4 Minmax theorems under the region of the graph of a multifunction

On some feasible region, the two functions version of minimax theorems does not hold again. However, by restricting to a proper range, the minimax theorem is down.

For instance, let the set $A = \{(x,y) \in [0,1]\times[0,1]; 0 \le y \le \frac{1}{2}x\}\backslash\{(0,0)\}$, and g be a real-valued function on $[0,1]\times[0,1]$, defined by

$$g(x,y) = \begin{cases} 0 & if\ (x,y) \in A \cup \{(0,1)\} \\ 1 & otherwise \end{cases}.$$

It is easy to see that

$$\inf_{y\in Y}\sup_{x\in X} g(x,y) = 1 > 0 = \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

If T is a multifunction on $[0,1]$ defined by

$$T(x) = \begin{cases} \left[x, x+\frac{1}{2}\right] & if\ x < \frac{1}{2} \\ [x,1] & if\ x \ge \frac{1}{2} \end{cases},$$

then we have (see Example 4.4)

$$\inf_{y\in Y}\sup_{x\in T^{-1}(y)} g(x,y) = 1 = \sup_{x\in X}\inf_{y\in T(x)} g(x,y).$$

This motivates us to define X-*quasiconcave* sets below. Recall that for a multifunction $T : X \longrightarrow Y$, the set $H_T = \{g : X\times Y \to \mathbb{R}\}$ is said to be a X-*quasiconcave* of T if for all $g \in H_T$, and $x, x_1, x_2 \in X$, there exists $x_3 \in X$, $h \in H_T$ such that

$$h(x_3,y) \ge \max\{g(x_1,y), g(x_2,y)\}, \quad \forall\ y \in T(x).$$

Theorem 4.1. *Let X be a nonempty compact convex set of a Hausdorff topological vector space, and Y be a nonempty set. Let $T : X \to Y$ be a multifunction having nonempty images, H_T a X-quasiconcave set of T, and f be a real-valued function defined on $X\times Y$ satisfied the following properties:*
(0) $\sup_X f(x,y) \le \sup_X g(x,y)$ *for all $y \in Y$ and $g \in H_T$;*
(i) T is upper semicontinuous on X;
(ii) For each $x \in X$, $y \in T(X)$ and $g \in H_T$, $g(x,\cdot)$ is lower semicontinous on $T(X)$ and $g(\cdot,y)$ is quasiconcave on X.
Then for any $\lambda \in \mathbb{R}$, there exists some $g \in H_T$ satisfying we have the following alternative:
Either $\sup_{g\in H_T}\sup_{x\in X}\inf_{y\in T(x)} g(x,y) \ge \lambda$,
or there exists $y_0 \in T(X)$ such that $f(x,y_0) \le \lambda$ for all $x \in T^{-1}(y_0)$.

Proof. *For each $\lambda \in \mathbb{R}$, let*

$$U_g(y) = \{x \in X; g(x,y) > \lambda\},$$

and

$$V_g(x) = \{y \in Y; g(x,y) > \lambda\}.$$

Fixed $\lambda \in \mathbb{R}$. We may assume that

$$\forall\ y \in T(X) \quad \exists\ x_0 \in T^{-1}(y)\ \ s.t.\ f(x_0,y) > \lambda. \tag{17}$$

For $g \in H_T$, let $S_g : X \to X$ be defined as $S_g(x) = \cap_{y \in T(x)} U_g(y)$.

We want to show that

(α) $S_g(x) \neq \emptyset$ for some $g \in H_T$.

Fixed $x \in X$, let

$$A = \{V_\varphi(z) \cap T(x); z \in X, \varphi \in H_T\}.$$

Obviously, the set A is a nonempty partially ordered set with the inclusion relation of subsets of Y. It is easy to see that any totally ordered subset of A has an upper bound. By Zorn's Lemma, there exists a maximal element $V_g(\hat{x}) \cap T(x)$ of A, that is, $\hat{x} \in X$ and $g \in H_T$ satisfied that for any $z \in X$, $\varphi \in H_T$

$$if \quad V_g(\hat{x}) \cap T(x) \subset V_\varphi(z) \cap T(x) \quad then \quad V_\varphi(z) \cap T(x) = V_g(\hat{x}) \cap T(x). \tag{18}$$

