname{Ker}\tau' \subseteq \operatorname{Im}\sigma'$ and so E' is exact.

Next, we show that the diagram (1) is commutative, i.e., $\beta\sigma = \sigma'\alpha$ and $\tau'\beta = 1_C\tau$.

- Because $\beta\sigma(a) - \sigma'\alpha(a) = [(0, \sigma(a)) + N] - [(\alpha(a), 0) + N] = (-\alpha(a), \sigma(a)) + N = N$ for all $a \in A$, $\beta\sigma = \sigma'\alpha$.
- Because $\tau'\beta(b) = \tau'((0, b) + N) = \tau(b) = 1_C\tau(b)$ for all $b \in B$, $\tau'\beta = 1_C\tau$.

Hence the diagram (1) is commutative, i.e., $\lambda(\alpha, \beta, 1_C) : E \to E'$ is a morphism in the category $\mathcal{E}$.

Finally, we show the uniqueness of the pair (λ, E'). Suppose that $E_1' : 0 \to A' \xrightarrow{\sigma_1'} B_1' \xrightarrow{\tau_1'} C \to 0$ is a short exact sequence and $\beta_1 : B \to B_1'$ is an R-module homomorphism such that $\lambda_1(\alpha, \beta_1, 1_C) : E \to E_1^{'}$ is a morphism in $\mathcal{E}$, i.e., the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, "\alpha"] & B \arrow[r, "\tau"] \arrow[d, "\beta_1"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E_1^{'} : 0 \arrow[r] & A' \arrow[r, "\sigma_1^{'}"] & B_1^{'} \arrow[r, "\tau_1^{'}"] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative. Consider the R-module homomorphism $\phi : A' \oplus B \to B_1^{'}$ defined by $\phi(a', b) = \sigma_1^{'}(a') + \beta_1(b)$ for all $(a', b) \in A' \oplus B$. Note that for all $a \in A$, $\phi(-\alpha(a'), \sigma(a)) = -\sigma_1^{'}(\alpha(a)) + \beta_1(\sigma(a)) = 0$, since $\sigma_1^{'}\alpha = \beta_1\sigma$. Thus $N \subseteq \operatorname{Ker}\phi$ and so ϕ induces an R-module homomorphism $\delta : B' = (A' \oplus B)/N \to B_1^{'}$ with $\delta((a', b) + N) = \sigma_1^{'}(a') + \beta_1(b)$ for all $(a', b) + N \in B'$. We claim that $(1_{A'}, \delta, 1_C) : E' \to E_1^{'}$ is a morphism in $\mathcal{E}$, i.e., the diagram

$$\begin{tikzcd} E' : 0 \arrow[r] & A' \arrow[r, "\sigma'"] \arrow[d, "1_{A'}"] & B' \arrow[r, "\tau'"] \arrow[d, "\delta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E_1^{'} : 0 \arrow[r] & A' \arrow[r, "\sigma_1^{'}"] & B_1^{'} \arrow[r, "\tau_1^{'}"] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative. We only need to show $\delta\sigma' = \sigma_1^{'}$ and $\tau_1^{'}\delta = \tau'$.

- Because $\delta\sigma'(a') = \delta(a', 0) = \sigma_1^{'}(a')$ for all $a' \in A'$, $\delta\sigma' = \sigma_1^{'}$.

- Note that for all $(a', b) + N \in B'$,

$$\begin{aligned} \tau_1^{'}\delta((a', b) + N) &= \tau_1^{'}(\sigma'(a') + \beta_1(b)) \\ &= \tau_1^{'}\sigma_1^{'}(a') + \tau_1^{'}\beta_1(b) \\ &= 0 + \tau(b) \qquad\qquad \text{(because } \tau_1^{'}\sigma_1^{'} = 0 \text{ and } \tau_1^{'}\beta_1 = \tau) \\ &= \tau'((a', b) + N)\ . \end{aligned}$$

  Thus $\tau_1^{'}\delta = \tau'$.

Hence E is congruent to E' and the proof is complete. $\square$

Lemma 2.1.4. *Let* $E : 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *be a short exact sequence. Suppose* $\gamma : C'' \to C$ *is an* R*-module homomorphism. Then there is a short exact sequence* $E'' : 0 \to A \xrightarrow{\sigma''} B'' \xrightarrow{\tau''} C'' \to 0$ *and an* R*-module homomorphism* $\beta : B'' \to B$ *such that* *the diagram*

$$\begin{tikzcd}
E'' : 0 \arrow[r] & A \arrow[r, "\sigma''"] \arrow[d, "1_A"] & B'' \arrow[r, "\tau''"] \arrow[d, "\beta"] & C'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
E : 0 \arrow[r] & A \arrow[r, "\sigma"] & B \arrow[r, "\tau"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative, i.e., $\theta(1_A, \beta, \gamma) : E'' \to E$ *is a morphism in* $\mathcal{E}$*. Moreover, the pair* (θ, E'') *is unique up to a congruence of* E''*.*

Proof. Similarly as Lemma 2.1.3, we want to find an R-module B'' and R-module homomorphisms σ'', τ'', β such that $E'' : 0 \to A \xrightarrow{\sigma''} B'' \xrightarrow{\tau''} C'' \to 0$ is exact and the diagram

$$\begin{tikzcd}
E'' : 0 \arrow[r] & A \arrow[r, "\sigma''"] \arrow[d, "1_A"] & B'' \arrow[r, "\tau''"] \arrow[d, "\beta"] & C'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
E : 0 \arrow[r] & A \arrow[r, "\sigma"] & B \arrow[r, "\tau"] & C \arrow[r] & 0
\end{tikzcd} \tag{2}$$

is commutative. Take $B'' = \{(b, c'') \in B \oplus C'' \mid \tau(b) = \gamma(c'')\}$, which is a submodule of $B \oplus C''$. Let $\overline{\sigma}$ be the composition of the R-module homomorphisms $A \xrightarrow{\sigma} B \to B \oplus C''$, i.e., $\overline{\sigma} : A \to B \oplus C''$ is the R-module homomorphism defined by $\overline{\sigma}(a) = (\sigma(a), 0)$ for all $a \in A$. Since E is exact, $\tau\sigma(a) = 0 = \gamma(0)$, so $(\sigma(a), 0) \in B''$. Therefore, we can define $\sigma'' : A \to B''$ by $\sigma''(a) = (\sigma(a), 0)$ for all $a \in A$. Also, let $\tau'' : B'' \to C''$ be the composition of the natural homomorphisms $B'' \to B \oplus C'' \to C''$, i.e., $\tau''(b, c'') = c''$ for all $(b, c'') \in B''$, and let $\beta : B'' \to B$ be the composition of the natural homomorphisms $B'' \to B \oplus C'' \to B$, i.e., $\beta(b, c'') = b$ for all $(b, c'') \in B''$.

We first show that E'' is exact.

- Because σ is one-to-one, it is clear that σ'' is also one-to-one.

- For each $c'' \in C''$, since τ is onto and $\gamma(c'') \in C$, $\gamma(c'') = \tau(b)$ for some $b \in B$. Then $(b, c'') \in B''$ and $\tau''(b, c'') = c''$. Hence τ'' is onto.

- Because $\tau''\sigma''(a) = \tau''(\sigma(a), 0) = 0$ for all $a \in A$, $\operatorname{Im}\sigma'' \subseteq \operatorname{Ker}\tau''$.

- Let $(b, c'') \in \operatorname{Ker}\tau'' \subseteq B''$. Then $c'' = \tau''(b, c'') = 0$. Also by the definition of B'', we have $\tau(b) = \gamma(c'') = \gamma(0) = 0$, i.e., $b \in \operatorname{Ker}\,\tau$. Since E is exact, $b = \sigma(a)$ for some $a \in A$. Therefore, $(b, c'') = (\sigma(a), 0) = \sigma''(a) \in \operatorname{Im}\sigma''$. Hence $\operatorname{Ker}\tau'' \subseteq \operatorname{Im}\sigma''$.

Therefore, E'' is exact.

Next, we show that the diagram (2) is commutative, i.e., $\beta\sigma'' = \sigma 1_A$ and $\gamma\tau'' = \tau\beta$.

- $\beta\sigma'' = \sigma 1_A$, because $\beta\sigma''(a) = \beta(\sigma(a), 0) = \sigma(a) = \sigma 1_A(a)$ for all $a \in A$.

- Note that if $(b, c'') \in B''$, $\tau(b) = \gamma(c'')$. Thus $\gamma\tau''(b, c'') - \tau\beta(b, c'') = \gamma(c'') - \tau(b) = 0$ for all $(b, c'') \in B''$, and so $\gamma\tau'' = \tau\beta$.

Hence the diagram (2) is commutative, i.e., $\theta(1_A, \beta, \gamma) : E'' \to E$ is a morphism in the category $\mathcal{E}$.

Finally, we show the uniqueness of the pair (θ, E''). Suppose $E_1'' : 0 \to A \xrightarrow{\sigma_1''} B_1'' \xrightarrow{\tau_1''} C'' \to 0$ is a short exact sequence and $\beta_1 : B'' \to B$ is an R-module homomorphism such that $\theta_1(1_A, \beta_1, \gamma) : E_1{}'' \to E$ is a morphism in $\mathcal{E}$, i.e., the diagram

$$\begin{tikzcd}
E_1'' : 0 \arrow[r] & A \arrow[r, "\sigma_1''"] \arrow[d, "1_A"] & B_1'' \arrow[r, "\tau_1''"] \arrow[d, "\beta_1"] & C'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
E : 0 \arrow[r] & A \arrow[r, "\sigma"] & B \arrow[r, "\tau"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative. Since $\tau(\beta_1(b_1'')) = \gamma(\tau_1''(b_1''))$ for all $b_1'' \in B_1''$, the map $\delta : B_1'' \to B''$ defined by $\delta(b_1'') = (\beta_1(b_1''), \tau_1''(b_1''))$ for all $b_1'' \in B_1''$ is well-defined. We claim that $(1_A, \delta, 1_{C''})$: $E_1'' \to E''$ is a morphism in $\mathcal{E}$, i.e., the diagram

$$\begin{tikzcd}
E_1'' : 0 \arrow[r] & A \arrow[r, "\sigma_1''"] \arrow[d, "1_A"] & B_1'' \arrow[r, "\tau_1''"] \arrow[d, "\delta"] & C'' \arrow[r] \arrow[d, "1_{C''}"] & 0 \\
E'' : 0 \arrow[r] & A \arrow[r, "\sigma''"] & B'' \arrow[r, "\tau''"] & C'' \arrow[r] & 0
\end{tikzcd}$$

is commutative. We only need to show $\delta\sigma_1'' = \sigma''$ and $\tau_1'' = \tau''\delta$.

- For all $a \in A$, we have

$$\begin{aligned}
\delta\sigma_1''(a) &= \delta(\sigma_1''(a)) \\
&= (\beta_1(\sigma_1''(a)), \tau_1''(\sigma_1''(a))) \\
&= (\beta_1\sigma_1''(a), 0) \quad \text{(since } E_1{}'' \text{ is exact)} \\
&= (\sigma(a), 0) \qquad \text{(since } \theta_1(1_A, \beta_1, \gamma) : E_1{}'' \to E \text{ is a morphism in } \mathcal{E}) \\
&= \sigma''(a)\ .
\end{aligned}$$

  Thus $\delta\sigma_1'' = \sigma''$.

- $\tau''\delta(b_1'') = \tau''(\beta_1(b_1''), \tau_1''(b_1'')) = \tau_1''(b_1'')$ for all $b_1'' \in B_1''$, so $\tau''\delta = \tau_1''$.

Hence $E_1{}''$ is congruent to E'' and the proof is complete. $\square$

From Lemma 2.1.3 and Lemma 2.1.4, we know that (λ, E') and (θ, E'') are uniquely determined by α and γ, respectively. We denote E' by αE and E'' by $E\gamma$.

2.2 Connected pairs

Definition 2.2.1. *A* ***connected pair*** (S, E_*, T) *is a pair of additive functors* S, T *together with a function which assigns each short exact sequence* $E: 0 \to A \to B \to C \to 0$ *an* R*-module homomorphism* $E_*: S(C) \to T(A)$ *such that for each morphism* $(\alpha, \beta, \gamma): E \to E'$ *in the category* $\mathcal{E}$*, where* $E: 0 \to A \to B \to C \to 0$ *and* $E': 0 \to A' \to B' \to C' \to 0$ *are short exact sequences of* R*-modules, the diagram*

$$\begin{tikzcd} S(C) \arrow[r, "E_*"] \arrow[d, "S(\gamma)"'] & T(A) \arrow[d, "T(\alpha)"] \\ S(C') \arrow[r, "(E')_*"] & T(A') \end{tikzcd}$$

is commutative.

Proposition 2.2.2. *Let* (S, E_*, T) *be a connected pair and let* $E: 0 \to A \to B \to C \to 0$ *and* $E': 0 \to A \to B' \to C \to 0$ *be two short exact sequences.*

(1) *Suppose* E *is congruent to* E'*. Then* $E_* = (E')_*$*.*

(2) *Suppose* $\alpha: A \to A'$ *is an* R*-module homomorphism. Then* $(\alpha E)_* = T(\alpha)E_*$*.*

(3) *Suppose* $\gamma: C' \to C$ *is an* R*-module homomorphism. Then* $(E\gamma)_* = E_*S(\gamma)$*.*

Proof. For (1), because E is congruent to E', there exists an R-module homomorphism $\delta: B \to B'$ such that $(1_A, \delta, 1_C): E \to E'$ is a morphism in $\mathcal{E}$, i.e., the diagram

$$\begin{tikzcd} E: 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & B \arrow[r] \arrow[d, "\delta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E': 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative. Therefore the diagram

$$\begin{tikzcd} S(C) \arrow[r, "E_*"] \arrow[d, "1_{S(C)}=S(1_C)"'] & T(A) \arrow[d, "T(1_A)=1_{T(A)}"] \\ S(C) \arrow[r, "(E')_*"] & T(A) \end{tikzcd}$$

is commutative, since (S, E_*, T) is a connected pair. Hence we have $E_* = 1_{T(A)}E_* = (E')_* 1_{S(C)} = (E')_*$.

For (2), by Lemma 2.1.3, there is a morphism $(\alpha, \beta, 1_C) : E \to \alpha E$ in the category $\mathcal{E}$. Therefore $(\alpha E)_* = (\alpha E)_* S(1_C) = T(\alpha)E_*$, since (S, E_*, T) is a connected pair.

Similarly for (3), by Lemma 2.1.4, there is a morphism $(1_A, \beta', \gamma) : E\gamma \to E$ in the category $\mathcal{E}$, so $(E\gamma)_* = T(1_A)(E\gamma)_* = E_*S(\gamma)$. □

Corollary 2.2.3.

(1) *Let* $E : 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *be a short exact sequence. Then* σE *and* $E\tau$ *are split.*

(2) *Let* $E_1 : 0 \to A_1 \xrightarrow{\sigma_1} B_1 \xrightarrow{\tau_1} C_1 \to 0$ *be a split short exact sequence. Suppose* (S, E_*, T) *is a connected pair. Then* $(E_1)_* = 0$.

Proof. For (1), we consider the split short exact sequence $E' : 0 \to B \xrightarrow{i} B \oplus C \xrightarrow{\pi} C \to 0$ and define $\beta : B \to B \oplus C$ by $\beta(b) = (b, \tau(b))$ for all $b \in B$. Using the fact that E is exact, it is not difficult to check that the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, "\sigma"] & B \arrow[r, "\tau"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ E' : 0 \arrow[r] & B \arrow[r, "i"] & B \oplus C \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd}$$

is commutative, i.e., $(\sigma, \beta, 1_C) : E \to E'$ is a morphism in $\mathcal{E}$. By Lemma 2.1.3, E' is congruent to σE. Hence σE is split by Remark 2.1.2. Similarly for $E\tau$, we consider the split short exact sequence $E'' : 0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ and define $\delta : A \oplus B \to B$ by $\delta(a, b) = \sigma(a) + b$ for all $(a, b) \in A \oplus B$. Because E is exact, it is also not difficult to check that the diagram

$$\begin{tikzcd}
E'' : 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "1_A"] & A \oplus B \arrow[r, "\pi"] \arrow[d, "\delta"] & B \arrow[r] \arrow[d, "\tau"] & 0 \\
E : 0 \arrow[r] & A \arrow[r, "\sigma"] & B \arrow[r, "\tau"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative. By Lemma 2.1.4, E'' is congruent to $E\tau$. Hence $E\tau$ is split by Remark 2.1.2.

Next, we show (2). Since E_1 is split, there is an R-module homomorphism $\delta_1 : B_1 \to A_1$ such that $\delta_1\sigma_1 = 1_{A_1}$. Consider the short exact sequence $E_2 : 0 \to A_1 \stackrel{1_{A_1}}{\to} A_1 \to 0 \to 0$. Then the diagram

$$\begin{tikzcd}
E_1 : 0 \arrow[r] & A_1 \arrow[r, "\sigma_1"] \arrow[d, "1_{A_1}"] & B_1 \arrow[r, "\tau_1"] \arrow[d, "\delta_1"] & C_1 \arrow[r] \arrow[d, "0"] & 0 \\
E_2 : 0 \arrow[r] & A_1 \arrow[r, "1_{A_1}"] & A_1 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

is commutative, i.e., $(1_{A_1}, \delta_1, 0) : E_1 \to E_2$ is a morphism in the category $\mathcal{E}$. Because (S, E_*, T) is a connected pair, the diagram

$$\begin{tikzcd}
S(C_1) \arrow[r, "(E_1)_*"] \arrow[d, "0=S(0)"'] & T(A_1) \arrow[d, "T(1_{A_1})=1_{T(A_1)}"] \\
S(0) \arrow[r, "(E_2)_*"] & T(A_1)
\end{tikzcd}$$

is commutative. Hence $(E_1)_* = 0$. $\square$

Remark 2.2.4. *By Proposition* 2.2.2 *and Corollary* 2.2.3*, for every short exact sequence* $0 \to A \stackrel{\sigma}{\to} B \stackrel{\tau}{\to} C \to 0$ *and for every connected pair* (S, E_*, T)*, we have a complex*

$$S(A) \stackrel{S(\sigma)}{\to} S(B) \stackrel{S(\tau)}{\to} S(C) \stackrel{E_*}{\to} T(A) \stackrel{T(\sigma)}{\to} T(B) \stackrel{T(\tau)}{\to} T(C)\ .$$

More precisely, $T(\sigma)E_* = (\sigma E)_* = 0$*, since* σE *is split;* $E_*S(\tau) = (E\tau)_* = 0$*, since* $E\tau$ *is split.*

Definition 2.2.5.

