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Reductions of Monomial Ideals in $k[x,y]_{(x,y)}$ and $k[x,y,z]_{(x,y,z)}$

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感謝詞

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葉柔吟謹誌

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Abstract

We consider monomial ideals in the two-dimensional localized polynomial ring $k[x, y]_{(x,y)}$ where k is an infinite field. In C-Y. Jean Chan and Jung-Chen Liu's paper, they determine a sufficient condition under which an ideal containing $x^a y^b + x^c y^d$ is a reduction of an ideal containing $x^a y^b$ and $x^c y^d$. In this thesis, we use another approach to prove the above result. Furthermore, we extend the sufficient condition to the three-dimensional localized polynomial ring $k[x, y, z]_{(x,y,z)}$ where k is an infinite field.

Keywords. *localized polynomial ring, monomial ideal, reduction.*

Contents

1 Introduction

While reading papers related to reductions of monomial ideals, it was brought to my attention that in V. C. Quiñonez's research report [Q1] and in C-Y. Jean Chan and Jung-Chen Liu's paper [CL], they had achieved similar corollaries ([Q1, Corollary 3.6] and [CL, Corollary 3.7]) regarding minimal reductions of monomial ideals in two-dimensional power series rings and two-dimensional localized polynomial rings from two different theorems ([Q1, Theorem 3.3] and [CL, Theorem 3.3]). In particular, in [CL], Chan and Liu consider monomial ideals in the two-dimensional localized polynomial ring $k[x, y]_{(x,y)}$ where k is an infinite field and determine a sufficient condition under which an ideal I is a reduction of the minimal monomial ideal containing I ([CL, Theorem 3.3]). For example, we can check straightforwardly from the graph whether an ideal containing $x^a y^b + x^c y^d$ is a reduction of an ideal containing $x^a y^b$ and $x^c y^d$. To me, this method is very direct and concrete. I wonder whether this sufficient condition can be extended to the three-dimensional localized polynomial ring $k[x, y, z]_{(x,y,z)}$. In addition to inspiration from reading Quiñonez's research reports, the experience of teaching in a senior high school also brings me new ideas.

In this thesis, we revisit the two-dimensional case in Section 3 and extend the results to the three-dimensional case in Section 4. More precisely, we use some property of vectors in the real plane, that I taught last semester, to prove Lemma 3.3 and Lemma 3.4, and use them to replace the key lemmas [CL, Lemma 3.5 and 3.6] for proving the main theorem ([CL, Theorem 3.3]) for the two-dimensional localized polynomial ring $k[x, y]_{(x,y)}$. We also give one example to demonstrate this algorithm. Moreover, since the property of vectors using in Section 3 still holds true in three-dimensional real space, we give a similar lemma (Lemma 4.2) and prove a theorem (Theorem 4.3) for the three-dimensional localized polynomial ring $k[x, y, z]_{(x,y,z)}$. As in Section 3, we provide a few examples to illustrate this algorithm at the end of Section 4.

2 Preliminaries

Let R be a commutative ring with identity and let $J \subseteq I$ be ideals of R.

1. I is integral over J, if for every element u in I, there exist elements a_i in J^i such that
$$u^n + a_1u^{n-1} + a_2u^{n-2} + \ldots + a_{n-1}u + a_n = 0.$$

2. J is a reduction of I, if $I^{m+1} = JI^m$ for some positive integer m.

When R is Noetherian, these two definitions are equivalent, i.e, I is integral over J if and only if J is a reduction of I.

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Given an ideal $I = (a_1, a_2, \ldots, a_s)$ in R, there are two rings that are used frequently in the study of reduction, namely the *Rees algebra* $R[It]$ of I and the *fiber cone* $R[It]/\mathfrak{m}R[It]$ of I. Note that

$$\begin{aligned} R[It] &= R \oplus I \oplus I^2 \oplus I^3 \oplus \ldots \\ &= R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \ldots \\ &= R[a_1t, a_2t, a_3t, \ldots, a_st] \\ &= \{f(a_1t, a_2t, \ldots, a_st) \mid f(x_1, x_2, \ldots, x_s) \in R[x_1, x_2, \ldots, x_s]\} \subseteq R[t]. \end{aligned}$$

Let J be a subideal of I, i.e., $J \subseteq I$. Then $R[Jt] = R \oplus Jt \oplus J^2t^2 \oplus \ldots \subseteq R[It]$, i.e., $R[Jt]$ is a subring of $R[It]$. Following directly from the definition, we have that for every element u in I,

$$u^n + b_1u^{n-1} + b_2u^{n-2} + \ldots + b_{n-1}u + b_n = 0 \text{ with } b_i \in J^i$$

if and only if

$$(ut)^n + (b_1t)(ut)^{n-1} + \ldots + (b_{n-1}t^{n-1})(ut) + b_nt^n = 0, \text{ with } b_it^i \in J^it^i \subseteq R[Jt].$$

From this observation, we see that I is integral over J if and only if $R[It]$ is integral over $R[Jt]$ as rings. In fact, a similar result holds for their fiber cone. Consider the homomorphism $\varphi : \dfrac{R[Jt]}{\mathfrak{m}R[Jt]} \longrightarrow \dfrac{R[It]}{\mathfrak{m}R[It]}$ induced by the inclusion map $R[Jt] \hookrightarrow R[It]$. We say $R[It]/\mathfrak{m}R[It]$ is integral over $R[Jt]/\mathfrak{m}R[Jt]$ provided that $R[It]/\mathfrak{m}R[It]$ is integral over $\operatorname{Im}\varphi$. Therefore, I is integral over J if and only if $R[It]/\mathfrak{m}R[It]$ is integral over $R[Jt]/\mathfrak{m}R[Jt]$.

Let $R = k[x,y]_{(x,y)}$ be the polynomial ring $k[x,y]$ localized at the maximal ideal (x,y) where k is an infinite field. Since k is a field, k is a Noetherian ring. This implies that $k[x,y]$ is Noetherian by Hilbert's Basis Theorem. Furthermore, a localization of a Noetherian ring is again Noetherian. Hence $k[x,y]_{(x,y)}$ is a Noetherian local ring. Similarly, $k[x,y,z]_{(x,y,z)}$ is also a Noetherian local ring.

Note that by multiplying a suitable unit to a generator, we see that every ideal in R is generated by polynomials in $k[x,y]$. For every element f in $k[x,y]$, f can be written as a linear combination of monomials, i.e., $f = \sum \eta_{ij}x^iy^j$ with $\eta_{ij} \in k$ where we assume no repeated like terms in the expression. We use the following notation for the collection of the finitely many monomials occurring in f

$$\Gamma(f) = \{x^iy^j \mid f = \sum \eta_{ij}x^iy^j \text{ and } \eta_{ij} \neq 0\}.$$

Let I be an ideal of R and suppose it is generated by $f_1, \ldots, f_m \in k[x,y]$. If I' is a monomial ideal containing I, then it is clear that I' contains $\Gamma(f_1) \cup \ldots \cup \Gamma(f_m)$. Hence the smallest monomial ideal containing I is generated by $\Gamma(f_1) \cup \ldots \cup \Gamma(f_m)$. We denote this monomial ideal by I^*. Note that since I^* is the smallest monomial ideal containing I, it does not depend on the choice of generators of I. A similar argument can be applied to ideals I in any localized polynomial rings $k[x_1, x_2, \ldots, x_n]_{(x_1,x_2,\ldots,x_n)}$ and define the ideal I^*. In particular, we discuss the case of $n = 3$ in Section 3, namely ideals in $k[x,y,z]_{(x,y,z)}$.

In the end of this section, we include two exercises and one property which are applied in later sections.

Let R be a commutative ring with identity and let I be an ideal of R. Consider the homomorphism of polynomial rings

$$\begin{array}{cccc} \varphi : & R[x] & \longrightarrow & (R/I)[x] \\ & a_0 + a_1x + \ldots + a_nx^n & \longmapsto & \overline{a_0} + \overline{a_1}x + \ldots + \overline{a_n}x^n \end{array}$$

where $\overline{a_i} = a_i + I \in R/I$. Then φ is a ring epimorphism and $\ker\varphi = IR[x]$. Thus, $R[x]/IR[x] \cong (R/I)[x]$.

