CHAPTER 5

Finite extension of $\mathbb{Q}_p$ of degree 3

5.1. Irreducible polynomials of degree 3

In this section we list some essential properties to help us discuss extensions of $\mathbb{Q}_p$ of degree 3.

Being different from extensions of $\mathbb{Q}_p$ of degree 2, extensions of $\mathbb{Q}_p$ of degree 3 is not always Galois. Hence besides determining the number of extensions of $\mathbb{Q}_p$ we will discuss what extensions are $\mathbb{Q}_p-$isomorphic. Lemma 5.1.1 gives a method using discriminant of an irreducible polynomial to check if an extension obtained by adjoining a root of the polynomial is a Galois extension or not.

LEMMA 5.1.1. *Let $f(x) = x^3 + ax^2 + bx + c$ be an irreducible polynomial over $\mathbb{Q}_p$ and α be a root of $f(x)$, then $\Delta_f = -4a^3 - 27b^2 + a^2(b^2 - 4ac) + 18abc$ and the Galois group of $f(x)$ is either S_3 or A_3. $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is Galois if and only if Δ_f is a square of an element in $\mathbb{Q}_p$.*

Proof. See [**3**, p.271] corollary 4.7 and proposition 4.8. □

For the same reason as extensions of degree 2, we separate into two cases: $p \neq 3$ and $p = 3$.

5.2. The case $p \neq 3$

In this section we look for the number of totally ramified extensions of degree 3 and describe these extensions in the case $p \neq 3$.

Since $p \neq 3$, by lemma 2.3.3 we have $\{f(x) = x^3 - a_0p,\ a_0 \in U_p\}$ generate all totally ramified extensions of $\mathbb{Q}_p$. For $f(x) = x^3 - up$, where $u \in U_p$, let α_i be roots of $f(x)$ for $i = 1, 2, 3$ and let $\delta = \min_{1 \le i \neq j \le 3}\{|\alpha_i - \alpha_j|_p\}$, which equals $|\sqrt{\Delta_f}|_p^{\frac{1}{3}} = |-27u^2p^2|_p^{\frac{1}{6}} = p^{-\frac{1}{3}}$. For $a_0 \in U_p$, we consider $\overline{a_0p} = \overline{ip}$ in M_p/M_p^2 with some $i \in \{1, 2, \cdots, p-1\}$. Let $f(x) = x^3 - a_0p$ and $g(x) = x^3 - ip$ Hence by lemma 2.4.6 we have

$$|f - g|_p = |a_0p - ip|_p \le \frac{1}{p^2} < \delta^3 = \frac{1}{p},$$

which implies for a root β of $x^3 - a_0p$, there exists a root θ of $x^3 - ip$ such that $\mathbb{Q}_p(\theta) = \mathbb{Q}_p(\beta)$. Hence we conclude that $\{f(x) = x^3 - ip, i \in \{1, \cdots, p-1\}\}$ generate all totally ramified extensions of $\mathbb{Q}_p$ of degree 3.

In theorem 5.2.1 we will further calculate the exact number of totally ramified extensions of $\mathbb{Q}_p$.

THEOREM 5.2.1. *There are exactly three totally ramified extensions of* $\mathbb{Q}_p$ *of degree* 3. *Furthermore, if* $3\ |p-1$, *then these extensions are Galois and are not* $\mathbb{Q}_p$*-isomorphic with each other; if* $3 \nmid p-1$, *then these extensions are not Galois and are* $\mathbb{Q}_p$*-isomorphic with each other.*

Proof. For $f(x) = x^3 - a_0p$, we have $\Delta_f = -27a_0^2p^2$. Hence $\mathbb{Q}_p(\sqrt[3]{a_0p})/\mathbb{Q}_p$ is Galois if and only if $(\frac{-3}{p}) = 1$. We separate the proof into two cases:

<u>Case1</u>: $3\ |p-1$. Then $(\frac{-3}{p}) = (\frac{-1}{p})(\frac{3}{p}) = 1$ and every totally ramified extension of $\mathbb{Q}_p$ is Galois. Because $3\ |p-1$, there exists an element a of $(\mathbb{Z}/p\mathbb{Z})^*$ such that $| < a > | = 3$. Let $H = \{u^3; u \in (\mathbb{Z}/p\mathbb{Z})^*\}$, $(\mathbb{Z}/p\mathbb{Z})^*/H \simeq < a >$ and hence has order 3. Denote these three coset of H by H, aH, a^2H.