(1) *Let* S *and* S' *be two additive functors. A* ***natural transformation*** $f : S \to S'$ *is a function that assigns every* R*-module* C *an* R*-module homomorphism* $f(C)$:

$S(C) \to S'(C)$ *such that if* $\gamma : C \to C_1$ *is an* R*-module homomorphism, then the diagram*

$$\begin{tikzcd} S(C) \arrow[r, "S(\gamma)"] \arrow[d, "f(C)"'] & S(C_1) \arrow[d, "f(C_1)"] \\ S'(C) \arrow[r, "S'(\gamma)"] & S'(C_1) \end{tikzcd}$$

is commutative.

(2) *Let* (S, E_*, T) *and* $(S', {E_*}', T')$ *be two connected pairs, and let* $E : 0 \to A \to B \to C \to 0$ *be a short exact sequence of* R*-modules. A* ***morphism*** $(f, g) : (S, E_*, T) \to (S', {E_*}', T')$ ***of connected pairs*** *is a pair of natural transformations* $f : S \to S'$ *and* $g : T \to T'$ *such that the diagram*

$$\begin{tikzcd} S(C) \arrow[r, "E_*"] \arrow[d, "f(C)"'] & T(A) \arrow[d, "g(A)"] \\ S'(C) \arrow[r, "{E_*}'"] & T'(A) \end{tikzcd}$$

is commutative.

(3) *A connected pair* (S, E_*, T) *is said to be* ***right universal*** *if for every connected pair* $(S', {E_*}', T')$ *and for every natural transformation* $f : S \to S'$*, there is a unique natural transformation* $g : T \to T'$ *such that* $(f, g) : (S, E_*, T) \to (S', {E_*}', T')$ *is a morphism of connected pairs.*

In this thesis, we only use the definition of being right universal for connected pairs, so we skip the definition of being left universal for connected pairs.

Theorem 2.2.6. *Let* (S, E_*, T) *be a connected pair. Suppose that for every short exact sequence* $0 \to A \to I \to M \to 0$ *with* I *an injective* R*-module, the sequence* $S(I) \to S(M) \to T(A) \to 0$ *is exact. Then* (S, E_*, T) *is right universal.*

Proof. Let $(S', {E_*}', T')$ be a connected pair and let $f : S \to S'$ be a natural transformation. We want to show that there exists a unique natural transformation $g : T \to T'$ such that $(f, g) : (S, E_*, T) \to (S', {E_*}', T')$ is a morphism of connected pairs. We first show the existence of g. Let A be an R-module. Then A can be embedded into an injective R-module I, so we have a short exact sequence $E_A : 0 \to A \xrightarrow{i} I \xrightarrow{\pi} M \to 0$, where $M = I/i(A)$. By the assumption, $S(I) \xrightarrow{S(\pi)} S(M) \xrightarrow{(E_A)_*} T(A) \longrightarrow 0$ is exact. By Remark 2.2.4, $S'(I) \xrightarrow{S'(\pi)} S'(M) \xrightarrow{{(E_A)_*}'} T'(A) \xrightarrow{T'(i)} T'(I)$ is a complex. And because $f : S \to S'$ is a natural transformation, the diagram

$$\begin{tikzcd}
S(I) \arrow[r, "S(\pi)"] \arrow[d, "f(I)"'] & S(M) \arrow[d, "f(M)"] \\
S'(I) \arrow[r, "S'(\pi)"] & S'(M)
\end{tikzcd}$$

is commutative. Therefore we have a commutative diagram

$$\begin{tikzcd}
S(I) \arrow[r, "S(\pi)"] \arrow[d, "f(I)"] & S(M) \arrow[r, "(E_A)_*"] \arrow[d, "f(M)"] & T(A) \arrow[r] & 0 \\
S'(I) \arrow[r, "S'(\pi)"] & S'(M) \arrow[r, "{(E_A)_*}'"] & T'(A) \arrow[r, "T'(i)"] & T'(I)
\end{tikzcd} ,$$

where the upper row is exact and the lower row is a complex. By the above commutative diagram, we have ${(E_A)_*}'f(M)S(\pi) = {(E_A)_*}'S'(\pi)f(I) = 0$, and so ${(E_A)_*}'f(M)[\mathrm{Ker}(E_A)_*]$ $= {(E_A)_*}'f(M)[\mathrm{Im}S(\pi)] = 0$. Thus $\mathrm{Ker}(E_A)_* \subseteq \mathrm{Ker}[{(E_A)_*}'f(M)]$. Moreover, because $T(A) \cong S(M)/\mathrm{Ker}(E_A)_*$, there exists an R-module homomorphism $g(A) : T(A) \to T'(A)$ such that

$$g(A)(E_A)_* = {(E_A)_*}'f(M). \tag{1}$$

Hence, for every R-module A, we assign A an R-module homomorphism $g(A) : T(A) \to T'(A)$ such that $g(A)(E_A)_* = {(E_A)_*}'f(M)$.

Next, we show that $g : T \to T'$ is a natural transformation. Let $\alpha : A \to A_1$ be an R-module homomorphism. We want to show that the diagram

$$\begin{tikzcd}
T(A) \arrow[r, "T(\alpha)"] \arrow[d, "g(A)"'] & T(A_1) \arrow[d, "g(A_1)"] \\
T'(A) \arrow[r, "T'(\alpha)"] & T'(A_1)
\end{tikzcd}$$

is commutative, i.e., $g(A_1)T(\alpha) = T'(\alpha)g(A)$. Similarly as above, we have a short exact sequence $E_{A_1} : 0 \to A_1 \xrightarrow{i_1} I_1 \xrightarrow{\pi_1} M_1 \to 0$, where I_1 is injective, and a commutative diagram

$$\begin{tikzcd}
S(I_1) \arrow[r, "S(\pi_1)"] \arrow[d, "f(I_1)"] & S(M_1) \arrow[r, "(E_{A_1})_*"] \arrow[d, "f(M_1)"] & T(A_1) \arrow[r] \arrow[d, "g(A_1)"] & 0 \\
S'(I_1) \arrow[r, "S'(\pi_1)"] & S'(M_1) \arrow[r, "{(E_{A_1})_*}'"] & T'(A_1) \arrow[r] & T'(I_1)
\end{tikzcd},$$

where the upper row is exact and the lower row is a complex. In particular, we have

$$(E_{A_1})_*{}'f(M_1) = g(A_1)(E_{A_1})_* \ . \tag{2}$$

We consider the diagram

$$\begin{tikzcd}
E_A : 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & I \arrow[r, "\pi"] & M \arrow[r] & 0 \\
E_{A_1} : 0 \arrow[r] & A_1 \arrow[r, "i_1"] & I_1 \arrow[r, "\pi_1"] & M_1 \arrow[r] & 0
\end{tikzcd}.$$

Note that because I_1 is injective, there exists an R-module homomorphism $\beta : I \to I_1$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "i_1\alpha"'] & I \arrow[dl, "\beta"] \\
& I_1 &
\end{tikzcd}$$

is commutative, i.e., $\beta i = i_1\alpha$. Also, since $\pi_1\beta i = \pi_1 i_1\alpha = 0$, $\operatorname{Im} i \subseteq \operatorname{Ker} \pi_1\beta$. Thus there exists an R-module homomorphism $\gamma : M \to M_1$ such that $\gamma\pi = \pi_1\beta$. Therefore, we have a commutative diagram

$$\begin{tikzcd}
E_A : 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & I \arrow[r, "\pi"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\gamma"] & 0 \\
E_{A_1} : 0 \arrow[r] & A_1 \arrow[r, "i_1"] & I_1 \arrow[r, "\pi_1"] & M_1 \arrow[r] & 0
\end{tikzcd},$$

i.e., $(\alpha, \beta, \gamma) : E_A \to E_{A_1}$ is a morphism in the category $\mathcal{E}$. Since (S, E_*, T) and $(S', E_*{}', T')$ are connected pairs,

$$T(\alpha)(E_A)_* = (E_{A_1})_* S(\gamma) \quad \text{and} \quad T'(\alpha)(E_A)_*{}' = (E_{A_1})_*{}' S'(\gamma) \ . \tag{3}$$

Moreover, because $f : S \to S'$ is a natural transformation,

$$S'(\gamma)f(M) = f(M_1)S(\gamma). \tag{4}$$

Thus we have

$$\begin{aligned}
T'(\alpha)g(A)(E_A)_* &= T'(\alpha)(E_A)_*{}'f(M) && \big(\text{by}(1)\big)\\
&= (E_{A_1})_*{}'S'(\gamma)f(M) && \big(\text{by}(3)\big)\\
&= (E_{A_1})_*{}'f(M_1)S(\gamma) && \big(\text{by}(4)\big)\\
&= g(A_1)(E_{A_1})_*S(\gamma) && \big(\text{by}(2)\big)\\
&= g(A_1)T(\alpha)(E_A)_* && \big(\text{by}(3)\big)\ .
\end{aligned}$$

Hence $T'(\alpha)g(A) = g(A_1)T(\alpha)$, since $(E_A)_*$ is onto.

Next we show that $(f, g) : (S, E_*, T) \to (S', E_*{}', T')$ is a morphism of connected pairs, i.e., if $E : 0 \to A \to B \to C \to 0$ is a short exact sequence, then the diagram

$$\begin{tikzcd}
S(C) \arrow[r, "E_*"] \arrow[d, "f(C)"'] & T(A) \arrow[d, "g(A)"] \\
S'(C) \arrow[r, "E_*{}'"] & T'(A)
\end{tikzcd}$$

is commutative. So far we already show that for each R-module A with the special short exact sequence $E_A : 0 \to A \to I \to M \to 0$, the diagram

$$\begin{tikzcd}
S(M) \arrow[r, "(E_A)_*"] \arrow[d, "f(M)"'] & T(A) \arrow[d, "g(A)"] \\
S'(M) \arrow[r, "(E_A)_*{}'"] & T'(A)
\end{tikzcd}$$

is commutative. Let $E : 0 \to A \to B \to C \to 0$ be a short exact sequence. We consider the diagram

$$\begin{tikzcd}
E : 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & B \arrow[r] & C \arrow[r] & 0 \\
E_A : 0 \arrow[r] & A \arrow[r] & I \arrow[r] & M \arrow[r] & 0\,.
\end{tikzcd}$$

Because I is injective and because the map $B \to C$ is onto, similar as above, there exist R-module homomorphisms $\mu : B \to I$ and $\nu : C \to M$ such that $(1_A, \mu, \nu) : E \to E_A$ is a morphism in the category $\mathcal{E}$, i.e, the diagram

$$\begin{tikzcd}
E : 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & B \arrow[r] \arrow[d, "\mu"] & C \arrow[r] \arrow[d, "\nu"] & 0 \\
E_A : 0 \arrow[r] & A \arrow[r] & I \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

is commutative. By the fact that (S, E_*, T) and $(S', E_*{}', T')$ are connected pairs, we have $E_* = (E_A)_* S(\nu)$ and $E_*{}' = (E_A)_*{}' S'(\nu)$. Moreover, $f(M)S(\nu) = S'(\nu)f(C)$, since f is a natural transformation. Therefore,

$$g(A)E_* = g(A)(E_A)_* S(\nu) = (E_A)_*{}' f(M)S(\nu) = (E_A)_*{}' S'(\nu)f(C) = E_*{}' f(C).$$

Hence (f, g) is a morphism of connected pairs.

Finally, we show that g is unique. Suppose that $g_1 : T \to T'$ is a natural transformation such that $(f, g_1) : (S, E_*, T) \to (S', E_*{}', T')$ is a morphism of connected pairs, i.e., $g_1(A)E_* = E_*{}' f(C)$. Then $g_1(A)E_* = E_*{}' f(C) = g(A)E_*$. Thus $g_1(A) = g(A)$ for every R-module A, since E_* is onto. Hence $g_1 = g$ and this proof is complete. $\square$

2.3 Connected sequences and universal connected sequences

Definition 2.3.1.

(1) *Let $\{T^n\}_{n\geq 0}$ be a family of additive functors and let $\{E^n\}_{n\geq 0}$ be a family of functions. A* ***connected sequence*** *$\{T^n, E^n\}_{n\geq 0}$ is a sequence $\{\cdots, T^n, E^n, T^{n+1}, \cdots\}$ such that each pair (T^n, E^n, T^{n+1}) is a connected pair for all $n \geq 0$.*

(2) *A connected sequence $\{T^n, E^n\}_{n\geq 0}$ is said to be* ***universal*** *if for every connected sequence $\{T'^n, E'^n\}_{n\geq 0}$ and for every natural transformation $f^0 : T^0 \to T'^0$, there is a unique family of natural transformations $\{f^n\}_{n\geq 1}$ such that $f^{n+1}E^n = E'^n f^n$ for all $n \geq 0$, i.e., for each short exact sequence of R-modules $E : 0 \to A \to B \to C \to 0$, the diagram*

$$\begin{array}{ccc} T^n(C) & \xrightarrow{E^n} & T^{n+1}(A) \\ {\scriptstyle f^n(C)}\downarrow & & \downarrow{\scriptstyle f^{n+1}(A)} \\ T'^n(C) & \xrightarrow{E'^n} & T'^{n+1}(A) \end{array}$$

is commutative.

Remark 2.3.2. *A connected sequence $\{T^n, E^n\}_{n\geq 0}$ assigns every short exact sequence $E : 0 \to A \to B \to C \to 0$ a complex*

$$\cdots \to T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{E^n} T^{n+1}(A) \to \cdots .$$

This follows from the definition of connected sequences and Remark 2.2.4.

Theorem 2.3.3. *Let $\{T^n, E^n\}_{n\geq 0}$ be a connected sequence. Suppose that for every short exact sequence $0 \to A \to I \to M \to 0$ with I an injective R-module, the sequence $T^n(I) \to T^n(M) \to T^{n+1}(A) \to 0$ is exact for all $n \geq 0$. Then $\{T^n, E^n\}_{n\geq 0}$ is universal.*

Proof. Let $\{T'^n, E'^n\}_{n\geq 0}$ be a connected sequence and let $f^0 : T^0 \to T'^0$ be a natural transformation. We want to show that there exists a unique family of natural transformations $\{f^n\}_{n\geq 1}$ such that $f^{n+1}E^n = E'^n f^n$ for all $n \geq 0$. First note that for every $n \geq 0$, (T^n, E^n, T^{n+1}) is a connected pair. Since for every short exact sequence $0 \to A \to I \to M \to 0$ with I an injective R-module, the sequence $T^n(I) \to T^n(M) \to T^{n+1}(A) \to 0$ is exact for all $n \geq 0$, by Theorem 2.2.6, (T^n, E^n, T^{n+1}) is right universal for all $n \geq 0$.

Now, we show the existence of $\{f^n\}_{n\geq 1}$ by induction on n. For $n = 1$, because we have a right universal connected pair (T^0, E^0, T^1) and a natural transformation $f^0 : T^0 \to T'^0$, by Definition 2.2.5, there is a unique natural transformation $f^1 : T^1 \to T'^1$ such that $(f^0, f^1) : (T^0, E^0, T^1) \to (T'^0, E'^0, T'^1)$ is a morphism of connected pairs, i.e., for every short exact sequence of R-modules $E : 0 \to A \to B \to C \to 0$, the diagram

$$\begin{tikzcd} T^0(C) \arrow[r, "E^0"] \arrow[d, "f^0(C)"'] & T^1(A) \arrow[d, "f^1(A)"] \\ T'^0(C) \arrow[r, "E'^0"] & T'^1(A) \end{tikzcd}$$

is commutative, i.e., $f^1E^0 = E'^0 f^0$.

Next, suppose $n \geq 2$ and suppose for $k = 1, 2, \cdots, n-1$, there exist natural transformations $f^k : T^k \to T'^k$ such that $f^k E^{k-1} = E'^{k-1} f^{k-1}$. Because the connected pair (T^{n-1}, E^{n-1}, T^n) is right universal and $f^{n-1} : T^{n-1} \to T'^{n-1}$ is a natural transformation, again by Definition 2.2.5, there is a unique natural transformation $f^n : T^n \to T'^n$ such that $(f^{n-1}, f^n) : (T^{n-1}, E^{n-1}, T^n) \to (T'^{n-1}, E'^{n-1}, T'^n)$ is a morphism of connected pairs, i.e., $f^n E^{n-1} = E'^{n-1} f^{n-1}$. Hence $\{f^n\}_{n\geq 1}$ is a family of natural transformations such that $f^{n+1}E^n = E'^n f^n$ for all $n \geq 0$.

Finally, we show the uniqueness of $\{f^n\}_{n\geq 1}$. In the above proof, we know that f^{n+1} is uniquely determined by f^n for all $n \geq 0$. Since f^0 is given, the proof is complete. $\square$

In the next corollary, we use Theorem 2.3.3 to give some other conditions that guarantees a connected sequence $\{T^n, E^n\}_{n\geq 0}$ to be universal. Later in this thesis, we will use the following corollary to prove that certain connected sequences are universal.

Corollary 2.3.4. *Let $\{T^n, E^n\}_{n\geq 0}$ be a connected sequence and suppose that*

- *$T^n(I) = 0$ for all injective R-modules I and $n > 0$, and that*
- *for every short exact sequence $E : 0 \to A \to B \to C \to 0$, the assigned complex*

$$\cdots \to T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{E^n} T^{n+1}(A) \to \cdots$$

  is exact.

Then $\{T^n, E^n\}_{n\geq 0}$ is universal.

