國立臺灣師範大學數學系碩士班碩士論文

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Continuous Selections For Almost Lower Semicontinuous Multifunctions

研 究 生：黃 建 豪

中 華 民 國 九 十 八 年 六 月

Continuous Selections For Almost Lower Semicontinuous Multifunctions

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Abstract

In this paper, we obtain several new continuous selection theorems for almost lower semicontinuous multifunctions T on a paracompact topological space X, in the general noncompact and/or nonconvex settings. We consider three interesting topics in the selection theory; each of these topics deals with a broad class of selection problems. One is to introduce and analyze some well known selection theorems. Based on Deutsch-Kenderov theorem and an equicontinuous property, we first establish a generalized selection theorem for the multifunctions, even without requiring lower semicontinuity on T, but merely an almost lower semicontinuous multifunction. Secondly, we establish some relationships between abstract convexity and the selection property. Under a mild condition of one point extension property, we show that a C-set structure on a metric space without convexity still has the continuous selection property. Finally, we modify our selection theorems by adjusting a closed subset Z of X with its covering dimension $dim_X Z \leq 0$. These results derived here generalize and unify various earlier ones from classic continuous selection theory.

Keywords. Continuous selection, ϵ-approximate selection, lower semicontinuous, partition of unity, almost lower semicontinuous, equicontinuous property(ECP), C-space, C-set, LC-metric space, one point extension property, covering dimension.

2000 AMS subject classifications. 47H05, 54C20, 54C60, 54C65, 54E50, 55M10.

§1 Introduction and Preliminaries.

Continuous selection plays an important role in optimization theory, especially in the proof of existence of fixed points for a multifunction, see for example [3,4,5,6,10,13,15,17]. In this paper, we develop two different approaches to establish some new selection theorems for *almost lower semicontinuous* multifunctions, which are weaker than usual lower semicontinuity. Beyond the convexity and compactness, the results derived here extend various earlier ones from classic continuous selection theory, as will be indicated below.

Let X and Y be two topological spaces. A *multifunction* T from X to Y, written as $T : X \longrightarrow Y$, is simply a function which assigns each point x of X to a (possibly empty) subset $T(x)$ of Y. We shall say T is *lower semicontinuous* (*l.s.c.*) at x, provided for any $y \in T(x)$ and any neighborhood V_y of y, there is a neighborhood V_x of x such that $T(z) \cap V_y \neq \emptyset$ for all $z \in V_x$. T is *lower semicontinuous*, if T is *l.s.c.* at each $x \in X$.

It is known in [10] that if X is paracompact and Y is a Banach space, then every *l.s.c.* multifunction $T : X \longrightarrow Y$ having nonempty closed convex images admits a continuous selection f; that is, $f : X \longrightarrow Y$ is a continuous single-valued function such that $f(x) \in T(x)$ for each $x \in X$.

A multifunction $T : X \longrightarrow Y$ is said to have *local intersection property* (*l.i.p.*), if for each $x \in X$ with $T(x) \neq \emptyset$, there exists a neighborhood V_x of x such that

$$\bigcap_{z \in V_x} T(z) \neq \emptyset.$$

When Y is a metric space with a metric d, we may define the $\epsilon-$neighborhood of a subset A of Y by

$$B_\epsilon(A) := \{y \in Y \mid d(y, A) \leq \epsilon\},$$

where $d(y, A) := \inf\{d(x, y) \mid x \in A\}$. In particular, if Y is a normed linear space equipped with a norm $\|\cdot\|$, then

$$d(y, A) := \inf\{\|y - x\| \mid x \in A\}.$$

The *convex hull, closure*, and *interior* of A shall be denoted by *coA*, *clA*, and *intA*, respectively.

In 1996, Wu and Shen [15] proved that if X is paracompact, Y is a nonempty subset of a Hausdorff topological space, and $T : X \longrightarrow Y$ is a multifunction having *l.i.p.* and nonempty convex images, then it admits a continuous selection.

A multifunction $T : X \longrightarrow Y$ is *almost lower semicontinuous* (*a.l.s.c.*) at x, if for any $\epsilon > 0$, there exists a neighborhood V_x of x such that

$$\bigcap_{z \in V_x} B_\epsilon(T(z)) \neq \emptyset.$$

In 1983, Deutsch and Kenderov [5] proved that if X is paracompact, Y is a normed linear space, and $T : X \longrightarrow Y$ is a multifunction having nonempty convex images, then T is *a.l.s.c.* if and only if for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection f; that is, $f : X \longrightarrow Y$ is a continuous single-valued function such that $f(x) \in B_\epsilon(T(x))$ for each $x \in X$.