We further assume that R is local with maximal ideal $\mathfrak{m}$ and let $\varphi : A \to B$ be an isomorphism of extension rings of R. Consider the epimorphism

$$\psi : A \xrightarrow{\varphi} B \xrightarrow{\pi} B/\mathfrak{m}B$$

where π is the canonical epimorphism. Since $\ker \pi = \mathfrak{m}B$, $\ker \psi = \varphi^{-1}(\mathfrak{m}B) = \mathfrak{m}A$. Then $A/\mathfrak{m}A \cong B/\mathfrak{m}B$.

Lastly, we present a property of vectors that will be applied in later proofs. Suppose O, P, Q are three non-collinear points and suppose C is a point such that $\overrightarrow{OC} = \alpha\overrightarrow{OP} + \beta\overrightarrow{OQ}$ with $\alpha, \beta \in \mathbb{R}$. Then P, Q, C are collinear if and only if $\alpha + \beta = 1$. On the other hand, if C is not on the line through P and Q, then $\alpha + \beta \neq 1$. More precisely, $\alpha + \beta < 1$ if and only if C and O are on the same side of $\overleftrightarrow{PQ}$; equivalently, $\alpha + \beta > 1$ if and only if C and O are on different two sides of $\overleftrightarrow{PQ}$. Furthermore, if C lies inside the two-dimensional region bounded by the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then α and β are both positive. Therefore, it can be concluded from the above properties that $\alpha + \beta = 1$ with $0 \leq \alpha, \beta \leq 1$ if and only if C is on the line segment $\overline{PQ}$.

3 Reductions in $k[x,y]_{(x,y)}$

In this section, we let $R = k[x,y]_{(x,y)}$, where k is an infinite field, and let I be an ideal of R generated by $f_1, \ldots, f_m \in k[x,y]$. We revisit [CL, Theorem 3.3] which states a sufficient condition, in terms of monomials in $\Gamma(f_1) \cup \ldots \cup \Gamma(f_m)$, for I to be a reduction of I^*. We will give a slightly different proof for this theorem, which allows us to generalize this theorem to the three-dimensional localized polynomial ring $k[x,y,z]_{(x,y,z)}$. In order to approach this theorem, we include several lemmas first. In particular, we use Lemma 3.3 and Lemma 3.4, which we prove by applying a property of vectors in the real plane, to replace [CL, Lemma 3.5 and 3.6].

Theorem 3.1. *([CL, Theorem 3.3]) Let $R = k[x,y]_{(x,y)}$ and $|k| = \infty$. Let I be an ideal of R generated by $f_1, \ldots, f_m \in k[x,y]$. Assume that the following is true: for all $i = 1, 2, \ldots, m$ and for any two distinct monomials $x^a y^b$ and $x^c y^d$ in $\Gamma(f_i)$ with $c < a$ and $b < d$, there exists $x^r y^s \in \Gamma(f_j)$ for some j such that the point (r,s) lies on the left hand side of the line through (a,b) and (c,d). Then I is a reduction of I^*.*

Prior to proving this theorem, we discuss several supporting lemmas. For completeness, we state the following lemma ([CL, Lemma 3.4]) and provide the proof.

Lemma 3.2. *([CL, Lemma 3.4]) Let $k[u_1, u_2, \ldots, u_n]$ be a k-algebra and consider its k-subalgebra $k[\eta_1 u_1 + \eta_2 u_2 + \ldots + \eta_n u_n]$ for nonzero $\eta_1, \ldots, \eta_n$ in k. For all $i \neq j$, suppose there are positive integers α_{ij}, β_{ij} such that ${u_i}^{\alpha_{ij}} {u_j}^{\beta_{ij}} = 0$. Then $k[u_1, u_2, \ldots, u_n]$ is integral over $k[\eta_1 u_1 + \eta_2 u_2 + \ldots + \eta_n u_n]$.*

Proof. First, we prove the case where $\eta_1 = \eta_2 = \ldots = \eta_n = 1$ by induction on n. For $n = 1$, it's trivial that $k[u_1]$ is integral over $k[u_1]$. For the case $n = 2$, we consider the k-algebra $k[u_1, u_2]$ and its k-subalgebra $k[u_1 + u_2]$. By assumption, there are positive integers α_1, α_2 such that $u_1^{\alpha_1} u_2^{\alpha_2} = 0$. This implies $u_1^{\alpha_1} u_2^{\alpha_2+1} = 0$.

So we may assume that α_2 is odd. Then we compute as the following:

$$\begin{aligned} & u_1^{\alpha_1+\alpha_2} \\ = \ & u_1^{\alpha_1}u_2^{\alpha_2} + u_1^{\alpha_1+\alpha_2} = u_1^{\alpha_1}(u_2^{\alpha_2} + u_1^{\alpha_2}) = u_1^{\alpha_1}\{[(u_1+u_2)-u_1]^{\alpha_2} + u_1^{\alpha_2}\} \\ = \ & u_1^{\alpha_1}[(u_1+u_2)^{\alpha_2} - \binom{\alpha_2}{1}(u_1+u_2)^{\alpha_2-1}u_1 + \ldots + \binom{\alpha_2}{\alpha_2-1}(u_1+u_2)u_1^{\alpha_2-1} - u_1^{\alpha_2} + u_1^{\alpha_2}]. \end{aligned}$$

We conclude

$$u_1^{\alpha_1+\alpha_2} - \binom{\alpha_2}{\alpha_2-1}(u_1+u_2)u_1^{\alpha_1+\alpha_2-1} + \ldots + \binom{\alpha_2}{1}(u_1+u_2)^{\alpha_2-1}u_1^{\alpha_1+1} - (u_1+u_2)^{\alpha_2}u_1^{\alpha_1} = 0.$$

Thus u_1 is integral over $k[u_1+u_2]$. Then $k[u_1+u_2][u_1]$ is a finitely generated $k[u_1+u_2]$-module and so $k[u_1+u_2][u_1] = k[u_1+u_2, u_1] = k[u_1, u_2]$ is integral over $k[u_1+u_2]$. Assume $n \geq 3$ and suppose the assertion holds for all k-algebras with $n-1$ or less generators. For the k-algebra $k[u_1, u_2, \ldots, u_n]$ with n generators, we choose $\alpha = \max_{i,j}\{\alpha_{ij}, \beta_{ij}\}$, then we have $u_i^{\alpha}u_j^{\alpha} = 0$ for all $i \neq j$. Consider the k-algebras

$$k[u_1 + \ldots + u_{n-1} + u_n] \subseteq k[u_1 + \ldots + u_{n-1}, u_n] \subseteq k[u_1, \ldots, u_{n-1}, u_n].$$

It's enough to show that

(1)$k[u_1, \ldots, u_{n-1}, u_n]$ is integral over $k[u_1 + \ldots + u_{n-1}, u_n]$, and

(2)$k[u_1 + \ldots + u_{n-1}, u_n]$ is integral over $k[u_1 + \ldots + u_{n-1} + u_n]$.

For (1), since $u_i^{\alpha}u_j^{\alpha} = 0$ for all $i \neq j$, by the induction hypothesis, we have that $k[u_1, u_2, \ldots, u_{n-1}]$ is integral over $k[u_1+u_2+\ldots+u_{n-1}]$ and so $k[u_1, u_2, \ldots, u_{n-1}, u_n]$ is integral over $k[u_1 + u_2 + \ldots + u_{n-1}, u_n]$. For (2), consider the element

$$(u_1 + u_2 + \ldots + u_{n-1})^{(n-1)\alpha}u_n^{\alpha}.$$

Note that after expanding $(u_1 + u_2 + \ldots + u_{n-1})^{(n-1)\alpha}$, all terms are of the form $u_1^{\alpha_1}u_2^{\alpha_2}\ldots u_{n-1}^{\alpha_{n-1}}$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = (n-1)\alpha$. If $\alpha_i < \alpha$ for all i, then $\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} < \alpha(n-1)$, which can never hold. Hence at least one of the α_i is larger than or equal to α. Let $\alpha_k \geq \alpha$, then

$$(u_1^{\alpha_1}u_2^{\alpha_2}\ldots u_{n-1}^{\alpha_{n-1}})u_n^{\alpha} = (u_1^{\alpha_1}\ldots u_{n-1}^{\alpha_{n-1}})u_k^{\alpha_k}u_n^{\alpha} = 0.$$

Therefore, $(u_1 + u_2 + \ldots + u_{n-1})^{(n-1)\alpha} u_n^\alpha = 0$. Thus, by the case of $n = 2$, $k[u_1 + \ldots + u_{n-1}, u_n]$ is integral over $k[u_1 + \ldots + u_{n-1} + u_n]$. So it follows from (1) and (2) that $k[u_1, \ldots, u_n]$ is integral over $k[u_1 + \ldots + u_n]$. At last, we consider the general case. Note that $u_i^{\alpha_{ij}} u_j^{\beta_{ij}} = 0$ implies $(\eta_i u_i)^{\alpha_{ij}} (\eta_j u_j)^{\beta_{ij}} = 0$. By replacing u_ℓ by $\eta_\ell u_\ell$ for all $\ell = 1, 2, \ldots, n$, we have $k[\eta_1 u_1, \eta_2 u_2, \ldots, \eta_n u_n]$ is integral over $k[\eta_1 u_1 + \eta_2 u_2 + \ldots + \eta_n u_n]$. Since $\eta_1, \eta_2, \ldots, \eta_n$ are all units in k,

$$k[u_1, u_2, \ldots, u_n] = k[\eta_1 u_1, \eta_2 u_2, \ldots, \eta_n u_n].$$

The proof is complete. □

Next, we use a new approach to prove the following two technical lemmas.

