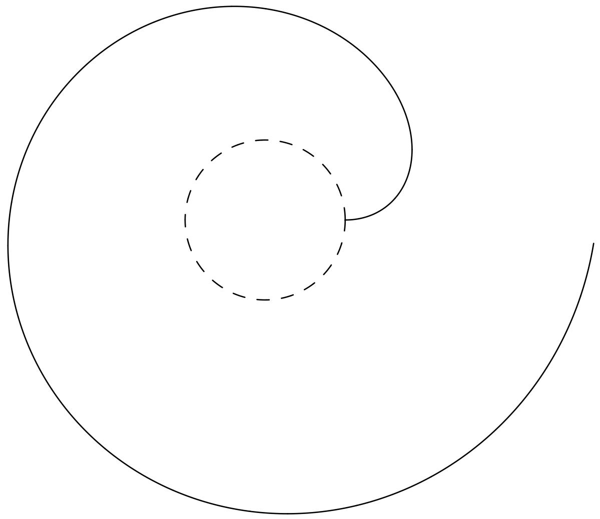

Figure 1: the corresponding curve of the type (A) solution

4 Geometric explanations and graphics

We study the steadily rotating spirals under the curvature flow $u = -\kappa$ from the kinematic model equation. This is equivalent to find a planar curve such that its tip rotates along a circle. In general, we can always draw a planar curve with a given curvature function parametrized by the arc length as follows.

Given a differentiable function $\kappa(s)$, $s \geq 0$, then the parametrized plane curve having $\kappa(s)$ as curvature is given by

$$\alpha(s) = \left(\int_0^s \cos\theta(\sigma)d\sigma + a, \int_0^s \sin\theta(\sigma)d\sigma + b \right), \tag{4.1}$$

where

$$\theta(\sigma) = \int_0^\sigma [-\kappa(\xi)]d\xi + \varphi, \tag{4.2}$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

In the case for $\omega > 0$, the tip is rotating along the core circle in the counterclockwise direction. Under our curvature flow $u = -\kappa$, the radius of the core circle is given by $\rho = \dfrac{|u(0)|}{\omega} = \dfrac{|\kappa_0|}{\omega}$. If $\kappa_0 < 0$, then the tangent of the tip is pointed outward to the center of the core circle, and the curvature of the curve is always negative (see Figure 1). Note that the core circle is reduced to a point, if $\kappa_0 = 0$. If $\kappa_0 > 0$, then the tangent of the tip is pointed inward to the center (see Figure 2), and the curvature of the curve would change sign once when $\kappa(s_0) = 0$ for some $s_0 > 0$. In any cases, the curvature $\kappa = \kappa(s)$ converges to 0 as s tends to infinite.

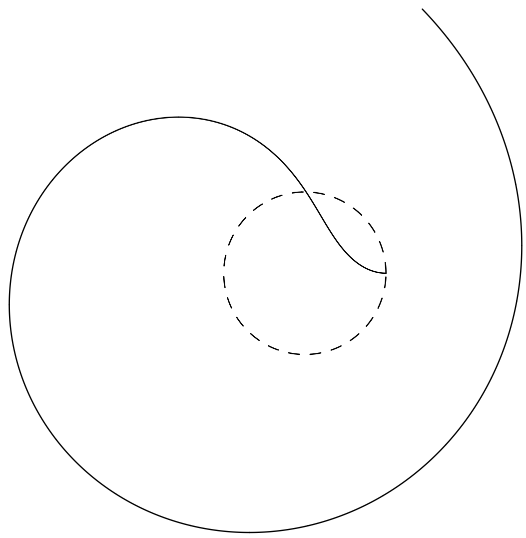

Figure 2: the corresponding curve of the type (B) solution

In the second case for $\omega < 0$, the tip is rotating along the core circle in the clockwise direction and the radius of the core circle is given by $\rho = \dfrac{|u(0)|}{-\omega} = \dfrac{|\kappa_0|}{-\omega}$. Similarly, the tangent of the tip is pointed outward if $\kappa_0 > 0$ and the curvature is always positive. If $\kappa_0 < 0$, then the tangent of the tip is pointed inward and the curvature would change sign once. The curvature always converges to zero.

In the case for $\omega = 0$, there is no rotation and the curve move along the normal direction which is the normal direction of the tip (see Figure 3). Conversely, such dynamics is also described by (1.1) as shown in the following proposition.

Proposition 4.1 *Let $z(s) \in \mathbb{R}^2$, $s \geq 0$, be the curve which has κ as curvature and moves along the normal direction of the tip with speed $-\kappa_0$, where κ_0 is the curvature of the tip. Then the curvature $\kappa(s)$ and the normal velocity $u(s)$ satisfy (1.1) with $\omega = 0$.*

Proof. Without loss of generality, by (4.1) and (4.2), we may let

$$z(s) = \left(\int_0^s \cos\theta(\sigma) d\sigma, \int_0^s \sin\theta(\sigma) d\sigma \right) \tag{4.3}$$

where

$$\theta(\sigma) = \int_0^\sigma [-\kappa(\xi)] d\xi. \tag{4.4}$$

Then, by (4.3), the normal vector

$$\vec{n}(s) = (-\sin\theta(s), \cos\theta(s)) \tag{4.5}$$

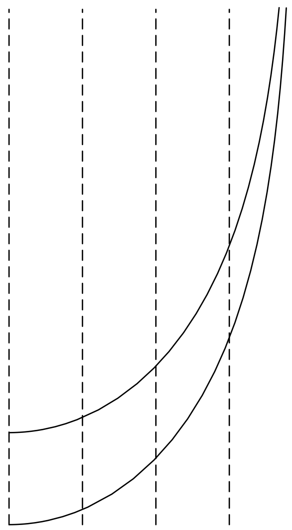

Figure 3: the corresponding curve of the type (F) solution

for all $s > 0$. Since the curve moves along the normal direction of the tip with speed κ_0, the family of the time dependent curves is given by

$$Z(s,t) = z(s) - \kappa_0 t \vec{n}(0) \tag{4.6}$$

where κ_0 is the curvature of the tip. Then, by (4.5) and (4.6), the normal velocity u is given by the normal projection

$$u(s) = \langle Z_t(s,t), \vec{n}(s) \rangle = -\kappa_0 \cos\theta(s). \tag{4.7}$$

Then, by (4.4) and (4.7),

$$u'(s) = \kappa_0 \sin\theta(s) \cdot \theta'(s) = -\kappa_0 \kappa(s) \sin\theta(s), \tag{4.8}$$

and

$$\int_0^s \kappa(\sigma) u(\sigma) d\sigma = \kappa_0 \int_0^s \cos\theta(\sigma) \cdot \theta'(\sigma) d\sigma = \kappa_0 \sin\theta(s). \tag{4.9}$$

Hence

$$u'(s) + \kappa(s) \int_0^s \kappa(\sigma) u(\sigma) d\sigma = 0. \tag{4.10}$$

This proves the proposition. □