Suppose $V_g(\hat{x}) \cap T(x) \neq T(x)$. Let $y \in T(x) \setminus V_g(\hat{x})$, by (17) we have some $x_0 \in T^{-1}(y)$ such that $f(x_0, y) > \lambda$. By (0), we have $\sup_X g(x, y) \geq \sup_X f(x, y) \geq f(x_0, y) > \lambda$. This implies some $x_1 \in X$ such that $g(x_1, y) > \lambda$. Since H_T is X-quasiconcave, for g, $\hat{x}$ and x_1 there exists $h \in H_T$, $x_3 \in X$ such that

$$h(x_3, y) \geq \max\{g(\hat{x}, y), g(x_1, y)\}\ , \quad \forall\ y \in T(x).$$

We have $V_g(\hat{x}) \cap T(x) \subsetneq V_h(x_3) \cap T(x)$, and get a contradiction from (18). This shows that $V_g(\hat{x}) \cap T(x) = T(x)$ and hence $T(x) \subset V_g(\hat{x})$. In other words, for all $y \in T(x)$, $g(\hat{x}, y) > \lambda$. It follows that $\hat{x} \in \cap_{y \in T(x)} U_g(y)$. Hence, $S_g(x) \neq \emptyset$ for some $g \in H_T$.

(β) $S_g(x)$ is convex.

Since $g(\cdot, y)$ is quasiconcave by (ii) for each $y \in T(X)$, $U_g(y)$ is convex. Hence $S_g(x)$ is also convex.

(γ) for any $z \in X$, $S_g^{-1}(z) = \{x \in X; g(z, y) > \lambda,\ \forall\ y \in T(x)\}$ is open.

Since $g(x, \cdot)$ is l.s.c, $U_g(x)$ is open. It follows from upper semicontinuity of T that

$$S_g^{-1}(z) = \{x \in X; T(x) \subset U_g(z)\} = T^+(U_g(z))$$

is open for each $z \in X$.

We already have some $g \in H_T$ satisfying (α), (β) and (γ). By Browder's Fixed Point Theorem, there exists some $x_1 \in X$ such that $x_1 \in S_g(x_1)$. That is,

$$g(x_1, y) > \lambda \ \ \forall\ y \in T(x_1) \Longrightarrow \inf_{y \in T(x_1)} g(x_1, y) > \lambda \Longrightarrow \sup_{x \in X} \inf_{y \in T(x)} g(x, y) \geq \lambda.$$

We conclude that

$$\sup_{g \in H_T} \sup_{x \in X} \inf_{y \in T(x)} g(x, y) \geq \lambda.$$

□

Theorem 4.2. *Let X be a nonempty compact convex set of a Hausdorff topological vector space, and Y be a nonempty set. Let $T : X \to Y$ be a multifunction having nonempty images, H_T a X-quasiconcave set for T, and f be a real-valued function defined on $X \times Y$ satisfied the following properties:*
(0) $\sup_X f(x,y) \le \sup_X g(x,y)$ *for all* $y \in Y$ *and* $g \in H_T$*;*
(i) T *is upper semicontinuous on* X*;*
(ii) *For each* $x \in X$, $y \in T(X)$ *and* $g \in H_T$, $g(x,\cdot)$ *is lower semicontinous on* $T(X)$ *and* $g(\cdot,y)$ *is quasiconcave on* X.
Then there exists some $g \in H_T$ *such that*

$$\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \sup_{g\in H_T} \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

Proof. *Let* $\lambda \in \mathbb{R}$ *and with*

$$\sup_{g\in H_T} \sup_{x\in X} \inf_{y\in T(x)} g(x,y) < \lambda.$$

Thus, by Theorem 4.1, there exists $y_0 \in T(X)$ *such that* $f(x,y_0) \le \lambda$ *for all* $x \in T^{-1}(y_0)$. *This implies that* $\sup_{x\in T^{-1}(y_0)} f(x,y_0) \le \lambda$ *and hence* $\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \lambda$. *It follows that*

$$\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \sup_{g\in H_T} \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