Lemma 3.3. *Let $(a, b), (c, d), (e, f) \in \mathbb{Z}^2_{\geq 0}$ with $a > c$ and $b < d$. If the point (e, f) lies within the triangular region with vertices $(a, b), (c, d), (0, 0)$, including all boundaries except the line segment connecting (a, b) and (c, d), then there exist nonnegative integers α, β and positive integers γ, δ such that*

$$(x^a y^b)^\alpha (x^c y^d)^\beta = x^\gamma y^\delta (x^e y^f)^{\alpha+\beta}.$$

Simply speaking, the above monomial equality holds if (e, f) is in the following shaded region:

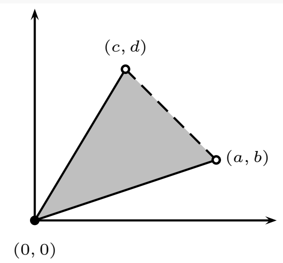


Figure 3.3

Proof. Suppose that (e, f) is in the assumed region. Then there exist $s, t \in \mathbb{R}$ with $0 \leq s, t < 1$ and $s + t < 1$ such that $(e, f) = s(a, b) + t(c, d)$. So we have

$$e = sa + tc,$$
$$f = sb + td.$$

The fact $a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}$ implies $s, t \in \mathbb{Q}$. Let $\ell = ad - bc > 0$, and set $\alpha = \ell s, \beta = \ell t$. Then we obtain that α and β are nonnegative integers and that

$$e(\alpha + \beta) = e\ell(s + t) < e\ell = (sa + tc)\ell = a\alpha + c\beta,$$
$$f(\alpha + \beta) = f\ell(s + t) < f\ell = (sb + td)\ell = b\alpha + d\beta.$$

By setting $\gamma = (a\alpha + c\beta) - e(\alpha + \beta)$ and $\delta = (b\alpha + d\beta) - f(\alpha + \beta)$, we have

$$(x^a y^b)^\alpha (x^c y^d)^\beta = x^\gamma y^\delta (x^e y^f)^{\alpha+\beta}$$

in which α, β are nonnegative integers and γ, δ are positive integers as desired. □

Lemma 3.4. *Let $(a, b), (c, d), (e, f) \in \mathbb{Z}^2_{\geq 0}$ with $a > c$ and $b < d$. If the point (e, f) lies within the triangular region bounded by the x-axis, the line through (a, b) and (c, d), and the line through (a, b) and $(0, 0)$, including the x-axis but excluding the other two boundaries, i.e., the shaded region in Figure 3.4.1,*

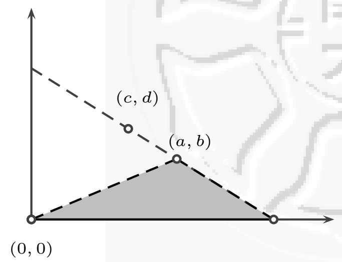


Figure 3.4.1.

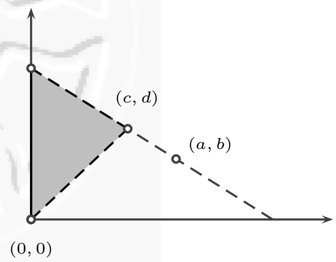


Figure 3.4.2.

then there exist positive integers $\alpha, \beta, \gamma, \delta$ such that

$$(x^a y^b)^{\alpha+\beta} = x^\gamma y^\delta (x^e y^f)^\alpha (x^c y^d)^\beta.$$

Symmetrically, if the point (e, f) is in the shaded region in Figure 3.4.2, then there exist positive integers $\alpha, \beta, \gamma, \delta$ such that

$$(x^c y^d)^{\alpha+\beta} = x^\gamma y^\delta (x^a y^b)^\alpha (x^e y^f)^\beta.$$

Proof. Note that if (e, f) is in the shaded region in Figure 3.4.1, then (a, b) is on the right hand side of the line through (c, d) and (e, f), and lies inside the region bounded by the ray from $(0,0)$ toward (c,d) and the ray from $(0,0)$ toward (e,f). This implies that there exist $s,t \in \mathbb{R}$ with $s,t > 0$ and $s+t > 1$ such that $(a,b) = s(e,f) + t(c,d)$. So we have

$$a = se + tc,$$
$$b = sf + td.$$

The fact $a,b,c,d,e,f \in \mathbb{Z}_{\geq 0}$ implies $s,t \in \mathbb{Q}$. Let $\ell = |ed - cf|$, and set $\alpha = \ell s$, $\beta = \ell t$. Then we obtain that α, β are positive integers and that

$$a(\alpha+\beta) = a\ell(s+t) > a\ell = (se+tc)\ell = e\alpha + c\beta,$$
$$b(\alpha+\beta) = b\ell(s+t) > b\ell = (sf+td)\ell = f\alpha + d\beta.$$

By setting $\gamma = a(\alpha+\beta) - (e\alpha + c\beta)$ and $\delta = b(\alpha+\beta) - (f\alpha + d\beta)$, we have

$$(x^a y^b)^{\alpha+\beta} = x^\gamma y^\delta (x^e y^f)^\alpha (x^c y^d)^\beta$$

in which $\alpha, \beta, \gamma, \delta$ are all positive integers as desired.

Symmetrically, if (e,f) is in the shaded region in Figure 3.4.2, (c,d) is on the right hand side of the line through (e,f) and (a,b), and lies inside the region bounded by the ray from $(0,0)$ toward (e,f) and the ray from $(0,0)$ toward (a,b). This implies that there exist $s,t \in \mathbb{R}$ with $s,t > 0$ and $s+t > 1$ such that $(c,d) = s(a,b) + t(e,f)$. So we have

$$c = sa + te,$$
$$d = sb + tf.$$

The fact $a,b,c,d,e,f \in \mathbb{Z}_{\geq 0}$ implies $s,t \in \mathbb{Q}$. Let $\ell = |af - eb|$, and set $\alpha = \ell s$, $\beta = \ell t$. Then we obtain that α, β are positive integers and that

$$c(\alpha+\beta) = c\ell(s+t) > c\ell = (sa+te)\ell = a\alpha + e\beta,$$
$$d(\alpha+\beta) = d\ell(s+t) > d\ell = (sb+tf)\ell = b\alpha + f\beta.$$

By setting $\gamma = c(\alpha+\beta) - (a\alpha + e\beta)$ and $\delta = d(\alpha+\beta) - (b\alpha + f\beta)$, we have

$$(x^c y^d)^{\alpha+\beta} = x^\gamma y^\delta (x^a y^b)^\alpha (x^e y^f)^\beta$$

in which $\alpha, \beta, \gamma, \delta$ are all positive integers as desired. □

Now, we are ready to prove the main theorem of this section, Theorem 3.1.