Proof. Let $E : 0 \to A \to I \to M \to 0$ be a short exact sequence with I an injective R-module. By assumption, we have an exact sequence

$$\cdots \to T^n(A) \to T^n(I) \to T^n(M) \xrightarrow{E^n} T^{n+1}(A) \to \cdots .$$

Since $T^n(I) = 0$ for all $n > 0$, we see that $T^n(I) \to T^n(M) \to T^{n+1}(A) \to 0$ is exact for all $n \geq 0$. By Theorem 2.3.3, $\{T^n, E^n\}_{n\geq 0}$ is universal. $\square$

Lemma 2.3.5. *Let $\{T^n, E^n\}_{n\geq 0}$ and $\{T'^n, E'^n\}_{n\geq 0}$ be two connected sequences. Suppose $\{T^n, E^n\}_{n\geq 0}$ and $\{T'^n, E'^n\}_{n\geq 0}$ are universal with $T^0 = T'^0$. Then $T^n(A) \cong T'^n(A)$ for all R-module A and $n \geq 0$.*

Proof. Because the connected sequence $\{T^n, E^n\}_{n\geq 0}$ is universal and $f^0 = 1 : T^0 \to T'^0$ is a natural transformation, by Definition 2.3.1, there exists a family of natural transformations $\{f^n\}_{n\geq 1}$ such that $f^{n+1}E^n = E'^n f^n$ for all $n \geq 0$, i.e., for each short exact sequence of R-modules $E : 0 \to A \to B \to C \to 0$, the diagram

$$\begin{tikzcd}
T^n(C) \arrow[r, "E^n"] \arrow[d, "f^n(C)"'] & T^{n+1}(A) \arrow[d, "f^{n+1}(A)"] \\
T'^n(C) \arrow[r, "E'^n"] & T'^{n+1}(A)
\end{tikzcd}$$

is commutative for all $n \geq 0$. Similarly, since the connected sequence $\{T'^n, E'^n\}_{n\geq 0}$ is universal and $f'^0 = 1 : T'^0 \to T^0$ is a natural transformation, there exists a family of natural transformations $\{f'^n\}_{n\geq 1}$ such that $f'^{n+1}{E_A}'^n = {E_A}^n f'^n$ for all $n \geq 0$, i.e., for each short exact sequence of R-modules $E : 0 \to A \to B \to C \to 0$, the diagram

$$\begin{tikzcd} T'^n(C) \arrow[r, "E'^n"] \arrow[d, "f'^n(C)"'] & T'^{n+1}(A) \arrow[d, "f'^{n+1}(A)"] \\ T^n(C) \arrow[r, "E^n"] & T^{n+1}(A) \end{tikzcd}$$

is commutative for all $n \geq 0$. Therefore, the diagram

$$\begin{tikzcd} T^n(C) \arrow[r, "E^n"] \arrow[d, "f^n(C)"'] & T^{n+1}(A) \arrow[d, "f^{n+1}(A)"] \\ T'^n(C) \arrow[r, "E'^n"] \arrow[d, "f'^n(C)"'] & T'^{n+1}(A) \arrow[d, "f'^{n+1}(A)"] \\ T^n(C) \arrow[r, "E^n"] & T^{n+1}(A) \end{tikzcd}$$

is commutative for all $n \geq 0$, i.e., $f'^{n+1}f^{n+1}E^n = E^n f'^n f^n$ for all $n \geq 0$. Hence for the connected sequence $\{T^n, E^n\}_{n\geq 0}$ and the natural transformation $1_{T^0} = f'^0 f^0 : T^0 \to T^0$, $\{f'^n f^n\}_{n\geq 1}$ is a family of natural transformations such that $(f'^{n+1}f^{n+1})E^n = E^n(f'^n f^n)$ for all $n \geq 0$. On the other hand, it is clear that $\{1_{T^n}\}_{n\geq 1}$ is also a family of natural transformations such that $1_{T^{n+1}}E^n = E^n\, 1_{T^n}$ for all $n \geq 0$. Since the connected sequence $\{T^n, E^n\}_{n\geq 0}$ is universal, $\{f'^n f^n\}_{n\geq 1}$=$\{1_{T^n}\}_{n\geq 1}$, i.e.,

$$f'^n f^n = 1_{T^n} \text{ for all } n \geq 1.$$

Similarly, because for each short exact sequence of R-modules $E : 0 \to A \to B \to C \to 0$, the diagram

$$\begin{tikzcd} T'^n(C) \arrow[r, "E'^n"] \arrow[d, "f'^n(C)"'] & T'^{n+1}(A) \arrow[d, "f'^{n+1}(A)"] \\ T^n(C) \arrow[r, "E^n"] \arrow[d, "f^n(C)"'] & T^{n+1}(A) \arrow[d, "f^{n+1}(A)"] \\ T'^n(C) \arrow[r, "E'^n"] & T'^{n+1}(A) \end{tikzcd}$$

is commutative for all $n \geq 0$, i.e., $f^{n+1}f'^{n+1}E'^n = E'^n f^n f'^n$ for all $n \geq 0$. Thus for the connected sequence $\{T'^n, E'^n\}_{n\geq 0}$ and the natural transformation $1_{T'^0} = f^0 f'^0$:

$T'^0 \to T'^0$, $\{f^n f'^n\}_{n\geq 1}$ is a family of natural transformations such that $(f^{n+1} f'^{n+1})E'^n = E'^n(f^n f'^n)$ for all $n \geq 0$. Again, it is clear that $\{1_{T'^n}\}_{n\geq 1}$ is a family of natural transformations such that $1_{T'^{n+1}} E'^n = E'^n 1_{T'^n}$ for all $n \geq 0$. Since the connected sequence $\{T'^n, E'^n\}_{n\geq 0}$ is universal, $\{f^n f'^n\}_{n\geq 1}$=$\{1_{T'^n}\}_{n\geq 1}$, i.e.,

$$f^n f'^n = 1_{T'^n} \text{ for all } n \geq 1.$$

Therefore, $f^n(A) : T^n(A) \to T'^n(A)$ is an isomorphism for all R-module A and $n \geq 0$. Hence $T^n(A) \cong T'^n(A)$ for all $n \geq 0$. $\square$

3 Local Cohomology and Čech Complexes

3.1 Local cohomology vs universal connected sequences

In this section, we will show that $\{H^n_{\mathbf{m}}(-), E^n\}_{n\geq 0}$ is a universal connected sequence. In order to do so, we need some properties of the right derived functors and cohomology.

Definition 3.1.1. ([3], 10.1) *Let $\mathcal{X}$ and $\mathcal{Y}$ be cochain complexes.*

(1) *A cochain map $f : \mathcal{X} \to \mathcal{Y}$ is said to be **null-homotopic** if for all n, there exist maps $s_n : X_n \to Y_{n-1}$ such that $\delta_n s_n + s_{n+1}\lambda_{n+1} = f_n$, where λ_n and δ_n are the boundary maps in $\mathcal{X}$ and $\mathcal{Y}$, respectively.*

(2) *Cochain maps $f, g : \mathcal{X} \to \mathcal{Y}$ are said to be **cochain homotopic**, and denoted by $f \sim g$, if $f - g$ is null homotopic .*

(3) *A cochain map $f : \mathcal{X} \to \mathcal{Y}$ is called a **homotopy equivalence** if there exists a cochain map $g : \mathcal{Y} \to \mathcal{X}$ such that $gf \sim 1_{\mathcal{X}}$ and $fg \sim 1_{\mathcal{Y}}$. In this case we say that $\mathcal{X}$ and $\mathcal{Y}$ are **homotopy equivalent**.*

Remark 3.1.2. ([3], 10.1) *It is well-known that if a cochain map $f : \mathcal{X} \to \mathcal{Y}$ is null-homotopic, then the induced map f^* on cohomology is the zero map.*

Let A be an R-module and let T be a left exact additive functor. The nth right derived functor R^nT associate to T is defined as the following.

- For an R-module A, $R^nT(A) = H^n(T(\mathcal{I}_A))$, where $\mathcal{I}_A$ is a cochain complex obtained by deleting A from an injective resolution $\mathcal{I}$ of A and $T(\mathcal{I}_A)$ is the cochain complex obtained by applying T to every term in $\mathcal{I}_A$.

- For an R-module homomorphism $f : A \to B$,

$$R^nT(f) = (T\hat{f})^* : H^n(T(\mathcal{I}_A)) \to H^n(T(\mathcal{I}_B))$$

  is the map on cohomology induced by the cochain map $T\hat{f} : T(\mathcal{I}_A) \to T(\mathcal{I}_B)$, where $\hat{f} : \mathcal{I}_A \to \mathcal{I}_B$ is a cochain map lifting f via the Comparison Theorem.

Proposition 3.1.3. ([3], 10.5)(***Comparison Theorem***) *Let A and B be R-modules. Let $h : A \to B$ be an R-module homomorphism. Suppose that*

$$\begin{array}{l} \mathcal{N} : 0 \longrightarrow A \xrightarrow{\phi_0} N_0 \xrightarrow{\phi_1} N_1 \longrightarrow \cdots \\ \qquad\quad\ \ \downarrow h \\ \mathcal{Q} : 0 \longrightarrow B \xrightarrow{\psi_0} Q_0 \xrightarrow{\psi_1} Q_1 \longrightarrow \cdots \end{array}$$

is a diagram of complexes with Q_n injective for each $n \geq 0$ and $\mathcal{N}$ exact. Then there exist maps $h_n : N_n \to Q_n$ making the diagram commute. In other words, there exists a cochain map $\hat{h} : \mathcal{N} \to \mathcal{Q}$ lifting h.

Note that if $\mathcal{I}$ and $\mathcal{I}'$ are two cochain complexes obtained by deleting A from injective resolutions of A, then $\mathcal{I}$ and $\mathcal{I}'$ are homotopy equivalent. Moreover, T carries homotopy equivalent cochain complexes to homotopy equivalent cochain complexes and carries chain homotopic cochain maps to chain homotopic cochain maps. Hence R^nT does not depend on the choice of injective resolutions.

Lemma 3.1.4. ([3], 11.9) *Let T be a left exact additive functor.*

(1) *R^nT is an additive functor for all $n \geq 0$.*

(2) *$R^0T(A) = T(A)$ and $R^0T(f) = T(f)$, for all R-modules A and R-module homomorphisms f.*

(3) *For each short exact sequence of R-modules $E : 0 \to A \to B \to C \to 0$, there is a long exact sequence of cohomology*

$$0 \to R^0T(A) \to R^0T(B) \to R^0T(C) \to R^1T(A) \to \cdots .$$

(4) *Let $E : 0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ be a split short exact sequence of R-modules. Then the sequence $T(E) : 0 \to T(A) \xrightarrow{T(\sigma)} T(B) \xrightarrow{T(\tau)} T(C) \to 0$ is exact.*

Now we know that R^nT is an additive functor for all $n \geq 0$ with $R^0T = T$. We let E^n denote the connecting maps $R^nT(C) \to R^{n+1}T(A)$ for all $n \geq 0$.

Lemma 3.1.5. ([4], 2.4.1) *Let $E : 0 \to \mathcal{K} \xrightarrow{\phi} \mathcal{L} \xrightarrow{\psi} \mathcal{M} \to 0$ be a short exact sequence of cochain complexes, where ϕ and ψ are cochain maps. Then there is a long exact sequence of cohomology*

$$\cdots \to H^n(\mathcal{K}) \xrightarrow{(\phi^n)^*} H^n(\mathcal{L}) \xrightarrow{(\psi^n)^*} H^n(\mathcal{M}) \longrightarrow H^{n+1}(\mathcal{K}) \to \cdots$$

where $(\phi^n)^*$ and $(\psi^n)^*$ are the maps of cohomology induced by the R-module homomorphisms $\phi^n : \mathcal{K}^n \to \mathcal{L}^n$ and $\psi^n : \mathcal{L}^n \to \mathcal{M}^n$, respectively, for all n.

We also let E^n denote the connecting map $H^n(\mathcal{M}) \to H^{n+1}(\mathcal{K})$ for all n.

Lemma 3.1.6. ([4], 2.4.2) *Let $E : 0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{M} \to 0$ and $E' : 0 \to \mathcal{K}' \to \mathcal{L}' \to \mathcal{M}' \to 0$ be two short exact sequences of cochain complexes. Suppose that there are three cochain maps $f : \mathcal{K} \to \mathcal{K}'$, $g : \mathcal{L} \to \mathcal{L}'$, and $h : \mathcal{M} \to \mathcal{M}'$ such that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, "f"] & \mathcal{L} \arrow[r] \arrow[d, "g"] & \mathcal{M} \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \mathcal{K}' \arrow[r] & \mathcal{L}' \arrow[r] & \mathcal{M}' \arrow[r] & 0 \end{tikzcd}$$

is commutative. Then the diagram

$$\begin{tikzcd} H^n(\mathcal{M}) \arrow[r, "E^n"] \arrow[d, "(h^n)^*"'] & H^{n+1}(\mathcal{K}) \arrow[d, "(f^{n+1})^*"] \\ H^n(\mathcal{M}') \arrow[r, "(E')^n"] & H^{n+1}(\mathcal{K}') \end{tikzcd}$$

is commutative for all n.

Proposition 3.1.7. *Let $0 \to A \xrightarrow{\sigma_0} B \xrightarrow{\tau_0} C \to 0$ be a short exact sequence of R-modules. Then there is a short exact sequence of complexes $0 \to \mathcal{I}_A \to \mathcal{I}_B \to \mathcal{I}_C \to 0$, where $\mathcal{I}_A, \mathcal{I}_B$, $\mathcal{I}_C$ are injective resolutions of A, B, C, respectively.*

Proof. First, we embed B and C into injective R-modules A_0 and C_0, respectively, and let $\phi_0 : B \to A_0$ and $h_0 : C \to C_0$ be embedding maps. Then the composite $f_0 = \phi_0\sigma_0 : A \to A_0$ is an R-module monomorphism, since ϕ_0 and σ_0 are both R-module monomorphisms. Now we define $g_0 : B \to A_0 \oplus C_0$ by $g_0(b) = (\phi_0(b), h_0\tau_0(b))$ for all

$b \in B$. Then it is not difficult to check that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\sigma_0"] \arrow[d, "f_0"] & B \arrow[r, "\tau_0"] \arrow[d, "g_0"] & C \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "i"] & A_0 \oplus C_0 \arrow[r, "\pi"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

is commutative. Moreover, because f_0 and h_0 are both one-to-one, g_0 is one-to-one, by the Five Lemma. Also, $A_0 \oplus C_0$ is injective since A_0 and C_0 are injective. Then the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\sigma_0"] \arrow[d, "f_0"] & B \arrow[r, "\tau_0"] \arrow[d, "g_0"] & C \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "i"] & A_0 \oplus C_0 \arrow[r, "\pi"] & C_0 \arrow[r] & 0
\end{tikzcd} \tag{1}$$

is commutative with exact rows and columns.

Secondly, by the Snake Lemma, we have the exact sequence

$$0 \to \operatorname{Ker} f_0 \to \operatorname{Ker} g_0 \to \operatorname{Ker} h_0 \to \operatorname{Coker} f_0 \to \operatorname{Coker} g_0 \to \operatorname{Coker} h_0 \to 0 \ .$$

Because h_0 is one-to-one, $0 \to \operatorname{Coker} f_0 \xrightarrow{\sigma_1} \operatorname{Coker} g_0 \xrightarrow{\tau_1} \operatorname{Coker} h_0 \to 0$ is exact, where σ_1 and τ_1 are the R-module homomorphisms induced by $i : A_0 \to A_0 \oplus C_0$ and $\pi : A_0 \oplus C_0 \to C_0$, respectively. Similar as above, there are injective R-modules A_1, C_1 and R-module homomorphisms $\overline{\phi_1}$, $\overline{f_1}$, $\overline{g_1}$, and $\overline{h_1}$ such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Coker} f_0 \arrow[r, "\sigma_1"] \arrow[d, "\overline{f_1}"] & \operatorname{Coker} g_0 \arrow[r, "\tau_1"] \arrow[d, "\overline{g_1}"] & \operatorname{Coker} h_0 \arrow[r] \arrow[d, "\overline{h_1}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "i"] & A_1 \oplus C_1 \arrow[r, "\pi"] & C_1 \arrow[r] & 0
\end{tikzcd} \tag{2}$$

is commutative with exact rows and columns. We take

$$f_1 = \overline{f_1}\, \pi_{A_0}, \quad g_1 = \overline{g_1}\, \pi_{A_0 \oplus C_0}, \quad \text{and} \quad h_1 = \overline{h_1}\, \pi_{C_0} \ ,$$

where $\pi_{A_0} : A_0 \to \operatorname{Coker} f_0$, $\pi_{A_0 \oplus C_0} : A_0 \oplus C_0 \to \operatorname{Coker} g_0$, and $\pi_{C_0} : C_0 \to \operatorname{Coker} h_0$ are the canonical epimorphisms. Since $\overline{f_1}$ is one-to-one and $\operatorname{Ker} \pi_{A_0} = \operatorname{Im} f_0$, $\operatorname{Ker} f_1 = \operatorname{Im} f_0$.

Similarly, we have $\operatorname{Ker} g_1 = \operatorname{Im} g_0$ and $\operatorname{Ker} h_1 = \operatorname{Im} h_0$. Hence, from the commutative diagrams (1) and (2), we see that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\sigma_0"] \arrow[d, "f_0"] & B \arrow[r, "\tau_0"] \arrow[d, "g_0"] & C \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "f_1"] & A_0 \oplus C_0 \arrow[r, "\pi"] \arrow[d, "g_1"] & C_0 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "i"] & A_1 \oplus C_1 \arrow[r, "\pi"] & C_1 \arrow[r] & 0
\end{tikzcd} \tag{3}$$

is commutative with exact rows and columns.