□

Corollary 4.3. *Let X be a nonempty compact convex set of a Hausdorff topological vector space, and Y be a nonempty set. Let $T : X \to Y$ be a multifunction having nonempty images, and f, g be two real-valued functions defined on $X \times Y$ satisfied the following properties:*
(0) $\sup_X f(x,y) \le \sup_X g(x,y)$ *for all* $y \in Y$;
(i) T *is upper semicontinuous on* X*;*
(ii) *For each* $x \in X$, $y \in T(X)$ *and* $g(x,\cdot)$ *is lower semicontinous on* $T(X)$ *and* $g(\cdot,y)$ *is quasiconcave on* X*;*
(iii) *For all* $x, x_1, x_2 \in X$, *there exists* $x_3 \in X$ *such that*

$$g(x_3,y) \ge \max\{g(x_1,y), g(x_2,y)\} \quad \forall\ y \in T(x).$$

Then

$$\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

Proof. *Let* $H_T = \{g\}$, *then by* (iii) H_T *is* X*-quasiconcave of* T. *By Theorem 4.1 we have* $\sup_{g\in H_T} \sup_{x\in X} \inf_{y\in T(x)} g(x,y) = \sup_{x\in X} \inf_{y\in T(x)} g(x,y)$. *Thus,*

$$\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

□

Example 4.4. *Let the set* $A = \{(x,y) \in [0,1]\times[0,1]; 0 \le y \le \frac{1}{2}x\}\backslash\{(0,0)\}$*, and* g *be a real-valued function defined on* $[0,1]\times[0,1]$ *by*

$$g(x,y) = \begin{cases} 0 & if\ (x,y) \in A \cup \{(0,1)\} \\ 1 & otherwise \end{cases}.$$

If T *is a multifunction on* $[0,1]$ *defined by*

$$T(x) = \begin{cases} \left[x, x+\frac{1}{2}\right] & if\ x < \frac{1}{2} \\ [x,1] & if\ x \ge \frac{1}{2} \end{cases}$$

figure 2

Then we have

$$\inf_{y\in Y}\sup_{x\in X} g(x,y) = 1 > 0 = \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

We will check the conditions of Corollary 4.3.

(i) *Clearly,* T *is u.s.c on* X*.*

(ii)*Let* $L^\beta(x) = \{y; g(x,y) \le \beta\}$,

$$\text{when} \quad \begin{array}{l} \beta \ge 1,\ L^\beta(x) = [0,1]. \\ 0 \le \beta < 1,\ L^\beta(x) = \begin{cases} \left[0, \frac{1}{2}x\right],\ if\ 0 < x \le 1 \\ \{(0,1)\},\ if\ x = 0 \end{cases}. \\ \beta < 0,\ L^\beta(x) = \emptyset. \end{array}$$

Hence, $g(x,\cdot)$ *is lower semicontinuous on* Y*.*

(iii)*Let* $U^\beta(y) = \{x; g(x,y) \ge \beta\}$*. Then*

$$\text{when} \quad \begin{array}{l} \beta > 1, U^\beta(y) = \emptyset. \\ 0 < \beta \le 1, U^\beta(y) = \begin{cases} \{(0,0)\},\ if\ y = 0 \\ [0,2y)\,,\ if\ 0 < y \le \frac{1}{2} \\ [0,1]\,,\ if\ \frac{1}{2} < y < 1 \\ (0,1]\,,\ if\ y = 1 \end{cases}. \\ \beta \le 0, U^\beta(y) = [0,1]. \end{array}$$

Hence, $g(\cdot,y)$ *is quasi-concave on* X*.*

(iv)*For all* $x, x_1, x_2 \in X$*, we have (by taking* $x_3 = x$*)*

$$g(x,y) = 1 \ge \max\{g(x_1,y), g(x_2,y)\}, \forall y \in T^{-1}(x).$$

By Corollary 4.3, we have

$$\inf_{y\in Y}\sup_{x\in T^{-1}(y)} g(x,y) \le \sup_{x\in X}\inf_{y\in T(x)} g(x,y).$$

In fact, $\inf_{y\in Y}\sup_{x\in T^{-1}(y)} g(x,y) = \sup_{x\in X}\inf_{y\in T(x)} g(x,y) = 0.$