Proof. We express the generators of I as the following:

$$f_1 = \sum_{j=1}^{n_1} \eta_{1j} x^{a_{1j}} y^{b_{1j}}, \ f_2 = \sum_{j=1}^{n_2} \eta_{2j} x^{a_{2j}} y^{b_{2j}}, \ \ldots, \ f_m = \sum_{j=1}^{n_m} \eta_{mj} x^{a_{mj}} y^{b_{mj}}$$

with $\eta_{ij} \neq 0$ in k. Then I^* is the ideal generated by $x^{a_{ij}} y^{b_{ij}}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Let $\mathfrak{m} = (x, y)R$ be the maximal ideal of R. In order to show that I is a reduction of I^*, it suffices to show that the fiber cone $R[I^*t]/\mathfrak{m}R[I^*t]$ of I^* is integral over the fiber cone $R[It]/\mathfrak{m}R[It]$ of I, that is, $R[I^*t]/\mathfrak{m}R[I^*t]$ is integral over $\Phi\left(\dfrac{R[It]}{\mathfrak{m}R[It]}\right)$ where $\Phi : \dfrac{R[It]}{\mathfrak{m}R[It]} \longrightarrow \dfrac{R[I^*t]}{\mathfrak{m}R[I^*t]}$ is the natural homomorphism induced by the inclusion map $R[It] \hookrightarrow R[I^*t]$.

Consider the polynomial ring

$$R[U_{ij}] = R[U_{ij} \mid i = 1, \ldots, m, \ j = 1, \ldots, n_i]$$

and the ring epimorphism

$$\begin{array}{rccc} \varphi & : & R[U_{ij}] & \longrightarrow & R[I^*t]. \\ & & U_{ij} & \longmapsto & x^{a_{ij}} y^{b_{ij}} t \end{array}$$

Then we have $R[I^*t] \cong R[U_{ij}]/\ker\varphi$ with $x^{a_{ij}} y^{b_{ij}} t \mapsto U_{ij} + \ker\varphi$ and

$$\frac{R[I^*t]}{\mathfrak{m}R[I^*t]} \cong \frac{R[U_{ij}]/\ker\varphi}{\mathfrak{m}(R[U_{ij}]/\ker\varphi)} \cong \frac{R[U_{ij}]}{(\mathfrak{m}R[U_{ij}] + \ker\varphi)}$$

with $x^{a_{ij}} y^{b_{ij}} t + \mathfrak{m}R[I^*t] \mapsto \big(U_{ij} + \ker\varphi\big) + \mathfrak{m}(R[U_{ij}]/\ker\varphi) \mapsto U_{ij} + (\mathfrak{m}R[U_{ij}] + \ker\varphi)$.

Since

$$\frac{R[U_{ij}]}{(\mathfrak{m}R[U_{ij}] + \ker\varphi)} \cong \frac{R[U_{ij}]/\mathfrak{m}R[U_{ij}]}{(\mathfrak{m}R[U_{ij}] + \ker\varphi)/\mathfrak{m}R[U_{ij}]}$$

with $U_{ij} + (\mathfrak{m}R[U_{ij}] + \ker\varphi) \mapsto \big(U_{ij} + \mathfrak{m}R[U_{ij}]\big) + \big((\mathfrak{m}R[U_{ij}] + \ker\varphi)/\mathfrak{m}R[U_{ij}]\big)$, we consider the epimorphism $\psi : R[U_{ij}] \longrightarrow (R/\mathfrak{m})[U_{ij}]$. Since $\ker\psi = \mathfrak{m}R[U_{ij}]$, ψ induces the isomorphism $\overline{\psi} : (R[U_{ij}]/\mathfrak{m}R[U_{ij}]) \longrightarrow (R/\mathfrak{m})[U_{ij}]$. Then we have

$$\frac{R[U_{ij}]/\mathfrak{m}R[U_{ij}]}{(\mathfrak{m}R[U_{ij}] + \ker\varphi)/\mathfrak{m}R[U_{ij}]} \cong \frac{(R/\mathfrak{m})[U_{ij}]}{\overline{\psi}\big((\mathfrak{m}R[U_{ij}] + \ker\varphi)/\mathfrak{m}R[U_{ij}]\big)}.$$

Since $\overline{\psi}\big((\mathfrak{m}R[U_{ij}]+\ker\varphi)/\mathfrak{m}R[U_{ij}]\big)=\psi(\mathfrak{m}R[U_{ij}]+\ker\varphi)=\psi(\ker\varphi)$,

$$\frac{(R/\mathfrak{m})[U_{ij}]}{\overline{\psi}\big((\mathfrak{m}R[U_{ij}]+\ker\varphi)/\mathfrak{m}R[U_{ij}]\big)}=\frac{(R/\mathfrak{m})[U_{ij}]}{\psi(\ker\varphi)}.$$

Let $u_{ij}=U_{ij}+\psi(\ker\varphi)$ in $(R/\mathfrak{m})[U_{ij}]/\psi(\ker\varphi)$ and identify $R/\mathfrak{m}$ with k. Then $\dfrac{(R/\mathfrak{m})[U_{ij}]}{\psi(\ker\varphi)}=k[u_{ij}]$. So we have

$$\frac{R[U_{ij}]/\mathfrak{m}R[U_{ij}]}{(\mathfrak{m}R[U_{ij}]+\ker\varphi)/\mathfrak{m}R[U_{ij}]}\cong k[u_{ij}]$$

with $\big(U_{ij}+\mathfrak{m}R[U_{ij}]\big)+\big((\mathfrak{m}R[U_{ij}]+\ker\varphi)/\mathfrak{m}R[U_{ij}]\big)\mapsto u_{ij}$. Therefore, we have an isomorphism $\dfrac{R[I^*t]}{\mathfrak{m}R[I^*t]}\cong k[u_{ij}]$ with $x^{a_{ij}}y^{b_{ij}}t+\mathfrak{m}R[I^*t]\mapsto u_{ij}$. Hence with the homomorphism

$$\begin{array}{llcll}
\Psi: & \dfrac{R[It]}{\mathfrak{m}R[It]} & \xrightarrow{\ \Phi\ } & \dfrac{R[I^*t]}{\mathfrak{m}R[I^*t]} & \cong \quad k[u_{ij}] \\
 & f_it+\mathfrak{m}R[It] & \longmapsto & \Phi\Big(\displaystyle\sum_{j=1}^{n_i}\eta_{ij}x^{a_{ij}}y^{b_{ij}}t+\mathfrak{m}R[It]\Big) & \\
 & & & =\left(\displaystyle\sum_{j=1}^{n_i}\eta_{ij}x^{a_{ij}}y^{b_{ij}}t\right)+\mathfrak{m}R[I^*t] & \\
 & & & =\displaystyle\sum_{j=1}^{n_i}\eta_{ij}\left(x^{a_{ij}}y^{b_{ij}}t+\mathfrak{m}R[I^*t]\right) & \longmapsto \quad \displaystyle\sum_{j=1}^{n_i}\eta_{ij}u_{ij}
\end{array}$$

where $R[It]=R[f_1t,f_2t,\dots,f_mt]$, $\Psi\left(\dfrac{R[It]}{\mathfrak{m}R[It]}\right)=k\big[\displaystyle\sum_{j=1}^{n_1}\eta_{1j}u_{1j},\dots,\sum_{j=1}^{n_m}\eta_{mj}u_{mj}\big]$. Thus, showing that $R[I^*t]/\mathfrak{m}R[I^*t]$ is integral over $R[It]/\mathfrak{m}R[It]$ is equivalent to showing that $k[u_{ij}]$ is integral over $k\big[\displaystyle\sum_{j=1}^{n_1}\eta_{1j}u_{1j},\dots,\sum_{j=1}^{n_m}\eta_{mj}u_{mj}\big]$. By Lemma 3.2, for each $i=1,\dots,m$, it's enough to prove that for all $\ell\neq j$, $u_{ij}^{\alpha_\ell}u_{i\ell}^{\beta_\ell}=0$ for some positive integers α_ℓ and β_ℓ. Note that U_{ij} (*resp.* $U_{i\ell}$) corresponds to $x^{a_{ij}}y^{b_{ij}}t$ (*resp.* $x^{a_{i\ell}}y^{b_{i\ell}}t$) in the epimorphism φ. Without loss of generality, we assume $a_{ij}\geq a_{i\ell}$. If $a_{ij}=a_{i\ell}$ and $b_{ij}<b_{i\ell}$ (*resp.* $b_{ij}>b_{i\ell}$), then $x^{a_{i\ell}}y^{b_{i\ell}}t=y^{b_{i\ell}-b_{ij}}(x^{a_{ij}}y^{b_{ij}}t)$ (*resp.* $x^{a_{ij}}y^{b_{ij}}t=y^{b_{ij}-b_{i\ell}}(x^{a_{i\ell}}y^{b_{i\ell}}t)$). This shows $U_{i\ell}-y^{b_{i\ell}-b_{ij}}U_{ij}\in\ker\varphi$ (*resp.* $U_{ij}-y^{b_{ij}-b_{i\ell}}U_{i\ell}\in\ker\varphi$) and so $U_{i\ell}\in\mathfrak{m}R[U_{ij}]+\ker\varphi$ (*resp.* $U_{ij}\in\mathfrak{m}R[U_{ij}]+\ker\varphi$). With the isomorphism $\dfrac{R[U_{ij}]}{(\mathfrak{m}R[U_{ij}]+\ker\varphi)}\cong k[u_{ij}]$, we obtain $u_{i\ell}=0$