A multifunction $T : X \longrightarrow Y$ is *sub-lower semicontinuous* (*sub-l.s.c*) at x, if for any $\epsilon > 0$, there exists a neighborhood V_x of x and a vector $y \in T(x)$ such that

$$y \in \bigcap_{z \in V_x} B_\epsilon(T(z)).$$

In [5], Deutsch and Kenderov also proved that if X is paracompact, and Y is a locally convex topological vector space, then a multifunction $T : X \longrightarrow Y$ is *sub-l.s.c.* if and only if for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection.

A multifunction $T : X \longrightarrow Y$ is *weakly Hausdorff lower semicontinuous* (H_w-*l.s.c.*) at x, if for any $\epsilon > 0$ and any neighborhood V_x of x, there are a neighborhood U_x of x (with $U_x \subset V_x$) and a vector $y \in U_x$ such that $T(y) \subset B_\epsilon(T(z))$ for all $z \in U_x$. T is called *weakly Hausdorff lower semicontinuous*, if T is H_w-*l.s.c.* at each $x \in X$.

In 1985, Blasi and Myjak proved in [2] that if X is paracompact and Y is a Banach space, then every H_w-*l.s.c.* multifunction $T : X \longrightarrow Y$ having nonempty closed convex images admits a continuous selection.

A multifunction $T : X \longrightarrow Y$ is *quasi-lower semicontinuous*(*q.l.s.c.*) at x, if for any $\epsilon > 0$ and any neighborhood V_x of x, there is a vector $y \in V_x$ such that for every $w \in T(y)$, there exists a neighborhood V_w of x for which $w \in B_\epsilon(T(z))$ for all $z \in V_w$. T is called *quasi-lower semicontinuous*, if T is *q.l.s.c.* at each $x \in X$.

In 1987, Gutev [7] proved that if X is paracompact and Y is a Banach space, then every *q.l.s.c.* multifunction $T : X \longrightarrow Y$ having nonempty closed convex images admits a continuous selection.

It should be noted [2,5,6,7,11,12,13,16] that all *l.s.c.*, H_w-*l.s.c.*, *q.l.s.c.* and *sub-l.s.c.* multifunctions are *a.l.s.c.*, but not conversely in general. Also, it is shown that an *a.l.s.c.* multifunction $T : X \longrightarrow Y$ does not admit a continuous selection in general; for example (see [12]), let $X = R$, $Y = R^2$, and

$$T(x) = \begin{cases} \{(t, xt) \mid t \in [0,1]\} & \text{, if } x \text{ is irrational;} \\ \{(t, 0) \mid t \in [0,1]\} & \text{, if } x \text{ is rational and } x \neq 0; \\ \{(1,0)\} & \text{, if } x = 0. \end{cases}$$

Then T is *a.l.s.c.*, but T has no continuous selection.

Let us list some basic known lemmas, which we will use in the following sections.

Lemma 1.1. Let X be a topological space, and Y be a metric space with metric d. For a multifunction $T : X \longrightarrow Y$, the following statements are equivalent:

(a) T is *l.s.c.*.

(b) For each open set G in Y, the set $T^-(G) := \{x \in X \mid T(x) \cap G \neq \emptyset\}$ is open.

(c) For each $x \in X$, $\epsilon > 0$, and $y \in T(x)$, there exists a neighborhood V_x of x such that $y \in intB_\epsilon(T(z))$ for all $z \in V_x$.

Lemma 1.2. Let (Y, d) be a metric space, $A \subset Y$, and $\alpha > 0,\ \beta > 0$. Then

(1) $B_\alpha(B_\beta(A)) \subset B_{\alpha+\beta}(A)$

(2) $B_\alpha(intB_\beta(A)) \subset intB_{\alpha+\beta}(A)$

Lemma 1.3. Let X be a topological space, (Y, d) a metric space, and $S : X \longrightarrow Y$ be an *a.l.s.c.* multifunction. For any fixed $\epsilon > 0$, let

$$U_y := \{x \in X \mid y \in B_\epsilon(S(x))\},\ \forall y \in Y,$$

then $\{intU_y \mid y \in Y\}$ is an open cover of X.