Thirdly, applying the Snake lemma to the diagram (2), because $\overline{h_1}$ is one-to-one, we have the exact sequence $0 \to \operatorname{Coker} \overline{f_1} \stackrel{\sigma_2}{\to} \operatorname{Coker} \overline{g_1} \stackrel{\tau_2}{\to} \operatorname{Coker} \overline{h_1} \to 0$, where σ_2 and τ_2 are the R-module homomorphisms induced by $i : A_1 \to A_1 \oplus C_1$ and $\pi : A_1 \oplus C_1 \to C_1$, respectively. Then similar as above, there are injective R-modules A_2, C_2 and R-module homomorphisms $\overline{\phi_2}$, $\overline{f_2}$, $\overline{g_2}$, and $\overline{h_2}$ such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Coker} \overline{f_1} \arrow[r, "\sigma_2"] \arrow[d, "\overline{f_2}"] & \operatorname{Coker} \overline{g_1} \arrow[r, "\tau_2"] \arrow[d, "\overline{g_2}"] & \operatorname{Coker} \overline{h_1} \arrow[r] \arrow[d, "\overline{h_2}"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "i"] & A_2 \oplus C_2 \arrow[r, "\pi"] & C_2 \arrow[r] & 0
\end{tikzcd} \tag{4}$$

is commutative with exact rows and columns. On the other hand, since $\operatorname{Im} f_1 = \operatorname{Im} \overline{f_1}$, $\operatorname{Coker} f_1 = \operatorname{Coker} \overline{f_1}$. Similarly, we have $\operatorname{Coker} g_1 = \operatorname{Coker} \overline{g_1}$, and $\operatorname{Coker} h_1 = \operatorname{Coker} \overline{h_1}$. We take

$$f_2 = \overline{f_2}\, \pi_{A_1}, \quad g_2 = \overline{g_2}\, \pi_{A_1 \oplus C_1}, \quad \text{and} \quad h_2 = \overline{h_2}\, \pi_{C_1}\,,$$

where $\pi_{A_1} : A_1 \to \operatorname{Coker} f_1 = \operatorname{Coker} \overline{f_1}$, $\pi_{A_1 \oplus C_1} : A_1 \oplus C_1 \to \operatorname{Coker} g_1 = \operatorname{Coker} \overline{g_1}$, and $\pi_{C_1} : C_1 \to \operatorname{Coker} h_1 = \operatorname{Coker} \overline{h_1}$ are the canonical epimorphisms. Similar as above, combining the commutative diagrams (3) and (4), we have the commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\sigma_0"] \arrow[d, "f_0"] & B \arrow[r, "\tau_0"] \arrow[d, "g_0"] & C \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "f_1"] & A_0 \oplus C_0 \arrow[r, "\pi"] \arrow[d, "g_1"] & C_0 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "f_2"] & A_1 \oplus C_1 \arrow[r, "\pi"] \arrow[d, "g_2"] & C_1 \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "i"] & A_2 \oplus C_2 \arrow[r, "\pi"] & C_2 \arrow[r] & 0
\end{tikzcd}
\tag{5}
$$

in which all rows and columns are exact.

Finally, continue the same discussion, then we will get injective resolutions $\mathcal{I}_A : 0 \to A \xrightarrow{f_0} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots$ of A, $\mathcal{I}_B : 0 \to B \xrightarrow{g_0} B_0 = A_0 \oplus C_0 \xrightarrow{g_1} B_1 = A_1 \oplus C_1 \xrightarrow{g_2} \cdots$ of B, and $\mathcal{I}_C : 0 \to C \xrightarrow{h_0} C_0 \xrightarrow{h_1} C_1 \xrightarrow{h_2} \cdots$ of C such that $0 \to \mathcal{I}_A \to \mathcal{I}_B \to \mathcal{I}_C \to 0$ is exact. The proof is complete. $\square$

Let $E : 0 \to A \xrightarrow{\sigma_0} B \xrightarrow{\tau_0} C \to 0$ and $E' : 0 \to A' \xrightarrow{\sigma_0'} B' \xrightarrow{\tau_0'} C' \to 0$ be two short exact sequences of R-modules. By Proposition 3.1.7, for the short exact sequence E, there exists a short exact sequence of complexes $0 \to \mathcal{I}_A \to \mathcal{I}_B \to \mathcal{I}_C \to 0$, where $\mathcal{I}_A$, $\mathcal{I}_B$, $\mathcal{I}_C$ are injective resolutions of A, B, C, respectively. In particular, we know that the nth level of the short exact sequence of complexes $0 \to \mathcal{I}_A \to \mathcal{I}_B \to \mathcal{I}_C \to 0$ is $0 \to A_n \xrightarrow{i} A_n \oplus C_n \xrightarrow{\pi} C_n \to 0$ for all $n \geq 0$. Similarly, for the short exact sequence E', there exists a short exact sequence of complexes $0 \to \mathcal{I}_{A'} \to \mathcal{I}_{B'} \to \mathcal{I}_{C'} \to 0$, where $\mathcal{I}_{A'}$, $\mathcal{I}_{B'}$, $\mathcal{I}_{C'}$ are injective resolutions of A', B', C', respectively. And the nth level of the short exact sequence of complexes $0 \to \mathcal{I}_{A'} \to \mathcal{I}_{B'} \to \mathcal{I}_{C'} \to 0$ is $0 \to {A_n}' \xrightarrow{i} {A_n}' \oplus {C_n}' \xrightarrow{\pi} {C_n}' \to 0$ for all $n \geq 0$. In the next proposition, we will show that if $(\alpha, \beta, \gamma) : E \to E'$ is a morphism in the category $\mathcal{E}$, then there exist three cochain maps $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ such that the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_A \arrow[r] \arrow[d, "\hat{\alpha}"] & \mathcal{I}_B \arrow[r] \arrow[d, "\hat{\beta}"] & \mathcal{I}_C \arrow[r] \arrow[d, "\hat{\gamma}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{A'} \arrow[r] & \mathcal{I}_{B'} \arrow[r] & \mathcal{I}_{C'} \arrow[r] & 0
\end{tikzcd}
$$

is commutative.

Proposition 3.1.8. *Let $E : 0 \to A \xrightarrow{\sigma_0} B \xrightarrow{\tau_0} C \to 0$ and $E' : 0 \to A' \xrightarrow{\sigma_0{}'} B' \xrightarrow{\tau_0{}'} C' \to 0$ be two short exact sequences of R-modules. Suppose that $(\alpha, \beta, \gamma) : E \to E'$ is a morphism in the category $\mathcal{E}$, i.e., the diagram*

$$\begin{tikzcd}
E : 0 \arrow[r] & A \arrow[r, "\sigma_0"] \arrow[d, "\alpha"] & B \arrow[r, "\tau_0"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
E' : 0 \arrow[r] & A' \arrow[r, "\sigma_0'"] & B' \arrow[r, "\tau_0'"] & C' \arrow[r] & 0
\end{tikzcd}$$

is commutative. Then there exist injective resolutions $\mathcal{I}_A$, $\mathcal{I}_B$, $\mathcal{I}_C$, $\mathcal{I}_{A'}$, $\mathcal{I}_{B'}$, $\mathcal{I}_{C'}$ of A, B, C, A', B', C', respectively, and three cochain maps $\hat{\alpha} : \mathcal{I}_A \to \mathcal{I}_{A'}$, $\hat{\beta} : \mathcal{I}_B \to \mathcal{I}_{B'}$, and $\hat{\gamma} : \mathcal{I}_C \to \mathcal{I}_{C'}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_A \arrow[r] \arrow[d, "\hat{\alpha}"] & \mathcal{I}_B \arrow[r] \arrow[d, "\hat{\beta}"] & \mathcal{I}_C \arrow[r] \arrow[d, "\hat{\gamma}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{A'} \arrow[r] & \mathcal{I}_{B'} \arrow[r] & \mathcal{I}_{C'} \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Proof. Recall that in the proof of Proposition 3.1.7, for the short exact sequence E, we construct injective resolutions

$$\mathcal{I}_A : 0 \to A \overset{f_0}{\to} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots$$
$$\mathcal{I}_B : 0 \to B \overset{g_0}{\to} B_0 = A_0 \oplus C_0 \xrightarrow{g_1} B_1 = A_1 \oplus C_1 \xrightarrow{g_2} \cdots$$
$$\mathcal{I}_C : 0 \to C \overset{h_0}{\to} C_0 \xrightarrow{h_1} C_1 \xrightarrow{h_2} \cdots$$

such that $0 \to \mathcal{I}_A \to \mathcal{I}_B \to \mathcal{I}_C \to 0$ is exact. In particular, we use two embedding maps $\phi_0 : B \to A_0$ and $h_0 : C \to C_0$ to obtain the maps f_0 and g_0; more precisely, $f_0 = \phi_0\sigma_0$ and $g_0 : B \to A_0 \oplus C_0$ is defined by $g_0(b) = (\phi_0(b), h_0\tau_0(b))$ for all $b \in B$. Similarly, for the short exact sequence E', we can construct injective resolutions

$$\mathcal{I}_{A'} : 0 \to A' \overset{f_0{}'}{\to} A_0{}' \xrightarrow{f_1{}'} A_1{}' \xrightarrow{f_2{}'} \cdots$$
$$\mathcal{I}_{B'} : 0 \to B' \overset{g_0{}'}{\to} B_0{}' = A_0{}' \oplus C_0{}' \xrightarrow{g_1{}'} B_1{}' = A_1{}' \oplus C_1{}' \xrightarrow{g_2{}'} \cdots$$
$$\mathcal{I}_{C'} : 0 \to C' \overset{h_0{}'}{\to} C_0{}' \xrightarrow{h_1{}'} C_1{}' \xrightarrow{h_2{}'} \cdots$$

such that $0 \to \mathcal{I}_{A'} \to \mathcal{I}_{B'} \to \mathcal{I}_{C'} \to 0$ is exact; we also use two embedding maps ${\phi_0}' : B' \to {A_0}'$ and ${h_0}' : C' \to {C_0}'$ to construct the maps ${f_0}'$ and ${g_0}'$; more precisely, ${\phi_0}'{\sigma_0}' = {f_0}'$ and ${g_0}' : B' \to {A_0}' \oplus {C_0}'$ is defined by ${g_0}'(b') = ({\phi_0}'(b'), h_0'\tau_0(b'))$ for all $b' \in B'$.

Because $\alpha : A \to A'$ is an R-module homomorphism, by the Comparison Theorem, there is a cochain map $\hat{\alpha} : \mathcal{I}_A \to \mathcal{I}_{A'}$, i.e., the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_0"] \arrow[d, "\alpha"] & A_0 \arrow[r, "f_1"] \arrow[d, "\alpha_0"] & A_1 \arrow[r, "f_2"] \arrow[d, "\alpha_1"] & \cdots \\
0 \arrow[r] & A' \arrow[r, "{f_0}'"] & {A_0}' \arrow[r, "{f_1}'"] & {A_1}' \arrow[r, "{f_2}'"] & \cdots
\end{tikzcd}$$

is commutative. Similarly, since $\gamma : C \to C'$ is an R-module homomorphism, there is a cochain map $\hat{\gamma} : \mathcal{I}_C \to \mathcal{I}_{C'}$, i.e, the diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "h_0"] \arrow[d, "\gamma"] & C_0 \arrow[r, "h_1"] \arrow[d, "\gamma_0"] & C_1 \arrow[r, "h_2"] \arrow[d, "\gamma_1"] & \cdots \\
0 \arrow[r] & C' \arrow[r, "{h_0}'"] & {C_0}' \arrow[r, "{h_1}'"] & {C_1}' \arrow[r, "{h_2}'"] & \cdots
\end{tikzcd}$$

is commutative. We want to find a cochain map $\hat{\beta} : \mathcal{I}_B \to \mathcal{I}_{B'}$, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_A \arrow[r] \arrow[d, "\hat{\alpha}"] & \mathcal{I}_B \arrow[r] \arrow[d, "\hat{\beta}"] & \mathcal{I}_C \arrow[r] \arrow[d, "\hat{\gamma}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{A'} \arrow[r] & \mathcal{I}_{B'} \arrow[r] & \mathcal{I}_{C'} \arrow[r] & 0
\end{tikzcd}$$

is commutative.

First, we want to find an R-module homomorphism $\beta_0 : A_0 \oplus C_0 \to {A_0}' \oplus {C_0}'$ such that the 3-dimensional diagram

$$
\begin{tikzcd}[column sep=small]
& 0 \arrow[rr] & & A \arrow[rr, "\sigma_0"] \arrow[dl, "\alpha"'] \arrow[dd, "f_0" near start] & & B \arrow[rr, "\tau_0"] \arrow[dl, "\beta"'] \arrow[dd, "g_0" near start] & & C \arrow[rr] \arrow[dl, "\gamma"'] \arrow[dd, "h_0" near start] & & 0 \\
0 \arrow[rr] & & A' \arrow[rr, "\sigma_0'" near start] \arrow[dd, "f_0'" near end] & & B' \arrow[rr, "\tau_0'" near start] \arrow[dd, "g_0'" near end] & & C' \arrow[rr] \arrow[dd, "h_0'" near end] & & 0 & \\
& 0 \arrow[rr] & & A_0 \arrow[rr] \arrow[dl, "\alpha_0"] & & A_0 \oplus C_0 \arrow[rr] \arrow[dl, "\beta_0"] & & C_0 \arrow[rr] \arrow[dl, "\gamma_0"] & & 0 \\
0 \arrow[rr] & & A_0' \arrow[rr] & & A_0' \oplus C_0' \arrow[rr] & & C_0' \arrow[rr] & & 0 &
\end{tikzcd}
\tag{1}
$$

is commutative. Since we already know that the upper level diagram and the left and the right vertical diagrams are commutative, it remains to show that the lower level diagram and the middle vertical diagram are commutative. First of all, because τ_0 is onto, for each $c \in C$, there exists $b \in B$ such that $c = \tau_0(b)$. If $c = \tau_0(b_1) = \tau_0(b_2)$ for b_1, $b_2 \in B$, then $\tau_0(b_1 - b_2) = 0$, and so $b_1 - b_2 \in \operatorname{Ker} \tau_0$. Since E is exact, $\operatorname{Ker} \tau_0 = \operatorname{Im} \sigma_0$ and so $b_1 - b_2 = \sigma_0(a)$ for some $a \in A$. Because ${\phi_0}'{\sigma_0}' = {f_0}'$ and $\phi_0\sigma_0 = f_0$, we have

$$
\begin{aligned}
& [\, {\phi_0}'\beta(b_1) - \alpha_0\phi_0(b_1)\,] - [\, {\phi_0}'\beta(b_2) - \alpha_0\phi_0(b_2)\,] \\
&= {\phi_0}'\beta(b_1 - b_2) - \alpha_0\phi_0(b_1 - b_2) \\
&= {\phi_0}'\beta(\sigma_0(a)) - \alpha_0\phi_0(\sigma_0(a)) \\
&= {\phi_0}'{\sigma_0}'\alpha(a) - \alpha_0\phi_0(\sigma_0(a)) && (\text{since } \beta\sigma_0 = {\sigma_0}'\alpha) \\
&= {f_0}'\alpha(a) - \alpha_0 f_0(a) && (\text{since } {\phi_0}'\sigma' = {f_0}' \text{ and } \phi_0\sigma_0 = f_0) \\
&= ({f_0}'\alpha - \alpha_0 f_0)(a) \\
&= 0\,.
\end{aligned}
$$

Thus for all $c \in C$, we can define a map $\mu_0 : C \to {A_0}'$ by $\mu_0(c) = \mu_0(\tau_0(b)) = {\phi_0}'\beta(b) - \alpha_0\phi_0(b)$ where $b \in B$ with $\tau_0(b) = c$. For all c_1, $c_2 \in C$, there exist b_1, $b_2 \in B$ such that $c_1 = \tau_0(b_1)$ and $c_2 = \tau_0(b_2)$. Then we have $c_1 + c_2 = \tau_0(b_1 + b_2)$ and then it is not difficult to check that μ_0 is an R-module homomorphism. Moreover, because ${A_0}'$ is injective, there exists an R-module homomorphism $\nu_0 : C_0 \to {A_0}'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "h_0"] \arrow[d, "\mu_0"'] & C_0 \arrow[dl, "\nu_0"] \\
& {A_0}' &
\end{tikzcd}$$

is commutative, i.e., $\nu_0 h_0 = \mu_0$. Therefore, we can define $\beta_0 : A_0 \oplus C_0 \to {A_0}' \oplus {C_0}'$ by $\beta_0(a_0, c_0) = (\alpha_0(a_0) + \nu_0(c_0), \gamma_0(c_0))$. It is not difficult to check that the lower level diagram in (1)

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r] \arrow[d, "\alpha_0"] & A_0 \oplus C_0 \arrow[r] \arrow[d, "\beta_0"] & C_0 \arrow[r] \arrow[d, "\gamma_0"] & 0 \\
0 \arrow[r] & {A_0}' \arrow[r] & {A_0}' \oplus {C_0}' \arrow[r] & {C_0}' \arrow[r] & 0
\end{tikzcd}$$

is commutative. Now we show that the middle vertical diagram in (1) is commutative, i.e., $\beta_0 g_0 = {g_0}'\beta$. For all $b \in B$, recall that $g_0(b) = (\phi_0(b), h_0\tau_0(b))$, so

$$\begin{aligned}
\beta_0 g_0(b) &= \beta_0(\phi_0(b), h_0\tau_0(b)) \\
&= (\alpha_0(\phi_0(b)) + \nu_0(h_0\tau_0(b)), \gamma_0(h_0\tau_0(b))) \\
&= (\alpha_0\phi_0(b) + \mu_0\tau_0(b), {h_0}'\gamma\tau_0(b)) && \text{(since } \nu_0 h_0 = \mu_0 \text{ and } \gamma_0 h_0 = {h_0}'\gamma) \\
&= ({\phi_0}'\beta(b), {h_0}'\gamma\tau_0(b)) && \text{(since } \mu_0(\tau_0(b)) = {\phi_0}'\beta(b) - \alpha_0\phi_0(b)) \\
&= ({\phi_0}'\beta(b), {h_0}'{\tau_0}'\beta(b)) && \text{(since } \gamma\tau_0 = {\tau_0}'\beta\text{)} \\
&= {g_0}'\beta(b)\ .
\end{aligned}$$

Hence $\beta_0 g_0 = {g_0}'\beta$, and we show that the 3-dimensional diagram (1) is commutative.