($resp.$ $u_{ij} = 0$) and so $u_{ij}u_{i\ell} = 0$. Similarly for $a_{ij} > a_{i\ell}$ and $b_{ij} \geq b_{i\ell}$, we have $x^{a_{ij}}y^{b_{ij}}t = x^{a_{ij}-a_{i\ell}}y^{b_{ij}-b_{i\ell}}(x^{a_{i\ell}}y^{b_{i\ell}}t)$. This shows $U_{ij} - x^{a_{ij}-a_{i\ell}}y^{b_{ij}-b_{i\ell}}U_{i\ell} \in \ker\varphi$ and so $U_{ij} \in \mathfrak{m}R[U_{ij}] + \ker\varphi$. That is $u_{ij} = 0$ and so $u_{ij}u_{i\ell} = 0$ is true.

The last case is that $a_{ij} > a_{i\ell}$ and $b_{ij} < b_{i\ell}$. By the assumption of the theorem, there exists $x^{a_{hs}}y^{b_{hs}} \in \Gamma(f_h)$ such that (a_{hs}, b_{hs}) lies on the left hand side of the line through (a_{ij}, b_{ij}) and $(a_{i\ell}, b_{i\ell})$ as shown in the following shaded region:

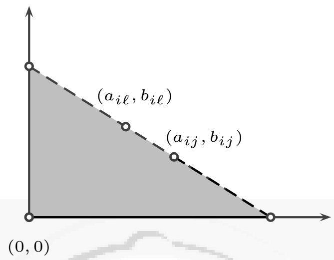


Figure 3.1.1.

We divide Figure 3.1.1. into the following three parts:

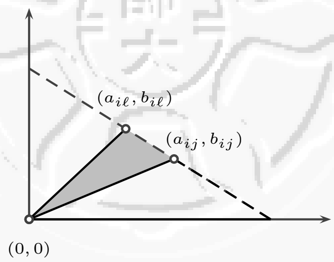


Figure 3.1.2.

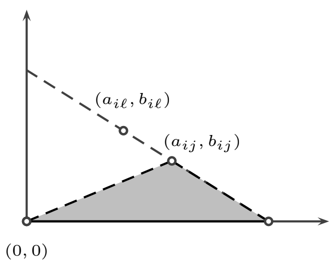


Figure 3.1.3.

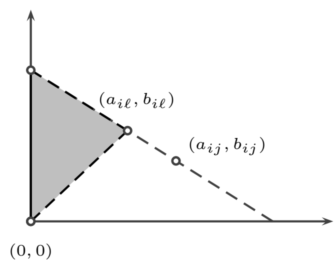


Figure 3.1.4.

If (a_{hs}, b_{hs}) lies in the shaded region in Figure 3.1.2., then, by Lemma 3.3, there exist nonnegative integers α, β and positive integers γ, δ such that

$$(x^{a_{ij}} y^{b_{ij}} t)^{\alpha} (x^{a_{i\ell}} y^{b_{i\ell}} t)^{\beta} = x^{\gamma} y^{\delta} (x^{a_{hs}} y^{b_{hs}} t)^{\alpha+\beta}.$$

Thus, we have $U_{ij}^{\alpha} U_{i\ell}^{\beta} - x^{\gamma} y^{\delta} U_{hs}^{\alpha+\beta} \in \ker \varphi$. That is $U_{ij}^{\alpha} U_{i\ell}^{\beta} \in \mathfrak{m}R[U_{ij}] + \ker \varphi$ and so $u_{ij}^{\alpha} u_{i\ell}^{\beta} = 0$. If (a_{hs}, b_{hs}) lies in the shaded region in Figure 3.1.3., then, by Lemma 3.4, there exist positive integers $\alpha, \beta, \gamma, \delta$ such that

$$(x^{a_{ij}} y^{b_{ij}} t)^{\alpha+\beta} = x^{\gamma} y^{\delta} (x^{a_{hs}} y^{b_{hs}} t)^{\alpha} (x^{a_{i\ell}} y^{b_{i\ell}} t)^{\beta},$$

and this implies $u_{ij}^{\alpha+\beta} = 0$ as above. Similarly, if (a_{hs}, b_{hs}) lies in the shaded region in Figure 3.1.4., we may apply Lemma 3.4 and get $u_{i\ell}^{\alpha+\beta} = 0$ for some positive integers α, β. These prove that for each $i = 1, \ldots, m$, $k[u_{i1}, \ldots, u_{in_i}]$ is integral over $k\big[\sum_{j=1}^{n_i} \eta_{ij} u_{ij}\big]$ and so $k[u_{ij}]$ is integral over $k\big[\sum_{j=1}^{n_1} \eta_{1j} u_{1j}, \ldots, \sum_{j=1}^{n_m} \eta_{mj} u_{mj}\big]$. Hence I is a reduction of I^* and the proof is completed. □

We end this section with an example which illustrates the algorithm provided in Theorem 3.1.

Example 3.5. Consider the ideal

$$I = \langle x^{12} + x^{10} y^2 + x^3 y^4 + x y^8, x^{10} y + x^5 y^3 + y^9, x^6 y^2 + x^2 y^8 \rangle$$

in $k[x, y]_{(x,y)}$ with $|k| = \infty$. Note that the monomials occurring in the first generator of I are $x^{12}, x^{10}y^2, x^3y^4, xy^8$. From Figure 3.5.1, we see that for every line through two of the above 4 monomials, either x^6y^2 or y^9 is on the left of the line. Similarly, from Figure 3.5.2, we see that for each of the three lines determined by the three monomials occurring in the second generator of I, either x^6y^2 or x^3y^4 is on the left of the line. From Figure 3.5.3, x^3y^4 is on the left of the line through the two monomials occurring in the third generator of I. Hence, by Theorem 3.1, $I^* = \langle x^{12}, x^{10}y, x^6y^2, x^5y^3, x^3y^4, xy^8, y^9 \rangle$ is integral over I.

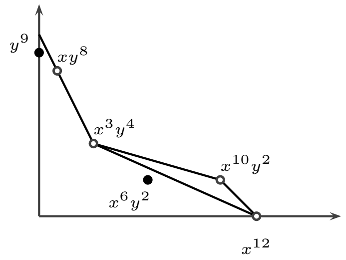


Figure 3.5.1.

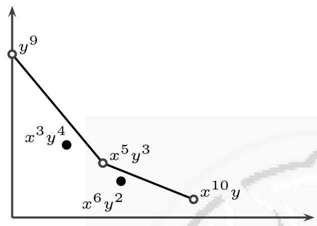


Figure 3.5.2.

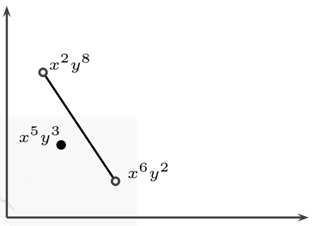


Figure 3.5.3.

4 Reductions in $k[x, y, z]_{(x,y,z)}$

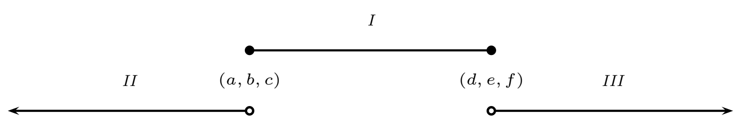
In V. C. Quiñonez's research report [Q1], although she proves a more general theorem on reductions of ideals ([Q1, Theorem 3.3]), it is not as concrete as Theorem 3.1 ([CL, Theorem 3.3]). Hence, we are interested in extending Theorem 3.1 to the three-dimensional case. In this section, we consider $R = k[x, y, z]_{(x,y,z)}$ where k is an infinite field and let I be an ideal of R generated by $f_1, \ldots, f_m \in k[x, y, z]$. Similarly to the proofs of Lemma 3.3 and Lemma 3.4, we apply the property of vectors in the three-dimensional real space to obtain Lemma 4.2. With the help of this new lemma, we are able to extend Theorem 3.1 to Theorem 4.3, which gives a sufficient condition for I to be a reduction of I^* in the ring $k[x, y, z]_{(x,y,z)}$.