Proof : For each $x \in X$, there is a neighborhood W_x of x such that $\bigcap_{z \in W_x} B_\epsilon(S(z)) \neq \emptyset$, say $y_x \in B_\epsilon(S(z))$ for all $z \in W_x$. That is,

$$x \in W_x \subset \{z \in X \mid y_x \in B_\epsilon(S(z))\}.$$

So, we have $x \in intU_y$ for some y, and hence $\{intU_y \mid y \in Y\}$ is an open cover of X. □

Lemma 1.4. Let $S : X \longrightarrow Y$ and $S_i : X \longrightarrow Y$ be *l.s.c* for each $i \in I$. Then

(1) $\overline{S}$ and coS are *l.s.c* (Here, $\overline{S}(x) := clS(x)$ and $coS(x) := co(S(x))$) ;

(2) $\bigcup_{i \in I} S_i$ is *l.s.c* ;

(3) if A is a nonempty closed set in X, and $\alpha : A \longrightarrow Y$ is a continuous function with $\alpha(x) \in S(x)$ for each $x \in A$, then the following multifunction $H : X \longrightarrow Y$ is *l.s.c.*, where

$$H(x) := \begin{cases} S(x) & , \text{if } x \in X \setminus A, \\ \{\alpha(x)\} & , \text{if } x \in A, \end{cases}$$

§2 Selection Theorems for Convex-valued Multifunctions.

In this section, we will establish some fundamental existence theorems of selections under a mild condition. Indeed, we shall deal with the case where the multifunction $T : X \longrightarrow Y$ is *a.l.s.c.* and each images $T(x)$ is convex. We begin with a generalization of Yannelis-Prabhakar's continuous selection theorem [15].

Theorem 2.1. Let X be paracompact, Y a normed linear space, and $T : X \longrightarrow Y$ be a multifunction. If there exists a multifunction $S : X \longrightarrow Y$ satisfying

(1) S is *a.l.s.c.*, and $S(x)$ is nonempty for each $x \in X$,

(2) $coS(x) \subset T(x)$ for each $x \in X$,

then for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection.

Proof : The proof is based on [5]. Since S is *a.l.s.c.*, coS is also *a.l.s.c.*. Thus, by [5, Theorem 2.4], coS admits a continuous ϵ-approximate selection f, and hence

$$f(x) \in B_\epsilon(coS(x)) \subset B_\epsilon(T(x)), \ \forall\ x \in X.$$

So, f is also a continuous ϵ-approximate selection of T. □

Corollary 2.2. Let X be paracompact, and Y be a normed linear space. If $T : X \longrightarrow Y$ is an *a.l.s.c.* multifunction such that every image $T(x)$ is nonempty and convex for each $x \in X$, then for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection.

For each $\epsilon > 0$ and a multifunction $S : X \longrightarrow Y$, we define

$$C_\epsilon(S) := \{f : X \longrightarrow Y \mid f \text{ is continuous, and } f(x) \in B_\epsilon(S(x)),\ \forall x \in X\}.$$

When S is *a.l.s.c.* with nonempty convex images, from Corollary 2.2, the set $C_\epsilon(S)$ is nonempty for each $\epsilon > 0$.

Proposition 2.3. For each $\epsilon > 0$, the multifunction $S_\epsilon : X \longrightarrow Y$, defined by

$$S_\epsilon(x) := \{f(x) \mid f \in C_\epsilon(S)\},\ \forall\ x \in X,$$

is *l.s.c.*. Moreover, if $S(x)$ is convex for each $x \in X$, then $S_\epsilon(x)$ is convex for each $x \in X$.

Proof : For each $x \in X$, and any open set G with $G \cap S_\epsilon(x) \neq \emptyset$, there exists $f \in C_\epsilon(S)$ such that $f(x) \in G$, which implies $x \in f^{-1}(G)$. Since f is continuous, $f^{-1}(G)$ is an open set, and hence there is a neighborhood N_x of x such that $x \in N_x \subset f^{-1}(G)$. Thus, for any $z \in N_x$, $f(z) \in G$, which implies

$$S_\epsilon(z) \cap G \neq \emptyset,\ \forall\ z \in N_x.$$

Moreover, for any $y_1, y_2 \in S_\epsilon(x)$, there exist $f_1, f_2 \in C_\epsilon(S)$ such that $f_1(x) = y_1, f_2(x) = y_2$. Now, for any $\lambda \in (0,1)$, it is clear that $\lambda f_1 + (1-\lambda) f_2$ is continuous, and since $S(z)$ is convex, the set $B_\epsilon(S(z))$ is also convex. Thus, we have

$$\lambda f_1(z) + (1-\lambda) f_2(z) \in B_\epsilon(S(z)),\ \forall z \in X,$$

it follows that

$$\lambda f_1 + (1-\lambda) f_2 \in C_\epsilon(S).$$

Therefore,

$$\lambda y_1 + (1-\lambda) y_2 = \lambda f_1(x) + (1-\lambda) f_2(x) \in S_\epsilon(x).$$