Secondly, because the 3-dimensional diagram (1) is commutative, the 3-dimensional diagram

$$
\begin{tikzcd}
& 0 \arrow[rr] & & A_0 \arrow[rr] \arrow[dl, "\alpha_0"'] \arrow[dd, "\pi_{A_0}"] & & A_0 \oplus C_0 \arrow[rr] \arrow[dl, "\beta_0"'] \arrow[dd, "\pi_{B_0}"] & & C_0 \arrow[rr] \arrow[dl, "\gamma_0"'] \arrow[dd, "\pi_{C_0}"] & & 0 \\
0 \arrow[rr] & & {A_0}' \arrow[rr] \arrow[dd, "\pi_{{A_0}'}"] & & {A_0}' \oplus {C_0}' \arrow[rr] \arrow[dd, "\pi_{{B_0}'}"] & & {C_0}' \arrow[rr] \arrow[dd, "\pi_{{C_0}'}"] & & 0 & \\
& 0 \arrow[rr] & & Coker\ f_0 \arrow[rr] \arrow[dl, "{\alpha_0}^*"] & & Coker\ g_0 \arrow[rr] \arrow[dl, "{\beta_0}^*"] & & Coker\ h_0 \arrow[rr] \arrow[dl, "{\gamma_0}^*"] & & 0 \\
0 \arrow[rr] & & Coker\ {f_0}' \arrow[rr] & & Coker\ {g_0}' \arrow[rr] & & Coker\ {h_0}' \arrow[rr] & & 0 &
\end{tikzcd}
\tag{2}
$$

is also commutative, where ${\alpha_0}^* : \operatorname{Coker} f_0 \to \operatorname{Coker} {f_0}'$, ${\beta_0}^* : \operatorname{Coker} g_0 \to \operatorname{Coker} {g_0}'$, ${\gamma_0}^* : \operatorname{Coker} h_0 \to \operatorname{Coker} {h_0}'$ are the R-module homomorphisms induced by α_0, β_0, γ_0, respectively. Similar as above, we can find an R-module homomorphism $\beta_1 : A_1 \oplus C_1 \to {A_1}' \oplus {C_1}'$ such that the 3-dimensional diagram

$$
\begin{tikzcd}
& 0 \arrow[rr] & & Coker\ f_0 \arrow[rr] \arrow[dl, "{\alpha_0}^*"'] \arrow[dd, "f_1"] & & Coker\ g_0 \arrow[rr] \arrow[dl, "{\beta_0}^*"'] \arrow[dd, "g_1"] & & Coker\ h_0 \arrow[rr] \arrow[dl, "{\gamma_0}^*"'] \arrow[dd, "h_1"] & & 0 \\
0 \arrow[rr] & & Coker\ {f_0}' \arrow[rr] \arrow[dd, "f_1'"] & & Coker\ {g_0}' \arrow[rr] \arrow[dd, "g_1'"] & & Coker\ {h_0}' \arrow[rr] \arrow[dd, "h_1'"] & & 0 & \\
& 0 \arrow[rr] & & A_1 \arrow[rr] \arrow[dl, "\alpha_1"] & & A_1 \oplus C_1 \arrow[rr] \arrow[dl, "\beta_1"] & & C_1 \arrow[rr] \arrow[dl, "\gamma_1"] & & 0 \\
0 \arrow[rr] & & A_1' \arrow[rr] & & A_1' \oplus C_1' \arrow[rr] & & C_1' \arrow[rr] & & 0 &
\end{tikzcd}
\tag{3}
$$

is commutative. Combining the 3-dimensional diagrams (1), (2), and (3), we have the commutative 3-dimensional diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & & \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl] \arrow[dd] & & B \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[rr] & & A' \arrow[rr] \arrow[dd] & & B' \arrow[rr] \arrow[dd] & & C' \arrow[rr] \arrow[dd] & & 0 & \\
 & 0 \arrow[rr] & & A_0 \arrow[rr] \arrow[dl] \arrow[dd] & & A_0 \oplus C_0 \arrow[rr] \arrow[dl] \arrow[dd] & & C_0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[rr] & & {A_0}' \arrow[rr] \arrow[dd] & & {A_0}' \oplus {C_0}' \arrow[rr] \arrow[dd] & & {C_0}' \arrow[rr] \arrow[dd] & & 0 & \\
 & 0 \arrow[rr] & & A_1 \arrow[rr] \arrow[dl] & & A_1 \oplus C_1 \arrow[rr] \arrow[dl] & & C_1 \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[rr] & & A_1' \arrow[rr] & & A_1' \oplus C_1' \arrow[rr] & & C_1' \arrow[rr] & & 0 &
\end{tikzcd}
\tag{4}
$$

Finally, continue the same discussion, then we will get R-module homomorphisms β_n for all $n \geq 2$ such that the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_A \arrow[r] \arrow[d, "\hat{\alpha}"] & \mathcal{I}_B \arrow[r] \arrow[d, "\hat{\beta}"] & \mathcal{I}_C \arrow[r] \arrow[d, "\hat{\gamma}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{A'} \arrow[r] & \mathcal{I}_{B'} \arrow[r] & \mathcal{I}_{C'} \arrow[r] & 0
\end{tikzcd}
$$

is commutative. This completes the proof. $\square$

Theorem 3.1.9. *Let T be a left exact additive functor. Then $\{R^nT, E^n\}_{n\geq 0}$ is a universal connected sequence.*

Proof. First, we show that $\{R^nT, E^n\}_{n\geq 0}$ is a connected sequence, i.e., $(R^nT, E^n, R^{n+1}T)$ is a connected pair for all $n \geq 0$. Let $(\alpha, \beta, \gamma) : E \to E'$ be a morphism in the category $\mathcal{E}$, where $E : 0 \to A \to B \to C \to 0$ and $E^{'} : 0 \to A^{'} \to B^{'} \to C^{'} \to 0$ are two short exact sequences of R-modules. We want to show that the diagram

$$
\begin{tikzcd}
R^nT(C) \arrow[r, "E^n"] \arrow[d, "R^nT(\gamma)"'] & R^{n+1}T(A) \arrow[d, "R^{n+1}T(\alpha)"] \\
R^nT(C') \arrow[r, "(E')^n"] & R^{n+1}T(A')
\end{tikzcd}
\tag{1}
$$

is commutative. By Proposition 3.1.8, there exist injective resolutions $\mathcal{I}_A$, $\mathcal{I}_B$, $\mathcal{I}_C$, $\mathcal{I}_{A'}$, $\mathcal{I}_{B'}$, $\mathcal{I}_{C'}$ of A, B, C, A', B', C', respectively, and three cochain maps $\hat{\alpha} : \mathcal{I}_A \to \mathcal{I}_{A'}$, $\hat{\beta} : \mathcal{I}_B \to \mathcal{I}_{B'}$, and $\hat{\gamma} : \mathcal{I}_C \to \mathcal{I}_{C'}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_A \arrow[r] \arrow[d, "\hat{\alpha}"] & \mathcal{I}_B \arrow[r] \arrow[d, "\hat{\beta}"] & \mathcal{I}_C \arrow[r] \arrow[d, "\hat{\gamma}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{A'} \arrow[r] & \mathcal{I}_{B'} \arrow[r] & \mathcal{I}_{C'} \arrow[r] & 0
\end{tikzcd}$$

is commutative. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & T(\mathcal{I}_A) \arrow[r] \arrow[d, "T(\hat{\alpha})"] & T(\mathcal{I}_B) \arrow[r] \arrow[d, "T(\hat{\beta})"] & T(\mathcal{I}_C) \arrow[r] \arrow[d, "T(\hat{\gamma})"] & 0 \\
0 \arrow[r] & T(\mathcal{I}_{A'}) \arrow[r] & T(\mathcal{I}_{B'}) \arrow[r] & T(\mathcal{I}_{C'}) \arrow[r] & 0
\end{tikzcd} \tag{2}$$

is commutative. Because the ith level of $0 \to \mathcal{I}_A \to \mathcal{I}_B \to \mathcal{I}_C \to 0$ is $0 \to A_i \to A_i \oplus C_i \to C_i \to 0$, which is a split short exact sequence, $0 \to T(A_i) \to T(A_i \oplus C_i) \to T(C_i) \to 0$ is exact by Lemma 3.1.4 (4). Similarly, since the ith level of $0 \to \mathcal{I}_{A'} \to \mathcal{I}_{B'} \to \mathcal{I}_{C'} \to 0$ is $0 \to A'_i \to A'_i \oplus C'_i \to C'_i \to 0$, which is a split short exact sequence, $0 \to T(A'_i) \to T(A'_i \oplus C'_i) \to T(C'_i) \to 0$ is exact. Thus both rows of the commutative diagram (2) are exact. Therefore, by Lemma 3.1.6, the diagram

$$\begin{tikzcd}
H^n(T(\mathcal{I}_C)) \arrow[r, "E^n"] \arrow[d, "(T(\gamma_n))^*"'] & H^{n+1}(T(\mathcal{I}_A)) \arrow[d, "(T(\alpha_{n+1}))^*"] \\
H^n(T(\mathcal{I}_{A'})) \arrow[r, "(E')^n"] & H^{n+1}(T(\mathcal{I}'_A))
\end{tikzcd} \tag{3}$$

is commutative. By the definition of right derived functors, the diagram (1) is just the diagram (3). Hence the diagram (1) is also commutative and so (R^nT, E^n, R^{n+1}) is a connected pair for all $n \geq 0$.

Next, we show that the connected sequence $\{R^nT, E^n\}_{n\geq 0}$ is universal. Note that for all injective R-modules I, $0 \to I \to I \to 0$ is an injective resolution of I. Since T is left exact, $0 \to T(I) \to T(I) \to T(0) = 0$ is exact and so $R^nT(I) = 0$ for all $n > 0$. On the other hand, by Lemma 3.1.4 (3), for each short exact sequence $0 \to A \to B \to C \to 0$, there is a long exact sequence of cohomology

$$0 \to R^0T(A) \to R^0T(B) \to R^0T(C) \to R^1T(A) \to \cdots .$$

Therefore, $\{R^nT, E^n\}_{n\geq 0}$ is universal by Corollary 2.3.4. This completes the proof. $\square$

Theorem 3.1.10. *$\{H^n_{\mathbf{m}}(-), E^n\}_{n\geq 0}$ is a universal connected sequence with the initial $H^0_{\mathbf{m}}(-) = \Gamma_{\mathbf{m}}(-)$.*

Proof. Because $\Gamma_{\mathbf{m}}(-)$ is a left exact additive functor, $\{H^n_{\mathbf{m}}(-), E^n\}_{n\geq 0}$ is a universal connected sequence by Theorem 3.1.9. Moreover, by Lemma 3.1.4 (2), we have that $H^0_{\mathbf{m}}(-) = \Gamma_{\mathbf{m}}(-)$. The proof is complete. □

3.2 Injective hulls

Definition 3.2.1. *Suppose $i : N \to M$ is an R-module embedding map.*

(1) *M is said to be an **essential extension** of N if $i^{-1}(U) \neq 0$ for all nonzero submodules U of M.*

(2) *An essential extension M of N is said to be proper if i is not onto.*

Remark 3.2.2. *If $i : N \to M$ is an R-module embedding map, we use the convention that $U \cap N$ denotes the pre-image of U in N, i.e., $U \cap N = i^{-1}(U)$. We also use the notation $N \subsetneq M$ to indicate that i is not onto.*

From Definition 3.2.1, we know that for every R-module N, N is an essential extension of N itself. In Proposition 3.2.4, we will show that N has no proper essential extension if and only if N is an injective R-module. Before we prove Proposition 3.2.4, we present a lemma that we need in the proof.

Lemma 3.2.3. *Given an R-module monomorphism $\phi : U \to V$ and an R-module homomorphism $f : U \to N$, we let $W = (V \oplus N)/C$, where*

$$C = \{(\phi(x), -f(x)) \in V \oplus N \mid x \in U\}$$

is a submodule of $V \oplus N$. Take $g : V \to W$ to be the composition of the natural homomorphisms $V \to V \oplus N \to W = (V \oplus N)/C$, i.e., $g(v) = (v, 0) + C$ for all $v \in V$; and take $\psi : N \to W$ to be the composition of the natural homomorphisms

$N \to V \oplus N \to W = (V \oplus N)/C$, i.e., $\psi(n) = (0, n) + C$ for all $n \in N$. Then ψ is one-to-one and the diagram

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "f"'] & V \arrow[d, "g"] \\ N \arrow[r, "\psi"] & W \end{tikzcd}$$

is commutative, i.e., $g\phi = \psi f$.

Proof. First, we show that ψ is one-to-one. Let $n \in \operatorname{Ker}\psi$. Then $(0, n) \in C$ and so $(0, n) = (\phi(x), -f(x))$ for some $x \in U$. Because ϕ is one-to-one, $x = 0$ and we get $n = -f(x) = -f(0) = 0$. Hence ψ is one-to-one.

Next, we show that $g\phi = \psi f$. Note that for all $x \in U$, since $(\phi(x), -f(x)) \in C$, we have $(\phi(x), 0) + C = (0, f(x)) + C$. Thus $g\phi(x) = (\phi(x), 0) + C = (0, f(x)) + C = \psi f(x)$ for all $x \in U$. Hence $g\phi = \psi f$. $\square$

Proposition 3.2.4. *Let N be an R-module. Then N has no proper essential extension if and only if N is an injective R-module.*

Proof. Suppose N has no proper essential extension. Now we show that N is injective, i.e., if $\phi : U \to V$ is an R-module monomorphism and $f : U \to N$ is an R-module homomorphism, then there exists an R-module homomorphism $\alpha : V \to N$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "\phi"] \arrow[d, "f"'] & V \arrow[dl, "\alpha"] \\ & N & \end{tikzcd}$$

is commutative. By Lemma 3.2.3, there exist an R-module W and R-module homomorphisms $g : V \to W$ and $\psi : N \to W$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "f"'] & V \arrow[d, "g"] \\ N \arrow[r, "\psi"] & W \end{tikzcd} \tag{1}$$

is commutative. Moreover, since $\psi : N \to W$ is a monomorphism, N can be thought as a submodule of W. In order to find an R-module homomorphism α, we consider the set

$\Sigma = \{D \mid D \text{ is a submodule of } W \text{ with } D \cap N = 0\}$. It not difficult to check that $(\Sigma, \subseteq)$ is a nonempty partially ordered set and that every chain in Σ has an upper bound. By the Zorn's Lemma, Σ has a maximal element. Let D be a maximal element in Σ. Since $D \cap N = 0$, the composition of the homomorphisms $N \xrightarrow{\psi} W \xrightarrow{\pi} W/D$ is one-to-one. So we can also think of N as a submodule of W/D. We want to show that W/D is an essential extension of N. Suppose W/D is not an essential extension of N. Then there exists a nonzero submodule W'/D of W/D such that $(W'/D) \cap N = 0$, where W' is a submodule of W with $D \subsetneq W'$. We claim that $W' \cap N = 0$. Let $x \in W' \cap N$. Then $x = \psi(n)$ for some $n \in N$ and we have $x + D = \pi(x) = \pi\psi(n) \in (W'/D) \cap N$. Because $(W'/D) \cap N = 0$, $x \in D$. Thus $x \in D \cap N = 0$. So we have $W' \cap N = 0$. Then $W' \in \Sigma$ with $D \subsetneq W'$. This contradicts the fact that D is a maximal element in Σ. Therefore, W/D is an essential extension of N. By assumption, $N = W/D$, i.e. $\pi\psi : N \to W/D$ is an isomorphism. Then there is an R-module homomorphism $\delta : W/D \to N$ such that $\delta(\pi\psi) = 1_N$, i.e., the diagram

$$\begin{tikzcd} N \arrow[r, "\psi"] & W \arrow[d, "\pi"] \\ & W/D \arrow[ul, "\delta"] \end{tikzcd} \tag{2}$$

is commutative. Combining the diagrams (1) and (2), we get the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "f"'] & V \arrow[d, "g"] \\ N \arrow[r, "\psi"] & W \arrow[d, "\pi"] \\ & W/D . \arrow[ul, "\delta"] \end{tikzcd}$$

Take $\alpha = \delta\pi g$, then we have $\alpha\phi = (\delta\pi g)\phi = \delta\pi\psi f = 1_N f = f$. Hence N is injective.

Conversely, suppose N is an injective R-module. We claim that N has no proper essential extension. Suppose that M is a proper essential extension of N. Because N is injective, N is a direct summand of M, i.e., there is a submodule M_1 of M such that $M_1 \cap N = 0$ and $M = N + M_1$. Since $N \subsetneq M$, $M_1 \neq 0$. This contradicts the fact that M is a proper essential extension of N. Hence N has no proper essential extension. $\square$

Definition 3.2.5. *Let M be an R-module. An injective R-module E is said to be an* ***injective hull*** *of M if E is an essential extension of M. We denote E by $E(M)$.*

Now we show the existence of injective hulls of M and some properties related to injective hulls of M.

Lemma 3.2.6. *Let M be an R-module.*

(1) *M has an injective hull.*

(2) *Let E be an injective hull of M and let I be an injective R-module. If $f : M \to I$ is an R-module monomorphism, then there exists an R-module monomorphism $\phi : E \to I$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[dl, "\phi"] \\ I & \end{tikzcd}$$

is commutative, where $i : M \to E$ is the embedding map.

(3) *If E and E' are injective hulls of M, then there exists an R-module isomorphism $\phi : E \to E'$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "i_1"'] & E \arrow[dl, "\phi"] \\ E' & \end{tikzcd}$$

is commutative, where i and i_1 are the embedding maps.