Notation 4.1. *Let $m = x^i y^j z^\ell$ be a monomial in $k[x, y, z]$. We use the following notation for the three-dimensional region corresponding to the ideal $\langle m \rangle$ subtracting the point m:*

$$M(m) = \{(a, b, c) \in \mathbb{R}^3_{\geq 0} \mid a \geq i, b \geq j, c \geq \ell\} \setminus \{(i, j, \ell)\}.$$

Except Lemma 3.3 and Lemma 3.4, the remaining part of the proof for Theorem 3.1 is still applicable in the three-dimensional localized polynomial ring $k[x, y, z]_{(x,y,z)}$. Therefore, we need some supporting lemma which is available in the three-dimensional case to take the place of Lemma 3.3 and Lemma 3.4 for the two-dimensional case. With such a lemma, the following Lemma 4.2, we can extend Theorem 3.1 to the three-dimensional case. Thus, it is a crucial part of this section.

Lemma 4.2. *Let $(a, b, c), (d, e, f), (g, h, i) \in \mathbb{Z}^3_{\geq 0}$. Suppose that the line through (a, b, c) and (d, e, f) intersects $M(x^g y^h z^i)$. Divide the line into three parts as the following:*



Case 1. *If part I intersects $M(x^g y^h z^i)$, then there exist nonnegative integers $\alpha, \beta, \gamma, \delta, \lambda$ such that γ, δ, λ are not all zero and that*

$$(x^a y^b z^c)^{\alpha}(x^d y^e z^f)^{\beta} = x^{\gamma} y^{\delta} z^{\lambda}(x^g y^h z^i)^{\alpha+\beta}.$$

Case 2. *If part II intersects $M(x^g y^h z^i)$, then there exist nonnegative integers $\alpha, \beta, \gamma, \delta, \lambda$ such that γ, δ, λ are not all zero and that*

$$(x^a y^b z^c)^{\alpha+\beta} = x^{\gamma} y^{\delta} z^{\lambda}(x^g y^h z^i)^{\alpha}(x^d y^e z^f)^{\beta}.$$

Case 3. *If part III intersects $M(x^g y^h z^i)$, then there exist nonnegative integers $\alpha, \beta, \gamma, \delta, \lambda$ such that γ, δ, λ are not all zero and that*

$$(x^d y^e z^f)^{\alpha+\beta} = x^{\gamma} y^{\delta} z^{\lambda}(x^a y^b z^c)^{\alpha}(x^g y^h z^i)^{\beta}.$$

Proof. Since the line through (a, b, c) and (d, e, f) intersects $M(x^g y^h z^i)$, there exist $s, t \in \mathbb{R}$ with $s + t = 1$ such that $s(a, b, c) + t(d, e, f) \in M(x^g y^h z^i)$. So we have

$$\begin{aligned} sa + td &\geq g, \\ sb + te &\geq h, \\ sc + tf &\geq i, \end{aligned}$$

and at lease one of the above inequalities is strict since $M(x^g y^h z^i)$ does not include the point (g, h, i). Note that $(s, t) \in \mathbb{R}^2$ is a solution of the system

$$\begin{cases} ax + dy \geq g, \\ bx + ey \geq h, \\ cx + fy \geq i, \\ x + y = 1. \end{cases}$$

Hence we know that the line $x + y = 1$ passes through the region determined by the system

$$\begin{cases} ax + dy \geq g, \\ bx + ey \geq h, \\ cx + fy \geq i. \end{cases}$$

If $x+y=1$ intersects the region at only one point (s,t), then (s,t) must be the intersection of two of the lines $ax+dy=g, bx+ey=h$, and $cx+fy=i$. Since $a,b,c,d,e,f,g,h,i \in \mathbb{Z}_{\geq 0}$, $s,t \in \mathbb{Q}$. If $x+y=1$ intersects the region at more than one point, then the intersection must contain a line segment and so we can find a point on this line segment with rational coordinates. In either case, we may assume $s,t \in \mathbb{Q}$. Let ℓ be a positive integer such that $s\ell, t\ell \in \mathbb{Z}$, then we have

$$\begin{aligned}(sa+td)\ell &\geq g\ell = g\ell(s+t),\\ (sb+te)\ell &\geq h\ell = h\ell(s+t),\\ (sc+tf)\ell &\geq i\ell = i\ell(s+t),\end{aligned}$$

and at lease one of the above inequalities is strict.

CASE 1: Since *part I* intersects $M(x^g y^h z^i)$, $0 \leq s,t \leq 1$. Then we can take $\alpha = \ell s$ and $\beta = \ell t$ to obtain that

$$\begin{aligned}a\alpha + d\beta &= (sa+td)\ell \geq g\ell(s+t) = g(\alpha+\beta),\\ b\alpha + e\beta &= (sb+te)\ell \geq h\ell(s+t) = h(\alpha+\beta),\\ c\alpha + f\beta &= (sc+tf)\ell \geq i\ell(s+t) = i(\alpha+\beta),\end{aligned}$$

where at lease one of the inequalities is strict. By setting

$$\begin{aligned}\gamma &= (a\alpha+d\beta) - g(\alpha+\beta),\\ \delta &= (b\alpha+e\beta) - h(\alpha+\beta),\\ \lambda &= (c\alpha+f\beta) - i(\alpha+\beta),\end{aligned}$$

we have

$$(x^a y^b z^c)^\alpha (x^d y^e z^f)^\beta = x^\gamma y^\delta z^\lambda (x^g y^h z^i)^{\alpha+\beta}$$

in which $\alpha,\beta,\gamma,\delta,\lambda$ are nonnegative integers and γ,δ,λ are not all zero.

CASE 2: Since *part II* intersects $M(x^g y^h z^i)$, $s>0$ and $t<0$. Then we can take $\alpha = \ell = \ell(s+t)$ and $\beta = -\ell t$ to obtain that

$$\begin{aligned}a(\alpha+\beta) &= a\ell s \geq g\ell - td\ell = g\alpha + d\beta,\\ b(\alpha+\beta) &= b\ell s \geq h\ell - te\ell = h\alpha + e\beta,\\ c(\alpha+\beta) &= c\ell s \geq i\ell - tf\ell = i\alpha + f\beta,\end{aligned}$$

and at lease one of the above inequalities is strict. By setting

$$
\begin{aligned}
\gamma &= a(\alpha+\beta) - (g\alpha + d\beta),\\
\delta &= b(\alpha+\beta) - (h\alpha + e\beta),\\
\lambda &= c(\alpha+\beta) - (i\alpha + f\beta),
\end{aligned}
$$

we have

$$(x^a y^b z^c)^{\alpha+\beta} = x^\gamma y^\delta z^\lambda (x^g y^h z^i)^\alpha (x^d y^e z^f)^\beta$$

in which $\alpha, \beta, \gamma, \delta, \lambda$ are nonnegative integers and γ, δ, λ are not all zero.

CASE 3: Since *part III* intersects $M(x^g y^h z^i)$, $s < 0$ and $t > 0$. Then we can take $\alpha = -\ell s$ and $\beta = \ell = \ell(s+t)$ to obtain that

$$
\begin{aligned}
d(\alpha+\beta) &= d\ell t \geq g\ell - sa\ell = a\alpha + g\beta,\\
e(\alpha+\beta) &= e\ell t \geq h\ell - sb\ell = b\alpha + h\beta,\\
f(\alpha+\beta) &= f\ell t \geq i\ell - sc\ell = c\alpha + i\beta,
\end{aligned}
$$

and at lease one of the above inequalities is strict. By setting

$$
\begin{aligned}
\gamma &= d(\alpha+\beta) - (a\alpha + g\beta),\\
\delta &= e(\alpha+\beta) - (b\alpha + h\beta),\\
\lambda &= f(\alpha+\beta) - (c\alpha + i\beta),
\end{aligned}
$$

we have

$$(x^d y^e z^f)^{\alpha+\beta} = x^\gamma y^\delta z^\lambda (x^a y^b z^c)^\alpha (x^g y^h z^i)^\beta$$

in which $\alpha, \beta, \gamma, \delta, \lambda$ are nonnegative integers and γ, δ, λ are not all zero. □

Now, we can extend the main theorem in Section 3 to the three-dimensional case. In the following Theorem 4.3, we give a sufficient condition for an ideal I to be a reduction of I^* in the localized polynomial ring $k[x,y,z]_{(x,y,z)}$.