This completes the proof. □

Remark 2.4. From the above proof, we see that the set $C_\epsilon(S)$ is convex, when S is a convex-valued multifunction. Also, by the definitions of $C_\epsilon(S)$ and S_ϵ, we have

(1) $C_{\epsilon_1}(S) \subset C_{\epsilon_2}(S)$, if $\epsilon_1 \leq \epsilon_2$, which implies $S_{\epsilon_1}(x) \subset S_{\epsilon_2}(x),\ \forall x \in X$.

(2) $S_\epsilon(x) \subset B_\epsilon(S(x)),\ \forall x \in X$.

Proposition 2.5. Suppose that for each $x \in X$, there exists some $\eta := \eta(x) > 0$ such that $B_\eta(S(x))$ is compact. Then for any $\epsilon > 0$ and $x \in X$, there is $\delta := \delta(x, \epsilon) > 0$ such that $S_\delta(x) \subset intB_{\frac{\epsilon}{2}}(S_0(x))$, where

$$S_0(x) := \bigcap_{\epsilon > 0} S_\epsilon(x),\ \forall x \in X.$$

Proof : Assume NOT, then there exist $\epsilon_0 > 0$, and $x_0 \in X$ such that for any $\delta > 0$, we always have

$$S_\delta(x_0) \not\subset intB_{\frac{\epsilon_0}{2}}(S_0(x_0)).$$

For this x_0, there exists $\eta > 0$ such that $B_\eta(S(x_0))$ is compact. Now, we take a sequence $\{\delta_n\}_{n=1}^\infty$ with $\delta_1 = \eta$ and $\delta_n \downarrow 0$ as $n \to \infty$ such that

$$S_{\delta_n}(x_0) \not\subset intB_{\frac{\epsilon_0}{2}}(S_0(x_0)),$$

i.e., for each n there exists y_n such that

$$y_n \in S_{\delta_n}(x_0) \text{ but } \ y_n \notin intB_{\frac{\epsilon_0}{2}}(S_0(x_0)).$$

Since $\delta_n \downarrow 0$, we have $\{y_n\}_{n=1}^\infty \subset S_\eta(x_0)$. Since $B_\eta(S(x_0))$ is compact, there exists a subsequence of $\{y_n\}$ which converges to some $y_0 \in B_\eta(S(x_0))$. WLOG, we may assume that $y_n \to y_0$. For each n, taking $f_n \in C_{\delta_n}(S)$ such that $f_n(x_0) = y_n$, and defining

$$g_n(x) := f_n(x) + y_0 - y_n,\ \forall x \in X,$$

we then have $g_n(x_0) = y_0$. Thus, for any $\epsilon > 0$, we can choose n sufficiently large such that $\delta_n < \frac{\epsilon}{2}$ and $d(y_n, y_0) < \frac{\epsilon}{2}$. Therefore, by Lemma 1.2,

$$g_n(x) \in B_{\delta_n}(S(x)) + (y_0 - y_n) \subset B_{\frac{\epsilon}{2}}(S(x)) + B_{\frac{\epsilon}{2}}(0) \subset B_\epsilon(S(x)),\ \forall\ x \in X,$$

and $y_0 = g_n(x_0) \in S_\epsilon(x_0),\ \forall \epsilon > 0$, which implies $y_0 \in S_0(x_0)$. But $y_n \in (intB_{\frac{\epsilon_0}{2}}(S_0(x_0)))^C$ implies $y_0 \in (intB_{\frac{\epsilon_0}{2}}(S_0(x_0)))^C$; this contradicts with $y_0 \in S_0(x_0)$. Hence the proof is complete. □

Remark 2.6. Form the proof of Proposition 2.5, we also obtain that $S_0(x) \neq \emptyset$ for each $x \in X$.

We shall say that a multifunction $S : X \longrightarrow Y$ has the *equicontinuous property* (ECP), provided that for each $x \in X$ and $\epsilon > 0$, there exist $\sigma := \sigma(\epsilon) > 0$, and a neighborhood N_x of x, such that

(1) $d(f(z), f(x)) < \frac{\sigma}{2},\ \forall z \in N_x,\ \forall f \in C_{\frac{\sigma}{2}}(S)$;

(2) $diam\,(\bigcap_{z\in N_x} B_\sigma(S(z))) \leq \epsilon$.