Proof. For (1), M can be embedded to an injective R-module I. We think of M as a submodule of I, i.e., $M \subseteq I$. Consider the set

$$\Sigma = \{E \mid E \subseteq I \text{ and } E \text{ is an essential extension of } M\}.$$

Then $(\Sigma, \subseteq)$ is a nonempty partially ordered set since $M \in \Sigma$. Now we show that every chain $\{E_i\}_{i \in I}$ in Σ has an upper bound. Take $N = \bigcup_{i \in I} E_i$. Then N is an R-module with $M \subseteq N \subseteq I$. Let N_1 be a nonzero submodule of N. Then there exists $x \in N_1$ such that $x \neq 0$. So $x \in E_i$ for some $i \in I$. Thus Rx is a nonzero submodule of E_i, and so

$Rx \cap M \neq 0$ since E_i is an essential extension of M. Therefore we have $N_1 \cap M \neq 0$. Thus N is an essential extension of M and so $N \in \Sigma$. Hence N is an upper bound of the chain $\{E_i\}_{i \in I}$. By the Zorn's Lemma, Σ has a maximal element. Let E be a maximal element in Σ. We claim that E is an injective hull of M. Because E is an essential extension of M, by Definition 3.2.4, it remains to show that E is an injective R-module. However, by Proposition 3.2.3, it is enough to show that E has no proper essential extension. Suppose that E' is a proper essential extension of E. Because I is injective, there exists an R-module homomorphism $\psi : E' \to I$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "i_1"] \arrow[d, "i_2"'] & E' \arrow[dl, "\psi"] \\ & I & \end{tikzcd}$$

is commutative, where i_1 and i_2 are embedding maps. Note that if $\operatorname{Ker}\psi \neq 0$, then $\operatorname{Ker}\psi \cap E \neq 0$ since E' is an essential extension of E. However for $a \in \operatorname{Ker}\psi \cap E$,

$$0 = \psi(a) = \psi(i_1(a)) = i_2(a) = a\,.$$

Therefore $\operatorname{Ker}\psi = 0$, i.e., ψ is one-to-one. Because $E \subsetneq E'$ and $\operatorname{Ker}\psi = 0$, $E \subsetneq \operatorname{Im}\psi \subseteq I$. Now we claim that $\operatorname{Im}\psi$ is an essential extension of M. We know that every nonzero submodule of $\operatorname{Im}\psi$ is of the form $\psi({E_1}')$, where $0 \subsetneq {E_1}' \subseteq E'$ since ψ is one-to-one. Then

$$\begin{array}{ll} \quad {E_1}' \cap E \neq 0 & \text{(since } E' \text{ is an essential extension of } E\text{)} \\ \Rightarrow (E_1' \cap E) \cap M \neq 0 & \text{(since } E \text{ is an essential extension of } M\text{)} \\ \Rightarrow \psi((E_1' \cap E) \cap M) \neq 0 & \text{(since } \psi \text{ is one-to-one)} \\ \Rightarrow \psi(E_1' \cap E) \cap M \neq 0 & \text{(since } \psi \text{ is one-to-one again)} \\ \Rightarrow \psi(E_1') \cap M \neq 0\,. & \end{array}$$

Thus $\operatorname{Im}\psi$ is an essential extension of M. Then $\operatorname{Im}\psi \in \Sigma$ with $E \subsetneq \operatorname{Im}\psi$ and this contradicts the fact that E is a maximal element in Σ. Hence E has no proper essential extension. Therefore, E is a injective hull of M.

For (2), since I is an injective R-module, there exists an R-module homomorphism

$\phi : E \to I$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[dl, "\phi"] \\ & I & \end{tikzcd}$$

is commutative. Because $\phi|_M = f$, $M \cap \operatorname{Ker} \phi = \operatorname{Ker} f = 0$. Then $\operatorname{Ker} \phi = 0$ since E is an essential extension of M.

For (3), by (2), there exists an R-module monomorphism $\phi : E \to E'$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[dl, "\phi"] \\ E' & \end{tikzcd}$$

is commutative. Thus $\operatorname{Im} \phi \cong E$. Because $\operatorname{Im} \phi$ is injective and $\operatorname{Im} \phi$ is a submodule of E', $\operatorname{Im} \phi$ is a direct summand of E', i.e., there exists a submodule ${E_1}'$ of E' such that ${E_1}' \cap \operatorname{Im} \phi = 0$ and $E' = \operatorname{Im} \phi + {E_1}'$. Moreover, we have ${E_1}' \cap M = 0$ since $M \subseteq \operatorname{Im} \phi$. Therefore, by the fact that E' is an essential extension of M, ${E_1}' = 0$. Hence $\operatorname{Im} \phi = E'$ and so ϕ is indeed an R-module isomorphism. $\square$

Definition 3.2.7. *Let M be an R-module.*

(1) $AssM = \{\mathbf{p} \in Spec(R) \mid$ *there is an* $R-$*module monomorphism* $f : R/\mathbf{p} \to M\}$. *If* $\mathbf{p} \in AssM$, $\mathbf{p}$ *is said to be an associate prime of* M.

(2) *M is said to be* ***decomposable*** *if there exist two nonzero submodules M_1, M_2 of M such that $M_1 \cap M_2 = 0$ and $M = M_1 \oplus M_2$. Otherwise, M is said to be* ***indecomposable****.*

In the next proposition, we will show that an R-module M is indecomposable injective if and only if $M \cong E(R/\mathbf{p})$ for some $\mathbf{p} \in \operatorname{Spec}(R)$. Before we prove Proposition 3.2.9, we present a lemma that we need in the proof.

Lemma 3.2.8. *Let M be a nonzero R-module. Then $AssM$ is nonempty.*

Proof. Since $M \neq 0$, we consider the set $\Sigma = \{Ann(m) \mid m \in M, m \neq 0\}$. Because R is a Noetherian ring, Σ has a maximal element. Let $Ann(x)$ be a maximal element in Σ. Now we show that $Ann(x)$ is a prime ideal. Suppose $ab \in Ann(x)$. Then $(ab)x = 0$. Thus we have $a(bx) = 0$, and so $a \in Ann(bx)$. Note that if $bx = 0$, then $b \in Ann(x)$. On the other hand, if $bx \neq 0$, then $Ann(bx) \in \Sigma$. However, we know that $Ann(x) \subseteq Ann(bx)$, so by maximality of $Ann(x)$, $Ann(x) = Ann(bx)$ and so $a \in Ann(x)$. Hence $Ann(x)$ is a prime ideal. Moreover, let $f : R \to M$ be the R-module homomorphism defined by $f(r) = rx$ for all $r \in R$. Because $Ann(x) = \operatorname{Ker} f$, there exists an R-module monomorphism $\phi : R/Ann(x) \to M$. Therefore, $Ann(x) \in AssM$ and the proof is complete. $\square$

Proposition 3.2.9. *A nonzero R-module M is indecomposable injective if and only if $M \cong E(R/\mathbf{p})$ for some $\mathbf{p} \in Spec(R)$.*

Proof. Suppose M is a nonzero indecomposable injective R-module. Because $M \neq 0$, $AssM$ is nonempty. Let $\mathbf{p} \in AssM$. Then $\mathbf{p} \in \operatorname{Spec}(R)$ and there is an R-module monomorphism $f : R/\mathbf{p} \to M$. By Lemma 3.2.6 (2), there exists an R-module monomorphism $\phi : E(R/\mathbf{p}) \to M$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & R/\mathbf{p} \arrow[r, "i"] \arrow[d, "f"'] & E(R/\mathbf{p}) \arrow[dl, "\phi"] \\ & M & \end{tikzcd}$$

is commutative. Because $E(R/\mathbf{p})$ is injective, there is a submodule M' of M such that $M \cong E(R/\mathbf{p}) \oplus M'$. By the fact that M is indecomposable, we have $M' = 0$ and so $M \cong E(R/\mathbf{p})$.

Conversely, suppose $M \cong E(R/\mathbf{p})$ for some $\mathbf{p} \in \operatorname{Spec}(R)$. Since $E(R/\mathbf{p})$ is injective, it remains to show that $E(R/\mathbf{p})$ is indecomposable. Suppose that $E(R/\mathbf{p})$ is decomposable. Then there exist nonzero submodules M_1 and M_2 of $E(R/\mathbf{p})$ such that $M_1 \cap M_2 = 0$ and $E(R/\mathbf{p}) = M_1 + M_2$. We take $N_1 = R/\mathbf{p} \cap M_1$ and $N_2 = R/\mathbf{p} \cap M_2$. Because $E(R/\mathbf{p})$ is an essential extension of $R/\mathbf{p}$, $N_1 \neq 0$ and $N_2 \neq 0$. On the other hand, since $R/\mathbf{p}$ is an integral domain, $N_1 N_2 \neq 0$ as ideals in $R/\mathbf{p}$. Because $N_1 N_2 \subseteq N_1 \cap N_2$, we have

$N_1 \cap N_2 \neq 0$ and so $M_1 \cap M_2 \neq 0$. This contradicts the fact that $M_1 \cap M_2 = 0$. Hence $E(R/\mathbf{p})$ is indecomposable. This completes the proof. $\square$

From Proposition 3.2.9, we know that every nonzero indecomposable injective R-module is of the form $E(R/\mathbf{p})$ for some $\mathbf{p} \in \mathrm{Spec}(R)$. In Proposition 3.2.11, we will show that every nonzero injective R-module is a direct sum of indecomposable injective R-modules. In the proof of Proposition 3.2.11, we will use the following lemma.

Lemma 3.2.10. *Let $\{M_j \mid j \in J\}$ be a family of R-modules. Then $\bigoplus_{j\in J} M_j$ is injective if and only if M_j is injective for every $j \in J$.*

Proposition 3.2.11. *Let I be a nonzero injective R-module. Then I is a direct sum of indecomposable injective R-modules.*

Proof. Because $I \neq 0$, $AssI$ is nonempty by Lemma 3.2.8. Let $\mathbf{p} \in AssI$, i.e., $\mathbf{p} \in \mathrm{Spec}(R)$ and there is an R-module monomorphism $f : R/\mathbf{p} \to I$. By Lemma 3.2.6 (2), there exists an R-module monomorphism $\phi : E(R/\mathbf{p}) \to I$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & R/\mathbf{p} \arrow[r, "i"] \arrow[d, "f"'] & E(R/\mathbf{p}) \arrow[dl, "\phi"] \\ & I & \end{tikzcd}$$

is commutative. Then we can consider $E(R/\mathbf{p})$ as a submodule of I. Let

$$\Sigma = \Big\{ S = \{E_j \mid j \in J\} \mid E_j \text{ is an indecomposable injective submodule of } I \text{ for every } j \in J \text{ and } \bigoplus_{j\in J} E_j = \sum_{j\in J} E_j \Big\}.$$

Since $E(R/\mathbf{p}) \subseteq I$, $S = \{E(R/\mathbf{p})\} \in \Sigma$ and so Σ is nonempty. Now we show that every chain in Σ has an upper bound. Let $\mathcal{C} = \{S_k \mid k \in K\}$ be a chain of Σ. We claim that $S = \bigcup_{k\in K} S_k$ is in Σ. Since every element in S is an indecomposable injective submodule of I, it remains to show $\bigoplus_{E\in S} E = \sum_{E\in S} E$, i.e., $E \cap \sum_{E'\in S, E'\neq E} E' = 0$ for all $E \in S$. Let $E \in S$ and let $a \in E \cap \sum_{E'\in S, E'\neq E} E'$. Then $a = \sum_{i=1}^{n} e_i^{'}$ for some $e_i^{'} \in E_i{'}$, $E_i{'} \in S\backslash\{E\}$. Because $\mathcal{C}$ is a chain, there exists $S_k \in \mathcal{C}$ such that $E \in S_k$ and $E_i{'} \in S_k$ for all $i = 1, 2, \cdots, n$. Thus $E \cap \sum_{i=1}^{n} E_i{'} = 0$ and so we have that $a = 0$. Therefore,

$E \cap \sum_{E' \in S, E' \neq E} E' = 0$ and so $\mathcal{C} = \{S_k \mid k \in K\}$ has an upper bound $S = \bigcup_{k \in K} S_k$. By the Zorn's Lemma, Σ has a maximal element. Let $S_J = \{E_j \mid j \in J\}$ be a maximal element in Σ. Now we show that $I = \bigoplus_{j \in J} E_j$. Because E_j is an injective submodule of I for every $j \in J$, $\bigoplus_{j \in J} E_j$ is an injective submodule of I by Lemma 3.2.10. Thus there exists a submodule I_1 of I such that $I_1 \cap \bigoplus_{j \in J} E_j = 0$ and $I = I_1 + \left(\bigoplus_{j \in J} E_j\right)$. By Lemma 3.2.10 again, I_1 is injective since I is injective and $I = I_1 \oplus \left(\bigoplus_{j \in J} E_j\right)$. We claim that $I_1 = 0$. Suppose $I_1 \neq 0$. By Lemma 3.2.8, $AssI_1$ is nonempty. Therefore there exists $\mathbf{p}_1 \in AssI$, i.e., $\mathbf{p}_1 \in \operatorname{Spec}(R)$ and there is an R-module monomorphism $f_1 : R/\mathbf{p}_1 \to I_1$. Moreover, because I_1 is injective, by Lemma 3.2.6 (2), there exists an R-module monomorphism $\phi_1 : E(R/\mathbf{p_1}) \to I_1$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & R/\mathbf{p}_1 \arrow[r, "i"] \arrow[d, "f_1"'] & E(R/\mathbf{p}_1) \arrow[dl, "\phi_1"] \\
& I_1 &
\end{tikzcd}$$

is commutative. Hence we can consider $E(R/\mathbf{p}_1)$ as a submodule of I_1. Because $I_1 \cap \left(\bigoplus_{j \in J} E_j\right) = 0$, $E(R/\mathbf{p}_1) \cap \left(\bigoplus_{j \in J} E_j\right) = 0$. Thus $S_J \cup \{E(R/\mathbf{p}_1)\} \in \Sigma$ and $S_J \subsetneq S_J \cup \{E(R/\mathbf{p}_1)\}$. This contradicts the fact that S_J is a maximal element in Σ. Hence $I_1 = 0$ and so $I = \bigoplus_{j \in J} E_j$. This completes the proof. $\square$

From Proposition 3.2.9 and Proposition 3.2.11, we see that a nonzero injective R-module I is a direct sum of $E(R/\mathbf{p})$ for some $\mathbf{p} \in \operatorname{Spec}(R)$. In the next lemma, we will show that $H^0(- \otimes \mathcal{C}) = \Gamma_{\mathbf{m}}(-)$, i.e., $H^0(A \otimes \mathcal{C}) \cong \Gamma_{\mathbf{m}}(A)$ for all R-module A.

Lemma 3.2.12. *Let A be an R-module and assume the ideal $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is $\mathbf{m}$-primary. Then $H^0(A \otimes \mathcal{C}) \cong \Gamma_{\mathbf{m}}(A)$, where $\mathcal{C}$ is the Čech complex with respect to the sequence $x_1, x_2, \ldots, x_n$.*

Proof. Because R is a Noetherian local ring with the maximal ideal $\mathbf{m}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is $\mathbf{m}$-primary, $\mathbf{m}^s \subseteq \mathbf{x} \subseteq \mathbf{m}$ for some $s > 0$. Then it is not difficult to check that

$$\Gamma_{\mathbf{m}}(A) = \{y \in A \mid \mathbf{x}^k y = 0 \text{ for some } k \geq 0\}.$$

Using the facts: $A \otimes R \cong A$, $A \otimes (\bigoplus_{i=1}^n R_{x_i}) \cong \bigoplus_{i=1}^n (A \otimes R_{x_i})$, and $A \otimes R_{x_i} \cong A_{x_i}$ for all $i = 1, 2, \cdots, n$, we see that

$$H^0(A \otimes \mathcal{C}) = \mathrm{Ker}\,(A \otimes R \longrightarrow A \otimes (\bigoplus_{i=1}^n R_{x_i})) \cong \mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i})\,.$$

Now we show that $\Gamma_{\mathbf{m}}(A) = \mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i})$.

- Let $y \in \Gamma_{\mathbf{m}}(A)$. Then $\mathbf{x}^k y = 0$ for some $k \geq 0$. So we have $x_i^k y = 0$ for all $i = 1, 2, \cdots, n$. Therefore, $(\frac{y}{1}, \frac{y}{1}, \cdots, \frac{y}{1}) = 0$ in $\bigoplus_{i=1}^n A_{x_i}$ and so $y \in \mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i})$. Hence $\Gamma_{\mathbf{m}}(A) \subseteq \mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i})$.
- Conversely, let $a \in \mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i})$. Then $(\frac{a}{1}, \frac{a}{1}, \cdots, \frac{a}{1}) = 0$ in $\bigoplus_{i=1}^n A_{x_i}$. Therefore, for each $i = 1, 2, \cdots, n$, there exists $t_i \in \mathbb{N}$ such that $x_i^{t_i} a = 0$. Take $t = \sum_{i=1}^n t_i$, then $\mathbf{x}^t a = 0$ and so we have that $a \in \Gamma_{\mathbf{m}}(A)$. Hence $\mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i}) \subseteq \Gamma_{\mathbf{m}}(A)$.

Therefore $\Gamma_{\mathbf{m}}(A) = \mathrm{Ker}\,(A \longrightarrow \bigoplus_{i=1}^n A_{x_i})$. Hence $H^0(A \otimes \mathcal{C}) \cong \Gamma_{\mathbf{m}}(A)$ and the proof is complete. $\square$

3.3 Čech complexes vs universal connected sequences

In Theorem 3.3.5, we will show that $\{H^n(- \otimes \mathcal{C}), E^n\}_{n \geq 0}$ is a universal connected sequence. Before we prove Theorem 3.3.5, we present some lemmas that we need in the proof.