Theorem 4.3. *Let $R = k[x,y,z]_{(x,y,z)}$ and $|k| = \infty$. Let I be an ideal of R generated by $f_1, \ldots, f_m \in k[x,y,z]$. Assume that the following is true: for all $i = 1, 2, \ldots, m$ and for any two distinct monomials $x^a y^b z^c$ and $x^d y^e z^f$ in $\Gamma(f_i)$, there exists $x^r y^s z^t \in \Gamma(f_j)$ for some j such that the line through (a,b,c) and (d,e,f) intersects $M(x^r y^s z^t)$. Then I is a reduction of I^*.*

Proof. We express the generators of I as the following:

$$f_1 = \sum_{j=1}^{n_1} \eta_{1j} x^{a_{1j}} y^{b_{1j}} z^{c_{1j}},\ f_2 = \sum_{j=1}^{n_2} \eta_{2j} x^{a_{2j}} y^{b_{2j}} z^{c_{2j}},\ \ \ldots,\ f_m = \sum_{j=1}^{n_m} \eta_{mj} x^{a_{mj}} y^{b_{mj}} z^{c_{mj}}$$

with $\eta_{ij} \neq 0$ in k. Then I^* is the ideal generated by $x^{a_{ij}} y^{b_{ij}} z^{c_{ij}}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Similar to the proof of Theorem 3.1, we consider the polynomial ring

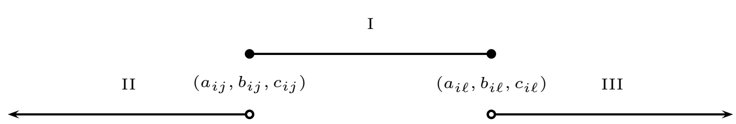
$$R[U_{ij}] = R[U_{ij} \mid i = 1, \ldots, m,\ j = 1, \ldots, n_i]$$

and the ring epimorphism

$$\begin{array}{rcccc} \varphi & : & R[U_{ij}] & \longrightarrow & R[I^*t]. \\ & & U_{ij} & \longmapsto & x^{a_{ij}} y^{b_{ij}} z^{c_{ij}} t \end{array}$$

Then we have $R[I^*t] \cong R[U_{ij}]/\ker\varphi$ and $\dfrac{R[I^*t]}{\mathfrak{m}R[I^*t]} \cong \dfrac{R[U_{ij}]}{(\mathfrak{m}R[U_{ij}] + \ker\varphi)}$, in which $\mathfrak{m} = (x, y, z)R$ is the maximal ideal of R.

Let u_{ij} denote the homomorphic image of U_{ij} in $R[U_{ij}]/(\mathfrak{m}R[U_{ij}] + \ker\varphi)$. In order to show that I is a reduction of I^*, it suffices to show that $R[I^*t]/\mathfrak{m}R[I^*t]$ is integral over $R[It]/\mathfrak{m}R[It]$. This is equivalent to showing that the k-algebra $k[u_{ij}]$ is integral over $k\big[\sum_{j=1}^{n_1} \eta_{1j} u_{1j}, \ldots, \sum_{j=1}^{n_m} \eta_{mj} u_{mj}\big]$. Thus, by Lemma 3.2, for each $i = 1, \ldots, m$, it's enough to prove that for all $\ell \neq j$, $u_{ij}^{\alpha_\ell} u_{i\ell}^{\beta_\ell} = 0$ for some positive integers α_ℓ and β_ℓ. Note that U_{ij} (*resp.* $U_{i\ell}$) corresponds to $x^{a_{ij}} y^{b_{ij}} z^{c_{ij}} t$ (*resp.* $x^{a_{i\ell}} y^{b_{i\ell}} z^{c_{i\ell}} t$) in the epimorphism φ. By the assumption of the theorem, there exists $x^{a_{hs}} y^{b_{hs}} z^{c_{hs}} \in \Gamma(f_h)$ such that the line through (a_{ij}, b_{ij}, c_{ij}) and $(a_{i\ell}, b_{i\ell}, c_{i\ell})$ intersects $M(x^{a_{hs}} y^{b_{hs}} z^{c_{hs}})$. We divide the line into the following three parts:



If part I intersects $M(x^{a_{hs}} y^{b_{hs}} z^{c_{hs}})$, then, by Lemma 4.2, there exist nonnegative integers $\alpha, \beta, \gamma, \delta, \lambda$ with γ, δ, λ not all zero such that

$$(x^{a_{ij}} y^{b_{ij}} z^{c_{ij}} t)^{\alpha} (x^{a_{i\ell}} y^{b_{i\ell}} z^{c_{i\ell}} t)^{\beta} = x^{\gamma} y^{\delta} z^{\lambda} (x^{a_{hs}} y^{b_{hs}} z^{c_{hs}} t)^{\alpha+\beta}.$$

Thus, we have $U_{ij}^{\alpha}U_{i\ell}^{\beta} - x^{\gamma}y^{\delta}z^{\lambda}U_{hs}^{\alpha+\beta} \in \ker\varphi$. That gives $U_{ij}^{\alpha}U_{i\ell}^{\beta} \in \mathfrak{m}R[U_{ij}] + \ker\varphi$ and $u_{ij}^{\alpha}u_{i\ell}^{\beta} = 0$. If part II intersects $M(x^{a_{hs}}y^{b_{hs}}z^{c_{hs}})$, then, by Lemma 4.2, there exist nonnegative integers $\alpha, \beta, \gamma, \delta, \lambda$ with γ, δ, λ not all zero such that

$$(x^{a_{ij}}y^{b_{ij}}z^{c_{ij}}t)^{\alpha+\beta} = x^{\gamma}y^{\delta}z^{\lambda}(x^{a_{hs}}y^{b_{hs}}z^{c_{hs}}t)^{\alpha}(x^{a_{i\ell}}y^{b_{i\ell}}z^{c_{i\ell}}t)^{\beta}$$

and this implies $u_{ij}^{\alpha+\beta} = 0$ as in the proof of Theorem 3.1. Similarly, if part III intersects $M(x^{a_{hs}}y^{b_{hs}}z^{c_{hs}})$, we may apply Lemma 4.2 and get $u_{i\ell}^{\alpha+\beta} = 0$ for some nonnegative integers α, β. These prove that for each $i = 1, \ldots, m$, $k[u_{i1}, \ldots, u_{in_i}]$ is integral over $k\big[\sum_{j=1}^{n_i} \eta_{ij}u_{ij}\big]$ and so $k[u_{ij}]$ is integral over $k\big[\sum_{j=1}^{n_1} \eta_{1j}u_{1j}, \ldots, \sum_{j=1}^{n_m} \eta_{mj}u_{mj}\big]$ for all i, j. Hence I is a reduction of I^*. The proof is completed. □

In the end of this section, we include three examples to illustrate the above algorithm. The first two examples Example 4.4 and Example 4.5 are adopted from [Q1]. Instead of using [Q1, Theorem 3.3], we apply Theorem 4.3 to these two examples.

Example 4.4. Consider the ideal $I = \langle x^3yz^2 + xy^3z, x^2y^2z + y^4z^2\rangle$ in $k[x, y, z]_{(x,y,z)}$ with $|k| = \infty$. Note that the monomials occurring in the first generator of I are x^3yz^2 and xy^3z. A point on the line through these two monomials can be written as $s(3, 1, 2) + t(1, 3, 1)$ with $s, t \in \mathbb{R}$ and $s + t = 1$. Take $s = \frac{1}{2}$ and $t = \frac{1}{2}$, then the point $(2, 2, \frac{3}{2}) \in M(x^2y^2z)$. In other words, the line through $(3, 1, 2)$ and $(1, 3, 1)$ intersects $M(x^2y^2z)$ as indicated in the following Figure 4.4.1.

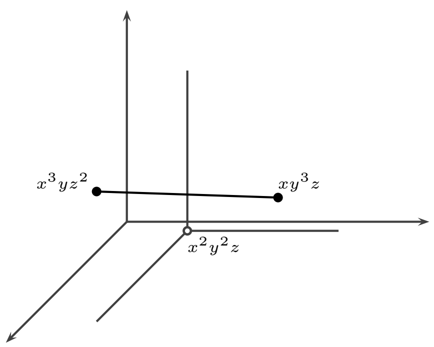


Figure 4.4.1.