Proposition 2.7. Suppose that for each $x \in X$, there is $\eta > 0$ such that $B_\eta(S(x))$ is compact. If S has the ECP, then for any $\epsilon > 0$,

$$S_{\frac{\sigma}{2}}(x) \subset intB_\epsilon(S_0(x)),\ \forall x \in X,$$

where $\sigma := \sigma(\frac{\epsilon}{2})$ is taken as in the definition of ECP.

Proof : For each $x \in X$, and any $y_1, y_2 \in S_{\frac{\sigma}{2}}(x)$, there exist $f_1, f_2 \in C_{\frac{\sigma}{2}}(S)$ such that $f_1(x) = y_1$ and $f_2(x) = y_2$. Thus, for $i = 1, 2$, we have

$$y_i = f_i(x) \in B_{\frac{\sigma}{2}}(f_i(z)) \subset B_{\frac{\sigma}{2}}(B_{\frac{\sigma}{2}}(S(z))) \subset B_\sigma(S(z)),\ \forall z \in N_x.$$

This yields

$$d(y_1, y_2) \leq diam(\bigcap_{z\in N_x} B_\sigma(S(z))) \leq \frac{\epsilon}{2}.$$

Applying Proposition 2.5, there exists $\delta > 0$ such that $S_\delta(x) \subset intB_{\frac{\epsilon}{2}}(S_0(x))$. If $\delta \geq \frac{\sigma}{2}$, it is clear that

$$S_{\frac{\sigma}{2}}(x) \subset S_\delta(x) \subset intB_{\frac{\epsilon}{2}}(S_0(x)) \subset intB_\epsilon(S_0(x)).$$

If $\delta < \frac{\sigma}{2}$, we shall claim that $S_{\frac{\sigma}{2}}(x) \subset B_{\frac{\epsilon}{2}}(S_\delta(x))$. Assume NOT, there exists $y_1 \in S_{\frac{\sigma}{2}}(x)$ but $y_1 \notin B_{\frac{\epsilon}{2}}(S_\delta(x))$, i.e., for any $y_2 \in S_\delta(x)$, $d(y_1, y_2) > \frac{\epsilon}{2}$. This is impossible because $y_2 \in S_\delta(x) \subset S_{\frac{\sigma}{2}}(x)$. Thus,

$$S_{\frac{\sigma}{2}}(x) \subset B_{\frac{\epsilon}{2}}(S_\delta(x)) \subset B_{\frac{\epsilon}{2}}(intB_{\frac{\epsilon}{2}}(S_0(x))) \subset intB_\epsilon(S_0(x)).$$

We complete the proof. □

Proposition 2.8. Under the same condition of Proposition 2.7, if $S(x)$ is convex for each $x \in X$, then the multifunction $S_0 : X \longrightarrow Y$ is *l.s.c.*, and $S_0(x)$ is nonempty and convex for each $x \in X$.

Proof : For any $\epsilon > 0$, by Proposition 2.7, we have $\sigma := \sigma(\frac{\epsilon}{4})$ such that

$$S_{\frac{\sigma}{2}}(x) \subset intB_{\frac{\epsilon}{2}}(S_0(x)), \ \forall x \in X.$$

Given any $y \in S_0(x)$, we have $y \in S_{\frac{\sigma}{2}}(x)$. Since $S_{\frac{\sigma}{2}}$ is *l.s.c.* at x, by Lemma 1.1, for $\frac{\epsilon}{2} > 0$, there exists a neighborhood V_x of x such that $y \in intB_{\frac{\epsilon}{2}}(S_{\frac{\sigma}{2}}(z)), \ \forall z \in V_x$. Thus,

$$y \in intB_{\frac{\epsilon}{2}}(intB_{\frac{\epsilon}{2}}(S_0(z))) \subset intB_{\epsilon}(S_0(z)), \ \forall z \in V_x.$$

This shows that S_0 is *l.s.c.*. Furthermore, since $S_\epsilon(x)$ is convex for each $x \in X$ and $\epsilon > 0$, it follows that $S_0(x)$ is also convex. □

Remark 2.9. When S_0 is *l.s.c.* and each image $S_0(x)$ is nonempty and convex, the multifunction $\overline{S_0} : X \longrightarrow Y$, defined by

$$\overline{S_0}(x) := cl(S_0(x)), \ \forall x \in X,$$

is also *l.s.c.*, and $\overline{S_0}(x)$ is nonempty, closed and convex for each $x \in X$.