Lemma 3.3.1. *Let M be an R-module and let S be a multiplicative closed set in R. Then as R_S-modules, $E(M)_S$ is an essential extension of M_S.*

Proof. It suffices show that $R_S x \cap M_S \neq 0$ for all nonzero $x \in E(M)_S$. Because $x \in E(M)_S$, $x = \frac{y}{s_1}$ for some $y \in E(M)$ and $s_1 \in S$. It is not difficult to check that

$$R_S\, x = R_S\, y\,.$$

Since $x \neq 0$ in $E(M)_S$, $ty \neq 0$ for all $t \in S$. We consider the set

$$\Sigma = \{Ann(sy) \mid s \in S\}\,.$$

Because R is a Noetherian ring, Σ has a maximal element. Let $Ann(ty)$ with $t \in S$ be a maximal element in Σ. Then we have that

$$R_S\, x = R_S\, y = R_S\, ty\,.$$

Since $E(M)$ is an essential extension of M and since $ty \neq 0$, $R(ty) \cap M = I(ty) \neq 0$, where $I = (M :_R ty)$ is an ideal of R. Again by the fact that R is a Noetherian ring, $I = (a_1, a_2, \cdots, a_n)$ for some elements $a_1, a_2, \cdots, a_n \in R$. Now we show that there exists a_i such that $s[a_i(ty)] \neq 0$ for all $s \in S$. Suppose that for each $i = 1, 2, \cdots, n$, there exists $s_i \in S$ such that $s_i[a_i(ty)] = 0$. Take $s = \prod_{i=1}^{n} s_i$, then $s[a_i(ty)] = 0$ for all $i = 1, 2, \cdots, n$. Thus we have that $a_i \in Ann(sty)$ for all $i = 1, 2, \cdots, n$. On the other hand, since $Ann(ty) \subseteq Ann(sty)$ and since $Ann(ty)$ is a maximal element in Σ, $Ann(ty) = Ann(sty)$. So we get $a_i \in Ann(ty)$ for all $i = 1, 2, \cdots, n$. Therefore, $I \subseteq Ann(ty)$, i.e., $I(ty) = 0$. It contradicts to $I(ty) \neq 0$, so there exists a_i such that $s[a_i(ty)] \neq 0$ for all $s \in S$. Hence $\frac{a_i(ty)}{1} \in (R(ty) \cap M)_S = R_S(ty) \cap M_S = R_S x \cap M_S$ and $\frac{a_i(ty)}{1} \neq 0$ in $E(M)_S$. Therefore, $R_S x \cap M_S \neq 0$ for all nonzero $x \in E(M)_S$. □

Lemma 3.3.2. *Let* $\mathbf{p} \in SpecR$ *and let* $y \in R$.

(1) *If* $y \in \mathbf{p}$, *then* $E(R/\mathbf{p})_y = 0$.

(2) *If* $y \notin \mathbf{p}$, *then* $yE(R/\mathbf{p}) = E(R/\mathbf{p})$.

Proof. For (1), because $y \in \mathbf{p}$, $(R/\mathbf{p})_y = 0$. By Lemma 3.3.1, $E(R/\mathbf{p})_y$ is an essential extension of $(R/\mathbf{p})_y$. Therefore $E(R/\mathbf{p})_y = 0$.

For (2), consider the R-module homomorphism $f : E(R/\mathbf{p}) \to E(R/\mathbf{p})$ defined by $f(a) = ya$ for all $a \in E(R/\mathbf{p})$. Now we claim that f is one-to-one, i.e., $\operatorname{Ker} f = 0$. Suppose $\operatorname{Ker} f \neq 0$. Because $\operatorname{Ker} f$ is a nonzero submodule of $E(R/\mathbf{p})$ and $E(R/\mathbf{p})$ is an essential extension of $R/\mathbf{p}$, $\operatorname{Ker} f \cap R/\mathbf{p} \neq 0$. However for $a \in \operatorname{Ker} f \cap R/\mathbf{p}$, $a = r + \mathbf{p}$ for some $r \in R$. Thus $0 = f(a) = f(r + \mathbf{p}) = yr + \mathbf{p}$, and so we have $yr \in \mathbf{p}$. Since $\mathbf{p}$ is a prime ideal and since $y \notin \mathbf{p}$, $r \in \mathbf{p}$, i.e., $a = r + \mathbf{p} = 0$ in $R/\mathbf{p}$. Hence $\operatorname{Ker} f \cap R/\mathbf{p} = 0$ and we get a contradiction. Thus $\operatorname{Ker} f = 0$, i.e., f is one-to-one. By the First Isomorphism Theorem, $E(R/\mathbf{p}) \cong \operatorname{Im} f = yE(R/\mathbf{p})$. Then $yE(R/\mathbf{p})$

is injective since $E(R/\mathbf{p})$ is injective. Moreover, because $yE(R/\mathbf{p})$ is a submodule of $E(R/\mathbf{p})$, $yE(R/\mathbf{p})$ is a direct summand of $E(R/\mathbf{p})$, i.e., there exists a submodule M_1 of $E(R/\mathbf{p})$ such that $E(R/\mathbf{p}) = yE(R/\mathbf{p}) \oplus M_1$. However, $E(R/\mathbf{p})$ is indecomposable, by Proposition 3.2.9, so $M_1 = 0$. Therefore, $yE(R/\mathbf{p}) = E(R/\mathbf{p})$. □

Remark 3.3.3. *In the proof of Lemma* 3.3.2(2)*, we have that the R-module homomorphism $f : E(R/\mathbf{p}) \to E(R/\mathbf{p})$, defined by $f(a) = ya$ for all $a \in E(R/\mathbf{p})$, is one-to-one, and* $\operatorname{Im} f = yE(R/\mathbf{p})$*. Therefore, for every $a \in E(R/\mathbf{p})$, there exists a unique element $b \in E(R/\mathbf{p})$ such that $a = yb$. Similarly, because $y \notin \mathbf{p}$ and $\mathbf{p}$ is prime, $y^s \notin \mathbf{p}$ for all $s \in \mathbb{N}$. Thus for every $a \in E(R/\mathbf{p})$ and for every $s \in \mathbb{N}$, there exists a unique element $b \in E(R/\mathbf{p})$ such that $a = y^s b$. In particular, if $a \in E(R/\mathbf{p})$ such that $y^s a = 0$, then $a = 0$.*

Notation 3.3.4. *In the proof of Theorem* 3.3.5*, we need to use some special notations.*

(1) *For $t \geqslant 1$ and $1 \leq i_1 < i_2 < \ldots < i_t \leq n$, we let $e_{i_1 i_2 \ldots i_t}$ to represent the component $R_{x_{i_1} x_{i_2} \cdots x_{i_t}}$ in $C^t = \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}}$. Hence, we can write*

$$C^t = \bigoplus\nolimits_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}} = \sum\nolimits_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}} e_{i_1 i_2 \ldots i_t}.$$

Similarly, we also write

$$\bigoplus\nolimits_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} E(R/\mathbf{p})_{x_{i_1} x_{i_2} \cdots x_{i_t}} = \sum\nolimits_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} E(R/\mathbf{p})_{x_{i_1} x_{i_2} \cdots x_{i_t}} e_{i_1 i_2 \ldots i_t},$$

where $\mathbf{p} \in Spec(R)$. We also use the convention that $e_{j_1 j_2 \ldots j_t} = e_{i_1 i_2 \ldots i_t}$ as long as $\{j_1, j_2, \ldots, j_t\} = \{i_1, i_2, \ldots, i_t\}$.

(2) *For two disjoint subsets X and Y of $\{1, 2, \ldots, n\}$, we let $\delta(X, Y) = (-1)^{|Z|}$, where $Z = \{(a, b) \in X \times Y \mid a < b\}$. Note that if $X = X_1 \cup X_2$ is a disjoint union, then $Z = \{(a, b) \in X \times Y \mid a < b\} = \{(a, b) \in X_1 \times Y \mid a < b\} \cup \{(a, b) \in X_2 \times Y \mid a < b\}$ is a disjoint union and so $\delta(X, Y) = \delta(X_1, Y) \cdot \delta(X_2, Y)$. Moreover, if $i, j \in \{1, 2, \ldots, n\}$ are distinct, then $\delta(\{i\}, \{j\}) \cdot \delta(\{j\}, \{i\}) = -1$. With this*

new notation, the component $R_{x_{i_1}\cdots x_{i_t}} \to R_{x_{j_1}x_{j_2}\cdots x_{j_{t+1}}}$*, that gives the differentiation* $d^t : C^t \to C^{t+1}$*, can be rewritten as*

$$\begin{cases} \delta(\{i_1, i_2, \ldots, i_t\}, \{j_s\}) \cdot nat : R_{x_{i_1}\cdots x_{i_t}} \to (R_{x_{i_1}\cdots x_{i_t}})_{x_{j_s}} & \text{if } \{j_1, \ldots, j_{t+1}\} = \{i_1, \ldots, i_t\} \cup \{j_s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $nat : R_{x_{i_1}\cdots x_{i_t}} \to (R_{x_{i_1}\cdots x_{i_t}})_{x_{j_s}}$ *is the natural* R*-module homomorphism defined by* $\frac{r}{(x_{i_1}\cdots x_{i_t})^l} \mapsto \frac{x_{j_s}^l r}{(x_{i_1}\cdots x_{i_t} x_{j_s})^l}$.

Theorem 3.3.5. *Let* $x_1, x_2, \cdots, x_n \in R$ *such that the ideal* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *is* $\mathbf{m}$*-primary and let* $\mathcal{C}$ *be the Čech complex with respect to the sequence* $x_1, x_2, \ldots, x_n$*. Then* $\{H^t(- \otimes \mathcal{C}), E^t\}_{t\geq 0}$ *is a universal connected sequence.*

Proof. First of all, we show that $\{H^t(- \otimes \mathcal{C}), E^t\}_{t\geq 0}$ is a connected sequence, i.e., $\big(H^t(- \otimes \mathcal{C}), E^t, H^{t+1}(- \otimes \mathcal{C})\big)$ is a connected pair for all $t \geq 0$. Let $(\alpha, \beta, \gamma) : E \to E'$ be a morphism in the category $\mathcal{E}$, where $E : 0 \to A \to B \to C \to 0$ and $E' : 0 \to A' \to B' \to C' \to 0$ are two short exact sequences of R-modules. For all $t = 0, 1, \cdots, n$, because the tth term C^t in $\mathcal{C}$ is a flat R-module, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes C^t & \longrightarrow & B \otimes C^t & \longrightarrow & C \otimes C^t & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha\otimes 1_{C^t}} & & \downarrow{\scriptstyle \beta\otimes 1_{C^t}} & & \downarrow{\scriptstyle \gamma\otimes 1_{C^t}} & & \\ 0 & \longrightarrow & A' \otimes C^t & \longrightarrow & B' \otimes C^t & \longrightarrow & C' \otimes C^t & \longrightarrow & 0 \end{array}$$

is commutative with both rows exact. In other words, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes \mathcal{C} & \longrightarrow & B \otimes \mathcal{C} & \longrightarrow & C \otimes \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha\otimes 1_{\mathcal{C}}} & & \downarrow{\scriptstyle \beta\otimes 1_{\mathcal{C}}} & & \downarrow{\scriptstyle \gamma\otimes 1_{\mathcal{C}}} & & \\ 0 & \longrightarrow & A' \otimes \mathcal{C} & \longrightarrow & B' \otimes \mathcal{C} & \longrightarrow & C' \otimes \mathcal{C} & \longrightarrow & 0 \end{array}$$

is commutative. By Lemma 3.1.6, we have that the diagram

$$\begin{array}{ccc} H^t(C \otimes \mathcal{C}) & \xrightarrow{E^t} & H^{t+1}(A \otimes \mathcal{C}) \\ {\scriptstyle H^t(\gamma\otimes 1_{\mathcal{C}})}\downarrow & & \downarrow{\scriptstyle H^{t+1}(\alpha\otimes 1_{\mathcal{C}})} \\ H^t(C' \otimes \mathcal{C}) & \xrightarrow{(E')^t} & H^{t+1}(A' \otimes \mathcal{C}) \end{array}$$

is commutative for all $t \geq 0$. Hence $\big(H^t(- \otimes \mathcal{C}), E^t, H^{t+1}(- \otimes \mathcal{C})\big)$ is a connected pair for all $t \geq 0$. Therefore, $\{H^t(- \otimes \mathcal{C}), E^t\}_{t\geq 0}$ is a connected sequence.

Next, we show that $\{H^t(-\otimes\mathcal{C}), E^t\}_{t\geq 0}$ is universal. Let $E : 0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. For all $t = 0, 1, \cdots, n$, because the tth term C^t in $\mathcal{C}$ is a flat R-module, the sequence

$$0 \longrightarrow A\otimes C^t \longrightarrow B\otimes C^t \longrightarrow C\otimes C^t \longrightarrow 0$$

is exact. Hence the sequence of complexes

$$0 \longrightarrow A\otimes \mathcal{C} \longrightarrow B\otimes \mathcal{C} \longrightarrow C\otimes \mathcal{C} \longrightarrow 0$$

is exact. By Lemma 3.1.5, there is a long exact sequence of cohomology

$$\cdots \to H^t(A\otimes\mathcal{C}) \to H^t(B\otimes\mathcal{C}) \to H^t(C\otimes\mathcal{C}) \xrightarrow{E^t} H^{t+1}(A\otimes\mathcal{C}) \to \cdots .$$

By Corollary 2.3.4, it remains to show that if I is an injective R-module, then $H^t(I\otimes\mathcal{C}) = 0$ for all $t > 0$. Let I be an injective R-module. From Proposition 3.2.9 and Proposition 3.2.11, we know that $I = \bigoplus_{j\in J} E(R/\mathbf{p}_j)$, where $\{\mathbf{p}_j \mid j \in J\}$ is a family of prime ideals of R. Note that $\big(\bigoplus_{j\in J} E(R/\mathbf{p}_j)\big)\otimes A \cong \bigoplus_{j\in J}\big(E(R/\mathbf{p}_j)\otimes A\big)$ for all R-module A. Hence, we only need to take care of the case where $I = E(R/\mathbf{p})$ for some $\mathbf{p}\in$ Spec(R). In other words, it suffices to show that if $\mathbf{p}\in \mathrm{Spec}(R)$, then $H^t(E(R/\mathbf{p})\otimes\mathcal{C}) = 0$ for all $t > 0$. We separate the discussion into two situations.

Suppose that $\mathbf{p} = \mathbf{m}$. Because the ideal $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is $\mathbf{m}$-primary, we have $x_{i_1}x_{i_2}\cdots x_{i_t} \in \mathbf{m}$ for $t \geqslant 1$ and $1 \leq i_1 < i_2 < \ldots < i_t \leq n$. By Lemma 3.3.2 (1), $E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} = 0$ for $t \geqslant 1$ and $1 \leq i_1 < i_2 < \ldots < i_t \leq n$. Hence for all $t \geqslant 1$, we have that

$$\begin{aligned} E(R/\mathbf{p})\otimes C^t &= E(R/\mathbf{p})\otimes\big(\textstyle\bigoplus_{1\leq i_1<i_2<\ldots<i_t\leq n} R_{x_{i_1}x_{i_2}\cdots x_{i_t}}\big) \\ &= \textstyle\bigoplus_{1\leq i_1<i_2<\ldots<i_t\leq n} E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} \\ &= 0. \end{aligned}$$

Therefore the complex $E(R/\mathbf{p})\otimes\mathcal{C}$ is

$$0 \to E(R/\mathbf{p}) \to 0 \to 0 \to \cdots$$

and so $H^t(E(R/\mathbf{p})\otimes\mathcal{C}) = 0$ for all $t > 0$.

Suppose that $\mathbf{p} \neq \mathbf{m}$. Because $\mathbf{m}$ is the unique maximal ideal of R, $\mathbf{p} \subsetneq \mathbf{m}$. Since $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is $\mathbf{m}$-primary, there exists x_j such that $x_j \notin \mathbf{p}$. Now we fix the index j and consider the natural R-module homomorphism $\iota : E(R/\mathbf{p}) \to E(R/\mathbf{p})_{x_j}$, i.e., $\iota(a) = \frac{a}{1}$ for all $a \in E(R/\mathbf{p})$. Note that

$$a \in \operatorname{Ker} \iota \Rightarrow \frac{a}{1} = 0 \text{ in } E(R/\mathbf{p})_{x_j} \Rightarrow {x_j}^k a = 0 \text{ in } E(R/\mathbf{p}) \Rightarrow a = 0 \text{ in } E(R/\mathbf{p}),$$

where the last implication follows from Remark 3.3.3. Hence the natural R-module homomorphism $\iota : E(R/\mathbf{p}) \to E(R/\mathbf{p})_{x_j}$ is one-to-one. Moreover, since $E(R/\mathbf{p})$ is an injective R-module, there exists an R-module homomorphism $g : E(R/\mathbf{p})_{x_j} \to E(R/\mathbf{p})$ such that $g\iota = 1_{E(R/\mathbf{p})}$. More precisely, by Remark 3.3.3, for all $s \in \mathbb{N}$ and for all $a \in E(R/\mathbf{p})$, there is a unique element $b \in E(R/\mathbf{p})$ such that ${x_j}^s b = a$. Thus for all $\frac{a}{{x_j}^s} \in E(R/\mathbf{p})_{x_j}$, $a = {x_j}^s b$ for some $b \in E(R/\mathbf{p})$ and so

$$g(\frac{a}{{x_j}^s}) = g(\frac{{x_j}^s b}{{x_j}^s}) = g(\frac{b}{1}) = g\iota(b) = b,$$

i.e., $g(\frac{a}{{x_j}^s}) = b$ where $b \in E(R/\mathbf{p})$ is such that $a = {x_j}^s b$. On the other hand, for all $t \geqslant 2$ and $1 \leq i_1 < i_2 < \ldots < i_{t-1} \leq n$ with $i_1, i_2, \ldots, i_{t-1} \neq j$, g induces an R-module homomorphism

$$g_{i_1, i_2, \ldots, i_{t-1}} : E(R/\mathbf{p})_{x_{i_1} x_{i_2} \cdots x_{i_{t-1}} x_j} = \big(E(R/\mathbf{p})_{x_j}\big)_{x_{i_1} x_{i_2} \cdots x_{i_{t-1}}} \quad \to \quad E(R/\mathbf{p})_{x_{i_1} x_{i_2} \cdots x_{i_{t-1}}}.$$