Finally we will prove that there exist exactly three totally ramified extensions: $\mathbb{Q}_p(\sqrt[3]{p}\,)$, $\mathbb{Q}_p(\sqrt[3]{ap}\,)$, $\mathbb{Q}_p(\sqrt[3]{a^2p}\,)$, which are not $\mathbb{Q}_p$-isomorphic. If $\mathbb{Q}_p(\sqrt[3]{up}\,) = \mathbb{Q}_p(\sqrt[3]{vp}\,)$ for $u, v \in \{1, a, a^2\}$, $u \neq v$. Then we have $\sqrt[3]{\frac{u}{v}} \in \mathbb{Q}_p(\sqrt[3]{vp}\,)$. By lemma 2.2.2 we have $\mathbb{Q}_p(\sqrt[3]{\frac{u}{v}}\,)/\mathbb{Q}_p$ is unramified and $\mathbb{Q}_p(\sqrt[3]{\frac{u}{v}}\,) = \mathbb{Q}_p(\sqrt[3]{vp}\,)$, which contradicts with the fact that $\mathbb{Q}_p(\sqrt[3]{vp}\,)$ is totally ramified. Hence we conclude that $\mathbb{Q}_p(\sqrt[3]{p}\,)$, $\mathbb{Q}_p(\sqrt[3]{ap}\,)$, $\mathbb{Q}_p(\sqrt[3]{a^2p}\,)$ are all distinct and not $\mathbb{Q}_p$-isomorphic.

<u>Case2</u>: $3 \nmid p - 1$. Then $(\frac{-3}{p}) = -1$ and every totally ramified extension of $\mathbb{Q}_p$ is not Galois. For $a, b \in (\mathbb{Z}/p\mathbb{Z})^*$ with $a \neq b$, $a^3 \neq b^3$, hence every element in $(\mathbb{Z}/p\mathbb{Z})^*$ is a cubic. By Hensel's Lemma we have $\sqrt[3]{i} \in \mathbb{Q}_p$ for $i = 1, 2, \cdots, p-1$ and hence $\mathbb{Q}_p(\sqrt[3]{ip}) = \mathbb{Q}_p(\sqrt[3]{p}\,)$. Let ζ be a primitive 3th root of unity in $\mathbb{Q}_p^{alg}$, then we have exactly three totally ramified extensions: $\mathbb{Q}_p(\sqrt[3]{p})$, $\mathbb{Q}_p(\zeta\sqrt[3]{p})$, $\mathbb{Q}_p(\zeta^2\sqrt[3]{p})$ which are $\mathbb{Q}_p$-isomorphic. $\square$

5.3. The case $p = 3$

In this section we will follow the formula in chapter 3 to calculate the number of all totally ramified extensions of $\mathbb{Q}_3$ of degree 3.