At the end of this section, we conclude a main selection theorem as follows.

Theorem 2.10. Let X be paracompact, Y a Banach space, and $T : X \longrightarrow Y$ be a multifunction with nonempty closed images. If there exists an *a.l.s.c. ECP* multifunction $S : X \longrightarrow Y$ satisfying

(1) each $S(x)$ is nonempty and convex, and $S(x) \subset T(x), \ \forall x \in X$,

(2) for each $x \in X$, $B_\eta(S(x))$ is compact for some $\eta > 0$,

then T admits a continuous selection.

Proof : Consider the multifunction $\overline{S_0}$, since for each $x \in X$,

$$S_0(x) := \bigcap_{\epsilon > 0} B_\epsilon(S(x)) \subset B_\epsilon(S(x)) \subset B_\epsilon(T(x)), \ \forall \epsilon > 0,$$

it follows that

$$\overline{S_0}(x) = cl(S_0(x)) \subset cl(\bigcap_{\epsilon > 0} B_\epsilon(T(x)) = cl(clT(x)) = T(x).$$

Moreover, from Remark 2.9, $\overline{S_0}$ is a *l.s.c.* multifunction with nonempty closed convex images. By Michael's selection theorem [10], there is a continuous selection $f : X \longrightarrow Y$ for $\overline{S_0}$. This implies that f is also a selection for T. □

§3 Selection Theorems for C-set-valued Multifunctions.

For any set Z, let $\langle Z\rangle$ denote the collection of all nonempty finite subsets of Z. In a topological space Y, a mapping $C : \langle Y\rangle \longrightarrow Y$ is called a *C-structure* on Y, if it satisfies

(1) for each $A \in \langle Y\rangle$, $C(A)$ is nonempty and contractible;

(2) for any $A, B \in \langle Y\rangle$ with $A \subset B$, $C(A) \subset C(B)$.

In this event, (Y, C) is called a *C-space*, and a subset Z of Y is called a *C-set*, if $C(A) \subset Z$ for each $A \in \langle Z\rangle$. A C-space (Y, C) is called a *LC-metric space*, if Y is a metric space satisfying all open balls are C-sets, and $intB_\epsilon(Z)$ is a C-set for any $\epsilon > 0$, whenever Z is a C-set in Y. Such a notion has been investigated in [1,7,8,9]. For example, any normed linear space Y, together with the C-structure $C(A) = coA$, is a LC-metric space.

Lemma 3.1. In a LC-metric space (Y, C), each singleton is a C-set; moreover,

$$C(\{y\}) = \{y\}, \ \forall y \in Y.$$

Proof : Notice that for each $y \in Y$, $\{y\} = \bigcap_{\epsilon>0} intB_\epsilon(y)$. Since all open balls and their intersection are C-sets, it follows that $\{y\}$ is a C-set; and hence, $C(\{y\}) = \{y\}$. □

Lemma 3.2. In LC-metric space (Y, C), clZ is a C-set whenever Z is a C-set in Y.

Proof : For any C-set Z in Y,

$$clZ = \bigcap_{n=1}^{\infty} intB_{\frac{1}{n}}(Z)$$

and each $intB_{\frac{1}{n}}(Z)$ is a C-set, so is clZ. □

Without convexity, in 1995, H. Ben-El-Mechaiekh [1] proved the following:

Theorem 3.3. Let X be paracompact, and Y be a complete LC-metric space. If $T : X \longrightarrow Y$ is a *l.s.c.* multifunction such that every image $T(x)$ is a nonempty closed C-set for each $x \in X$, then T admits a continuous selection.

In 2006, H. Kim and S. Lee [9] proved that

Theorem 3.4. Let X be paracompact, and Y be a LC-metric space. If $T : X \longrightarrow Y$ is an *a.l.s.c.* multifunction such that every image $T(x)$ is a nonempty C-set for each $x \in X$, then for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection.

Using the same idea as Theorem 2.1, we have immediately

Corollary 3.5. Let X be paracompact, Y a LC-metric space, and $T : X \longrightarrow Y$ be a multifunction. If there exists an $a.l.s.c.$ multifunction $S : X \longrightarrow Y$ satisfying

(1) S is $a.l.s.c.$,

(2) each $S(x)$ is a nonempty C-set, and $S(x) \subset T(x)$ for each $x \in X$,

then for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection.