Similarly, a point on the line through x^2y^2z and y^4z^2, which occur in the second generator of I, can be written as $s(2, 2, 1) + t(0, 4, 2)$ for some $s, t \in \mathbb{R}$ with

$s + t = 1$. By direct calculating, we see that the line intersects $M(xy^3z)$ at the point $\frac{1}{2}(2,2,1) + \frac{1}{2}(0,4,2)$, which we indicate in Figure 4.4.2. Hence, by Theorem 4.3, $I^* = \langle x^3yz^2, xy^3z, x^2y^2z, y^4z^2\rangle$ is integral over I.

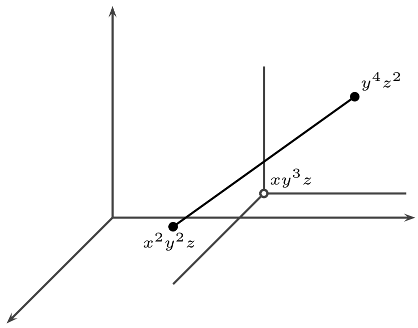


Figure 4.4.2.

Example 4.5. Consider the ideal $I = \langle x^4+y^2z, y^4+z^2x, z^4+x^2y\rangle$ in $k[x,y,z]_{(x,y,z)}$ with $|k| = \infty$. Note that the monomials occurring in the first generator of I are x^4, y^2z. A point on the line through the above two monomials can be written as $s(4,0,0) + t(0,2,1)$ for some $s,t \in \mathbb{R}$ with $s+t = 1$. Take $s = \frac{1}{2}$ and $t = \frac{1}{2}$, then the point $(2,1,\frac{1}{2})$ is contained in $M(x^2y)$. In other words, the line through x^4 and y^2z intersects $M(x^2y)$ as in the following Figure 4.5.1.

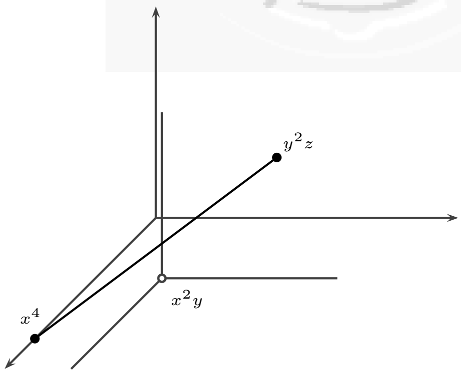


Figure 4.5.1.

Similarly, a point on the line through y^4 and z^2x, which occur in the second generator of I, can be written as $s(0,4,0)+t(1,0,2)$ where $s,t \in \mathbb{R}$ with $s+t = 1$. By calculating directly, we get that this line intersects $M(y^2z)$ at $\frac{1}{2}(0,4,0)+\frac{1}{2}(1,0,2)$ as in Figure 4.5.2. At last, we apply the same approach to the line through the two monomials occurring in the third generator of I and get that it intersects $M(z^2x)$ at $\frac{1}{2}(0,0,4)+\frac{1}{2}(2,1,0)$ as in Figure 4.5.3. Hence, by Theorem 4.3, $I^* = \langle x^4, y^2z, y^4, z^2x, z^4, x^2y\rangle$ is integral over I.

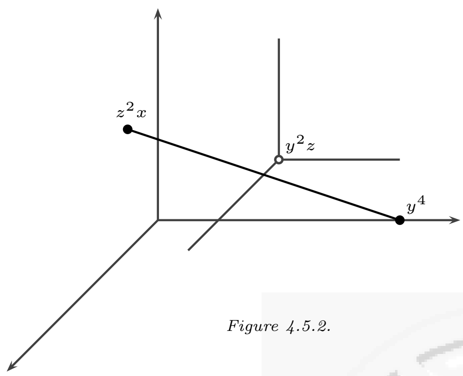


Figure 4.5.2.

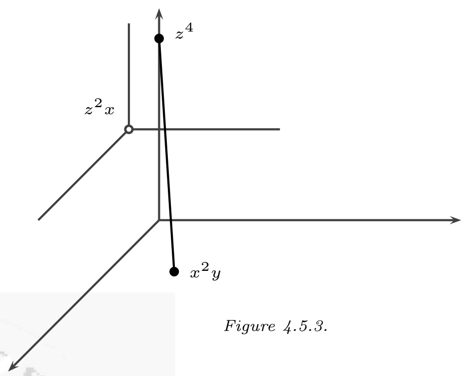


Figure 4.5.3.

At last, we give a slightly more complicated example. In particular, one of the two generators of the ideal I contains three monomials in this example.

Example 4.6. Let $I = \langle xy^2z^3 + x^6z^7, y^4z^4 + x^2y^3z^2 + x^3yz^5\rangle \subseteq k[x,y,z]_{(x,y,z)}$ with $|k| = \infty$. Note that the monomials occurring in the first generator of I are xy^2z^3 and x^6z^7. A point on the line through these two monomials can be written as $s(1,2,3)+t(6,0,7)$ with $s,t \in \mathbb{R}$ and $s+t=1$. Take $s=\frac{1}{2}$ and $t=\frac{1}{2}$, then the point $(\frac{7}{2},1,5) \in M(x^3yz^5)$. In other words, the line through $(1,2,3)$ and $(6,0,7)$ intersects $M(x^3yz^5)$ as indicated in the following Figure 4.6.1.

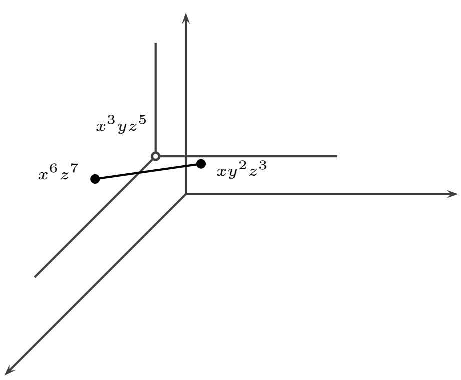


Figure 4.6.1.

Similarly, $y^4z^4, x^2y^3z^2$, and x^3yz^5 are the monomials that occur in the second generator of I. We indicate these points and the connecting line segments, and $M(xy^2z^3)$ in the following Figure 4.6.2.

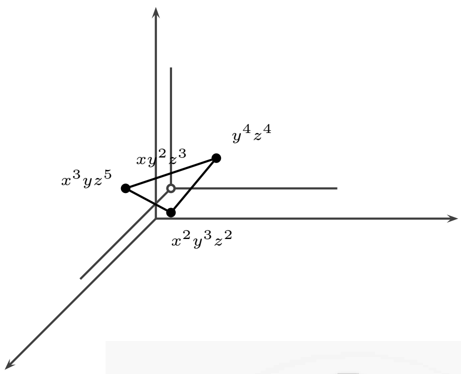


Figure 4.6.2.

By direct calculation, we know that for every line through two of the above three monomials intersects $M(xy^2z^3)$. More precisely,

- ⋆ the point $(1, \frac{7}{2}, 3) = \frac{1}{2}(0, 4, 4) + \frac{1}{2}(2, 3, 2)$ lies in both $M(xy^2z^3)$ and the line through y^4z^4 and $x^2y^3z^2$;
- ⋆ the point $(\frac{7}{3}, \frac{7}{3}, 3) = \frac{2}{3}(2, 3, 2) + \frac{1}{3}(3, 1, 5)$ lies in both $M(xy^2z^3)$ and the line through $x^2y^3z^2$ and x^3yz^5;
- ⋆ the point $(1, 3, \frac{13}{3}) = \frac{2}{3}(0, 4, 4) + \frac{1}{3}(3, 1, 5)$ lies in both $M(xy^2z^3)$ and the line through y^4z^4 and x^3yz^5.

Hence, by Theorem 4.3, $I^* = \langle xy^2z^3, x^6z^7, y^4z^4, x^2y^3z^2, x^3yz^5\rangle$ is integral over I.

In [CL, Corollary 3.7], which is an immediate application of [CL, Theorem 3.3], they find a minimal reduction of a given monomial ideal I in $k[x, y]_{(x,y)}$ where k is an infinite field. In fact, the original aims of our study are not only extending [CL, Theorem 3.3] to the three-dimensional case but also finding a minimal reduction of a given monomial ideal I in $k[x, y, z]_{(x,y,z)}$. But we are not able to do so yet. This issue can be discussed in a further research.

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