First we have the fact that the only integers satisfying $Ore'scondition$ are $j = 1, 2, 3$ and hence we calculate $\aleph(K_{3,j})$ in the following lemma:

LEMMA 5.3.1. *$\aleph(K_{3,1}) = 6$, $\aleph(K_{3,2}) = 6$, $\aleph(K_{3,3}) = 9$*

Proof. When $j = 1$ we have $c = 2$, $\aleph(D_{E_{3,1}}) = 12$, hence

$$\aleph(K_{3,1}) = \frac{3}{3^{3*2-3-1+1-2}2}\aleph(D_{E_{3,1}}) = 6.$$

When $j = 2$, we have $c = 3$, $\aleph(D_{E_{3,2}}) = 54$, hence

$$\aleph(K_{3,2}) = \frac{3}{3^{3*3-3-1+1-2}2}\aleph(D_{E_{3,2}}) = 6.$$

When $j = 3$, we have $c = 4$, $\aleph(D_{E_{3,2}}) = 108$, hence

$$\aleph(K_{3,3}) = \frac{3}{3^{3*4-3-1+1-2}2}\aleph(D_{E_{3,3}}) = 9.$$

□

Next we will separate into three cases: $j = 1$, $j = 2$, $j = 3$.

PROPOSITION 5.3.2. *Suppose $j = 1$. Then we can find only two Eisenstein polynomials which generate all elements in $K_{3,1}$. Furthermore, all elements in $K_{3,1}$ are not Galois over $\mathbb{Q}_p$.*

Proof. Consider two polynomials: $f_1(x) = x^3 + 3x + 3$ and $f_2(x) = x^3 + 6x + 3$, then $\Delta_{f_1} = -351$, $\Delta_{f_2} = -1107$. Hence the splitting field of $f_1(x)$ and $f_2(x)$ have $\mathbb{Q}_p(\sqrt{-3})$ and $\mathbb{Q}_p(\sqrt{3})$ as intermediate fields, respectively. This implies $f_1(x) = x^3 + 3x + 3$ and $f_2(x) = x^3 + 6x + 3$ generate distinct elements in $K_{3,1}$. Let α be a root of $f_1(x)$, β a root of $f_2(x)$. By lemma 5.3.1 we have exactly six elements in $K_{3,1}$ which are not Galois over $\mathbb{Q}_p$. Furthermore there are three elements $\mathbb{Q}_p$−isomorphic to $\mathbb{Q}_p(\alpha)$; three elements $\mathbb{Q}_p$−isomorphic to $\mathbb{Q}_p(\beta)$. □

PROPOSITION 5.3.3. *Suppose $j = 2$. Then we can find only four Eisenstein polynomials which generate all elements in $K_{3,2}$. Furthermore, there are three elements in $K_{3,2}$ which are not Galois over $\mathbb{Q}_p$; three elements in $K_{3,2}$ which are Galois over $\mathbb{Q}_p$.*

Proof. Consider two polynomials: $f_1(x) = x^3 + 3x^2 + 3$ and $f_2(x) = x^3 + 3x^2 + 6$, then $\Delta_{f_1} = -567$, $\Delta_{f_2} = -1620$. This implies the extension obtained by adjoining a root of $f_1(x)$ is not Galois over $\mathbb{Q}_p$ and the other one obtained by adjoining a root of $f_2(x)$ is Galois over $\mathbb{Q}_p$. Because $\aleph(K_{3,2}) = 6$, and for every non-Galois extension K of $\mathbb{Q}_p$ of degree 3 we can find other two distinct extensions of $\mathbb{Q}_p$ of degree 3 which are $\mathbb{Q}_p$-isomorphic, we conclude that there are three elements in $K_{3,2}$ which are not Galois over $\mathbb{Q}_p$; three elements in $K_{3,2}$ which are Galois over $\mathbb{Q}_p$. □

PROPOSITION 5.3.4. *Suppose $j = 3$. Then we can find only three Eisenstein polynomials which generate all elements in $K_{3,3}$. Furthermore, all elements in $K_{3,3}$ are not Galois over $\mathbb{Q}_p$.*

Proof. Because for $f(x) \in E_{3,3}$, $\upsilon_p(\Delta_f) = 5$, which implies $\sqrt{\Delta_f} \notin \mathbb{Q}_p$. Hence we conclude that all elements in $K_{3,3}$ are not Galois over $\mathbb{Q}_p$ and hence they can be generate by only three Eisenstein polynomials. □