As the same process in Section 2, we have the following propositions.

Proposition 3.6. Let Y be a LC-metric space. For each $\epsilon > 0$, the multifunction $S_\epsilon : X \longrightarrow Y$, defined as in Proposition 2.3, is $l.s.c.$.

Proposition 3.7. Let (Y, C) be a complete LC-metric space. Suppose that for each $x \in X$, there is some $\eta > 0$ such that $B_\eta(S(x))$ is compact. Then for any $\epsilon > 0$ and $x \in X$, there is $\delta := \delta(x, \epsilon) > 0$ such that $S_\delta(x) \subset intB_{\frac{\epsilon}{2}}(S_0(x))$, where $S_0(x)$ is as in Proposition 2.5.

Proof : As the proof in Proposition 2.5, we can obtain a sequence $\{y_n\}$ satisfying

(1) $y_n \in S_{\delta_n}(x_0)$ and $y_n \notin intB_{\frac{\epsilon_0}{2}}(S_0(x_0))$;

(2) y_n converges to some point $y_0 \in B_\eta(S(x))$.

Then, for any $\epsilon > 0$, we choose n sufficiently large such that $\delta_n < \frac{\epsilon}{2}$ and $d(y_n, y_0) < \frac{\epsilon}{2}$. Since $y_n \in S_{\delta_n}(x_0)$, taking $f_n \in C_{\delta_n}(S)$ such that $f_n(x_0) = y_n$, we have

$$f_n(x) \in B_{\delta_n}(S(x)) \subset B_{\frac{\epsilon}{2}}(S(x)),\ \forall x \in X.$$

Now, we define a multifunction $F : X \longrightarrow Y$ by

$$F(x) = \begin{cases} B_{\frac{\epsilon}{2}}(f_n(x)) & , \text{if } x \neq x_0, \\ \{y_0\} & , \text{if } x = x_0. \end{cases}$$

It is easy to check that F is $l.s.c.$, and each $F(x)$ is a C-set by Lemma 3.1 and 3.2. Therefore, F admits a continuous selection g_n, and we have

$$g_n(x) \in F(x) \subset B_{\frac{\epsilon}{2}}(f_n(x)) \subset B_\epsilon(S(x)),\ \forall x \in X,$$

which implies $g_n \in C_\epsilon(S)$, and hence

$$y_0 = g_n(x_0) \in S_\epsilon(x_0),\ \forall \epsilon > 0.$$

It follows that $y_0 \in S_0(x_0)$. But $y_n \in (intB_{\frac{\epsilon_0}{2}}(S_0(x_0)))^C$ implies $y_0 \in (intB_{\frac{\epsilon_0}{2}}(S_0(x_0)))^C$, which contradicts with $y_0 \in S_0(x_0)$, and hence we complete the proof. □

We observe that Proposition 2.7 is also valid if (Y, C) is a complete LC-metric space; however, Proposition 2.8 should be modified into a weaker form as follows.

Proposition 3.8. Under the same condition of Proposition 3.7, if S has the ECP and $S(x)$ is nonempty, closed for each $x \in X$, then the multifunction $S_0 : X \longrightarrow Y$ is *l.s.c.*, and $S_0(x) \subset S(x)$ for each $x \in X$.

Given an *a.l.s.c.* multifunction $S : X \longrightarrow Y$, we define

$$M := \{L : X \longrightarrow Y \mid L \text{ is } l.s.c., \text{ and } L(z) \subset S(z), \forall z \in X\}.$$

Under the condition of Proposition 3.8, $S_0 \in M$, and hence $M \neq \emptyset$. Now, we can define a partial order $\preceq$ on M by

$$L_1 \preceq L_2 \;\Leftrightarrow\; L_1(z) \subset L_2(z),\ \forall z \in X,$$

then $(M, \preceq)$ forms a partial order set. Thus, given any chain $\mathcal{C}$ in M, if we set the multifunction $\mathcal{L} : X \longrightarrow Y$ by

$$\mathcal{L}(z) := \bigcup_{L \in \mathcal{C}} L(z),$$

then $\mathcal{L}$ is *l.s.c.*, by Lemma 1.4(2), and hence is an upper bound of the chain $\mathcal{C}$. Therefore, by Zorn's Lemma, M has a maximal element. We may take one maximal element $S_0 \in M$, which will be used later.