Consider the identity cochain map

$$1 : E(R/\mathbf{p}) \otimes \mathcal{C} \to E(R/\mathbf{p}) \otimes \mathcal{C},$$

i.e., the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E(R/\mathbf{p}) \otimes C^{t-1} \arrow[r] \arrow[d, "1"] & E(R/\mathbf{p}) \otimes C^{t} \arrow[r] \arrow[d, "1"] & E(R/\mathbf{p}) \otimes C^{t+1} \arrow[r] \arrow[d, "1"] & \cdots \\
\cdots \arrow[r] & E(R/\mathbf{p}) \otimes C^{t-1} \arrow[r] & E(R/\mathbf{p}) \otimes C^{t} \arrow[r] & E(R/\mathbf{p}) \otimes C^{t+1} \arrow[r] & \cdots ,
\end{tikzcd} \tag{1}$$

where $C^t = \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}}$. Since

$$\begin{aligned}
E(R/\mathbf{p}) \otimes C^t &= E(R/\mathbf{p}) \otimes \big(\textstyle\bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}}\big) \\
&\cong \textstyle\bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} \big(E(R/\mathbf{p}) \otimes R_{x_{i_1} x_{i_2} \cdots x_{i_t}}\big) \\
&\cong \textstyle\bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq n} E(R/\mathbf{p})_{x_{i_1} x_{i_2} \cdots x_{i_t}} \ ,
\end{aligned}$$

the diagram (1) is just the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_{t-1}}} \arrow[r, "d^{t-1}"] \arrow[d, "1"] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} \arrow[r, "d^t"] \arrow[d, "1"] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_{t+1}}} \arrow[r] \arrow[d, "1"] & \cdots \\
\cdots \arrow[r] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_{t-1}}} \arrow[r, "d^{t-1}"] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} \arrow[r, "d^t"] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_{t+1}}} \arrow[r] & \cdots .
\end{tikzcd} \tag{2}$$

We let σ^1 be the composition of the canonical R-module homomorphism $\bigoplus_{i=1}^n E(R/\mathbf{p})_{x_i} \to E(R/\mathbf{p})_{x_j}$ and g, i.e., for all $\bigoplus_{i=1}^n \frac{a_i}{{x_i}^{s_i}} \in \bigoplus_{i=1}^n E(R/\mathbf{p})_{x_i}$, $\sigma^1\big(\bigoplus_{i=1}^n \frac{a_i}{{x_i}^{s_i}}\big) = g(\frac{a_j}{{x_j}^{s_j}})$, and let $\sigma^{n+1} : 0 \to E(R/\mathbf{p})_{x_1x_2\cdots x_n}$ be the zero map, and for each t with $2 \leqslant t \leqslant n$, we define the R-module homomorphism

$$\sigma^t : \textstyle\bigoplus_{1\leq i_1<i_2<\ldots<i_t\leq n} E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} \to \bigoplus_{1\leq i_1<i_2<\ldots<i_{t-1}\leq n} E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_{t-1}}}$$

by giving on the component $E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} \to E(R/\mathbf{p})_{x_{j_1}x_{j_2}\cdots x_{j_{t-1}}}$ to be

$$\begin{cases} \delta(\{j_1,\ldots,j_{t-1}\},\{j\}) \cdot g_{i_1,i_2,\ldots,i_{t-1}} & \text{if } \{i_1,\ldots,i_t\} = \{j_1,\ldots,j_{t-1}\} \cup \{j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we show that $d^{t-1}\sigma^t + \sigma^{t+1}d^t = 1$ for all $t > 0$. First, we show that $d^0\sigma^1 + \sigma^2 d^1 = 1$, i.e., the case of $t = 1$. Note that with the notation we mention in Notation 3.3.4, we have

$$\bigoplus_{i=1}^n E(R/\mathbf{p})_{x_i} = \sum_{i=1}^n E(R/\mathbf{p})_{x_i} e_i.$$

Hence, in order to show that $d^0\sigma^1 + \sigma^2 d^1 = 1$, we only need to show that for all $i = 1, 2, \ldots, n$, $(d^0\sigma^1 + \sigma^2 d^1)(\alpha e_i) = \alpha e_i$ for all $\alpha \in E(R/\mathbf{p})_{x_i}$. Let $\frac{a}{{x_i}^k} \in E(R/\mathbf{p})_{x_i}$. If $i \neq j$, then

$$d^0\sigma^1\big(\frac{a}{{x_i}^k} e_i\big) = 0.$$

Moreover, let $L = \{1, 2, \ldots, n\} \setminus \{i\}$. Then $j \in L$ and we have

$$\begin{aligned}
\sigma^2 d^1\big(\tfrac{a}{{x_i}^k} e_i\big) &= \sigma^2\big(\textstyle\sum_{w\in L} \delta(\{i\},\{w\}) \frac{{x_w}^k a}{(x_i x_w)^k} e_{iw}\big) \\
&= \sigma^2\big(\delta(\{i\},\{j\}) \tfrac{{x_j}^k a}{(x_i x_j)^k} e_{ij}\big) + \sigma^2\big(\textstyle\sum_{w\in L\setminus\{j\}} \delta(\{i\},\{w\}) \frac{{x_w}^k a}{(x_i x_w)^k} e_{iw}\big) \\
&= \delta(\{i\},\{j\}) \sigma^2\big(\tfrac{{x_j}^k a}{(x_i x_j)^k} e_{ij}\big) + 0 \\
&= \delta(\{i\},\{j\}) \cdot \big(\delta(\{i\},\{j\}) \tfrac{a}{{x_i}^k} e_i\big) \\
&= \tfrac{a}{{x_i}^k} e_i,
\end{aligned}$$

where the third equality follows form the fact that $j \notin \{i, w\}$ for all $w \in L \setminus \{j\}$. Hence $(d^0\sigma^1 + \sigma^2 d^1)(\frac{a}{{x_i}^k}e_i) = \frac{a}{{x_i}^k}e_i$. If $i = j$, let $b \in E(R/\mathbf{p})$ such that ${x_j}^k b = a$. Then

$$d^0\sigma^1\big(\frac{a}{{x_j}^k}e_j\big) = d^0(b) = \sum_{i=1}^{n} \frac{b}{1}e_i = \sum_{i=1}^{n} \frac{{x_i}^k b}{{x_i}^k}e_i.$$

Moreover, let $X = \{1, 2, \ldots, n\} \setminus \{j\}$. Then

$$\begin{aligned}
\sigma^2 d^1\big(\tfrac{a}{{x_j}^k}e_j\big) &= \sigma^2\big(\textstyle\sum_{w\in X} \delta(\{j\}, \{w\})\frac{{x_w}^k a}{(x_j x_w)^k}e_{jw}\big) \\
&= \textstyle\sum_{w\in X} \delta(\{j\}, \{w\})\sigma^2\big(\frac{{x_w}^k {x_j}^k b}{(x_j x_w)^k}e_{jw}\big) \\
&= \textstyle\sum_{w\in X} \delta(\{j\}, \{w\})\delta(\{w\}, \{j\})\frac{{x_w}^k b}{{x_w}^k}e_w \\
&= \textstyle\sum_{w\in X}(-1) \cdot \frac{{x_w}^k b}{{x_w}^k}e_w.
\end{aligned}$$

Hence $(d^0\sigma^1 + \sigma^2 d^1)(\frac{a}{{x_j}^k}e_j) = \frac{a}{{x_j}^k}e_j$. Therefore, $d^0\sigma^1 + \sigma^2 d^1 = 1$. On the other hand, for all $\frac{a}{(x_1x_2\cdots x_n)^k} \in E(R/\mathbf{p})_{x_1x_2\cdots x_n}$, let $b \in E(R/\mathbf{p})$ such that ${x_j}^k b = a$, then

$$\begin{aligned}
(d^{n-1}\sigma^n + \sigma^{n+1}d^n)\big(\tfrac{a}{(x_1x_2\cdots x_n)^k}e_{1,2,\ldots,n}\big) &= d^{n-1}\sigma^n\big(\tfrac{{x_j}^k b}{(x_1x_2\cdots x_n)^k}e_{1,2,\ldots,n}\big) \\
&= d^{n-1}\big(\delta(X, \{j\})\tfrac{b}{(x_1\cdots x_{j-1}x_{j+1}\cdots x_n)^k}e_{1,\ldots,j-1,j+1\ldots,n}\big) \\
&= \delta(X, \{j\}) \cdot \delta(X, \{j\})\tfrac{{x_j}^k b}{(x_1x_2\cdots x_n)^k}e_{1,2,\ldots,n} \\
&= \tfrac{a}{(x_1x_2\cdots x_n)^k}e_{1,2,\ldots,n}.
\end{aligned}$$

Hence $d^{n-1}\sigma^n + \sigma^{n+1}d^n = 1$. Therefore, we have $d^{t-1}\sigma^t + \sigma^{t+1}d^t = 1$ for $t = 1$ and $t = n$. Next, we show that $d^{t-1}\sigma^t + \sigma^{t+1}d^t = 1$ for all $2 \leq t \leq n-1$. Note that with the notation we mention in Notation 3.3.4(1), we have that

$$\textstyle\bigoplus_{1\leq i_1<i_2<\ldots<i_t\leq n} E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}} = \sum_{1\leq i_1<i_2<\ldots<i_t\leq n} E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}}e_{i_1i_2\ldots i_t}.$$

Hence, in order to show that $d^{t-1}\sigma^t + \sigma^{t+1}d^t = 1$, we only need to show that for all $1 \leq i_1 < i_2 < \ldots < i_t \leq n$, $(d^{t-1}\sigma^t + \sigma^{t+1}d^t)(\alpha e_{i_1i_2\ldots i_t}) = \alpha e_{i_1i_2\ldots i_t}$ for all $\alpha \in E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}}$. We separate our discussion into two cases.

Case 1: $j \notin \{i_1, i_2, \ldots, i_t\}$. Let $\frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} \in E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}}$. Since $j \notin \{i_1, i_2, \ldots, i_t\}$, we have

$$d^{t-1}\sigma^t\big(\frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k}e_{i_1i_2\ldots i_t}\big) = d^{t-1}(0) = 0.$$

Let $J = \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_t\}$. Then $j \in J$ and we have

$$\begin{aligned}
& \sigma^{t+1} d^t\big(\tfrac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t}\big) \\
= \;& \sigma^{t+1}\big(\textstyle\sum_{w\in J} \delta(\{i_1, i_2, \ldots, i_t\}, \{w\}) \tfrac{{x_w}^k a}{(x_{i_1}x_{i_2}\cdots x_{i_t}x_w)^k} e_{i_1 i_2 \ldots i_t w}\big) \\
= \;& \sigma^{t+1}\big(\delta(\{i_1, i_2, \ldots, i_t\}, \{j\}) \tfrac{{x_j}^k a}{(x_{i_1}x_{i_2}\cdots x_{i_t}x_j)^k} e_{i_1 i_2 \ldots i_t j}\big) \\
& + \sigma^{t+1}\big(\textstyle\sum_{w\in J\setminus\{j\}} \delta(\{i_1, i_2, \ldots, i_t\}, \{w\}) \tfrac{{x_w}^k a}{(x_{i_1}x_{i_2}\cdots x_{i_t}x_w)^k} e_{i_1 i_2 \ldots i_t w}\big) \\
= \;& \delta(\{i_1, i_2, \ldots, i_t\}, \{j\}) \sigma^{t+1}\big(\tfrac{{x_j}^k a}{(x_{i_1}x_{i_2}\cdots x_{i_t}x_j)^k} e_{i_1 i_2 \ldots i_t j}\big) + 0 \\
= \;& \delta(\{i_1, i_2, \ldots, i_t\}, \{j\}) \cdot \big(\delta(\{i_1, i_2, \ldots, i_t\}, \{j\}) \tfrac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t}\big) \\
= \;& \tfrac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t},
\end{aligned}$$

where the third equality follows form the fact that $j \notin \{i_1, i_2, \ldots, i_t, w\}$ for all $w \in J\setminus\{j\}$. Hence $(d^{t-1}\sigma^t + \sigma^{t+1}d^t)(\frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t}) = \frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t}$.

Case 2: $j \in \{i_1, i_2, \ldots, i_t\}$, i.e., $j = i_m$ for some $m \in \{1, 2, \ldots, t\}$. We let

$$U = \{i_1, i_2, \ldots, i_t\},\ U_m = U \setminus \{i_m\},\ V = \{1, 2, \ldots, n\} \setminus U, \text{ and } V_m = \{1, 2, \ldots, n\} \setminus U_m.$$

Let $\frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} \in E(R/\mathbf{p})_{x_{i_1}x_{i_2}\cdots x_{i_t}}$ and let $b \in E(R/\mathbf{p})$ such that ${x_j}^k b = a$. Then

$$\begin{aligned}
& d^{t-1}\sigma^t\big(\tfrac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t}\big) \\
= \;& d^{t-1}\big(\delta(U_m, \{j\}) \tfrac{b}{(x_{i_1}\cdots x_{i_{m-1}}x_{i_{m+1}}\cdots x_{i_t})^k} e_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_t}\big) \\
= \;& \delta(U_m, \{j\})\big(\textstyle\sum_{w\in V_m} \delta(U_m, \{w\}) \tfrac{{x_w}^k b}{(x_{i_1}\cdots x_{i_{m-1}}x_{i_{m+1}}\cdots x_{i_t}x_w)^k} e_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_t w}\big) \\
= \;& \delta(U_m, \{j\}) \cdot \delta(U_m, \{j\}) \tfrac{{x_j}^k b}{(x_{i_1}\cdots x_{i_{m-1}}x_{i_{m+1}}\cdots x_{i_t}x_j)^k} e_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_t j}\big) \\
& + \textstyle\sum_{w\in V_m\setminus\{j\}} \delta(U_m, \{j\}) \cdot \delta(U_m, \{w\}) \tfrac{{x_w}^k b}{(x_{i_1}\cdots x_{i_{m-1}}x_{i_{m+1}}\cdots x_{i_t}x_w)^k} e_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_t w} \\
= \;& \tfrac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t} + \textstyle\sum_{w\in V} \delta(U_m, \{j\})\delta(U_m, \{w\}) \tfrac{{x_w}^k b}{(x_{i_1}\cdots x_{i_{m-1}}x_{i_{m+1}}\cdots x_{i_t}x_w)^k} e_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_t w}
\end{aligned}$$

and

$$\begin{aligned}
& \sigma^{t+1} d^t\big(\tfrac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k} e_{i_1 i_2 \ldots i_t}\big) \\
= \;& \sigma^{t+1}\big(\textstyle\sum_{w\in V} \delta(U, \{w\}) \tfrac{{x_w}^k a}{(x_{i_1}x_{i_2}\cdots x_{i_t}x_w)^k} e_{i_1 i_2 \ldots i_t w}\big) \\
= \;& \textstyle\sum_{w\in V} \delta(U, \{w\}) \sigma^{t+1}\big(\tfrac{{x_w}^k {x_j}^k b}{(x_{i_1}x_{i_2}\cdots x_{i_t}x_w)^k} e_{i_1 i_2 \ldots i_t w}\big) \\
= \;& \textstyle\sum_{w\in V} \delta(U, \{w\}) \cdot \delta(U_m \cup \{w\}, \{j\}) \tfrac{{x_w}^k b}{(x_{i_1}\cdots x_{i_{m-1}}x_{i_{m+1}}\cdots x_{i_t}x_w)^k} e_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_t w}.
\end{aligned}$$

Note that for all $w \in V$, we have

$$\begin{aligned}
& \delta(U,\{w\}) \cdot \delta(U_m \cup \{w\},\{j\}) \\
= \;& \big(\delta(U_m,\{w\})\delta(\{j\},\{w\})\big) \cdot \big(\delta(U_m,\{j\})\delta(\{w\},\{j\})\big) \\
= \;& \delta(U_m,\{w\})\delta(U_m,\{j\})\delta(\{j\},\{w\})\delta(\{w\},\{j\}) \\
= \;& (-1) \cdot \delta(U_m,\{w\})\delta(U_m,\{j\}).
\end{aligned}$$

Hence $(d^{t-1}\sigma^t + \sigma^{t+1}d^t)(\frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k}e_{i_1 i_2 \ldots i_t}) = \frac{a}{(x_{i_1}x_{i_2}\cdots x_{i_t})^k}e_{i_1 i_2 \ldots i_t}$.

Thus $d^{t-1}\sigma^t + \sigma^{t+1}d^t = 1$ for all $t > 0$. Hence by Definition 3.1.1 (1), the cochain map

$$\begin{tikzcd}
\bigoplus E(R/\mathbf{p})_{x_i} \arrow[r] \arrow[d, "1"] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}} \arrow[r] \arrow[d, "1"] & \cdots \arrow[r] & \bigoplus E(R/\mathbf{p})_{x_{i_1}\cdots x_{i_{n-1}}} \arrow[r] \arrow[d, "1"] & E(R/\mathbf{p})_{x_1\cdots x_n} \arrow[r] \arrow[d, "1"] & 0 \\
\bigoplus E(R/\mathbf{p})_{x_i} \arrow[r] & \bigoplus E(R/\mathbf{p})_{x_{i_1}x_{i_2}} \arrow[r] & \cdots \arrow[r] & \bigoplus E(R/\mathbf{p})_{x_{i_1}\cdots x_{i_{n-1}}} \arrow[r] & E(R/\mathbf{p})_{x_1\cdots x_n} \arrow[r] & 0
\end{tikzcd}$$

is null homotopic. Hence by Remark 3.1.2, the induced map $1^* : H^t(E(R/\mathbf{p}) \otimes \mathcal{C}) \to H^t(E(R/\mathbf{p}) \otimes \mathcal{C})$ is the zero map for all $t > 0$. Therefore $H^t(E(R/\mathbf{p}) \otimes \mathcal{C}) = 0$ for all $t > 0$, and this completes the proof. $\square$

Theorem 3.3.6. *Let $x_1, x_2, \cdots, x_n \in R$ such that the ideal $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is $\mathbf{m}$-primary and let $\mathcal{C}$ be the Čech complex with respect to the sequence $x_1, x_2, \ldots, x_n$. Then $H^t_{\mathbf{m}}(A) \cong H^t(A \otimes \mathcal{C})$, for all R-modules A and $t \geq 0$.*

Proof. By Theorem 3.1.10, $\{H^t_{\mathbf{m}}(-), E^t\}_{t\geq 0}$ is a universal connected sequence with initial $H^0_{\mathbf{m}}(-) = \Gamma_{\mathbf{m}}(-)$. By Theorem 3.3.5, $\{H^t(- \otimes \mathcal{C}), E^t\}_{t\geq 0}$ is also a universal connected sequence. Moreover, from Lemma 3.2.12, we have that $H^0(- \otimes \mathcal{C}) = \Gamma_{\mathbf{m}}(-)$. Therefore, by Lemma 2.3.5, $H^t_{\mathbf{m}}(A) \cong H^t(A \otimes \mathcal{C})$ for all R-modules A and $t \geq 0$. $\square$

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