For a multifunction $S : X \longrightarrow Y$, we say that S has the *one point extension property*, provided that for each *l.s.c.* multifunction $L : X \longrightarrow Y$ with $L(z) \subset S(z)$, $\forall z \in X$, and for each $(x, a) \in G(S) \setminus G(L)$, there is a *l.s.c* multifunction $L^* : X \longrightarrow Y$ such that $(x, a) \in G(L^*)$ and

$$L(z) \subset L^*(z) \subset S(z),\ \forall z \in X.$$

For example, let Y be a complete LC-metric space, and $S : X \longrightarrow Y$ be a *l.s.c.* multifunction with closed C-set images. For each *l.s.c.* multifunction $L : X \longrightarrow Y$ with $L(z) \subset S(z)$, $\forall z \in X$, and for each $(x, a) \in G(S) \setminus G(L)$, we define $S^* : X \longrightarrow Y$ by

$$S^*(z) = \begin{cases} S(z) & , \text{if } z \neq x, \\ \{a\} & , \text{if } z = x. \end{cases}$$

Then S^* is also a *l.s.c.* multifunction with closed C-set images. By Theorem 3.3, S^* admits a continuous selection, say l. Moreover, since L is *l.s.c.* and l is a continuous single-valued function, the multifunction $L^* : X \longrightarrow Y$, defined by

$$L^*(z) := L(z) \cup l(z),\ \forall z \in X,$$

is also *l.s.c.*. Hence, we have $(x, a) \in G(L^*)$ and $L(z) \subset L^*(z) \subset S(z)$, $\forall z \in X$. Thus, this multifunction S has the *one point extension property*.

Proposition 3.9. Under the same condition of Proposition 3.8, if S has the *one point extension property*, then $S_0(x)$ is a closed C-set for each $x \in X$, where $S_0 : X \longrightarrow Y$ is taken a maximal element in M; that is, S_0 is $l.s.c.$, and $S_0(z) \subset S(z),\ \forall z \in X$.

Proof : Assume NOT, there is $x \in X$ such that $S_0(x)$ is not a C-set, i.e., $C(A) \not\subset S_0(x)$ for some $A \in \langle S_0(x) \rangle$. So, there exists an $a \in C(A)$ but $a \notin S_0(x)$. Note that

$$A \in \langle S_0(x) \rangle \subset \langle S(x) \rangle,$$

and $S(x)$ is a C-set; it follows that $a \in S(x)$. Thus, $(x, a) \in G(S) \setminus G(S_0)$. Since S has the *one point extension property*, there is a $l.s.c$ multifunction $S_0^* : X \longrightarrow Y$ such that $(x, a) \in G(S_0^*)$, and

$$S_0(z) \subset S_0^*(z) \subset S(z),\ \forall z \in X.$$

This contradicts to the maximality of S_0. Hence, $S_0(x)$ is a C-set for each $x \in X$. Moreover, we have known in Remark 2.6 that $\overline{S_0}$ is $l.s.c.$, and for each $z \in X$,

$$S_0(z) \subset clS_0(z) = \overline{S_0}(z).$$

Thus, by the maximality of S_0, we obtain $S_0(z) = cl(S_0(z)) = \overline{S_0}(z)$. This yields that S_0 has the closed images. □

Theorem 3.10. Let X be paracompact, Y a complete LC-metric space, and $T : X \longrightarrow Y$ be a multifunction. If there exists an $a.l.s.c.$ ECP multifunction $S : X \longrightarrow Y$ satisfying

(1) each $S(x)$ is a nonempty closed C-set, and $S(x) \subset T(x),\ \forall\ x \in X$,

(2) for each $x \in X$, $B_\eta(S(x))$ is compact for some $\eta > 0$,

(3) S has the *one point extension property*,

then T admits a continuous selection.

Proof. Notice that the multifunction S_0 defined in Proposition 3.9 is $l.s.c$, with nonempty closed C-set images. By Theorem 3.3, S_0 admits a continuous selection f, and hence

$$f(x) \in S_0(x) \subset S(x) \subset T(x),\ \forall x \in X.$$

We complete the proof. □

§4 Modified Continuous Selection Theorems.

In this section, we shall modify our selection theorems by adjusting a little closed set Z with $dim_X Z \leq 0$. Here $dim_X Z \leq 0$ means that $dim E \leq 0$ for every set $E \subset Z$, which is closed in X (where $dimE$ denotes the *covering dimension* of E). A fundamental theorem is the following, due to Michael and Pixley [11].

Theorem 4.1. Let X be paracompact, Y a Banach space, and Z be a subset of X, with $dim_X Z \leq 0$. If $T : X \longrightarrow Y$ is a *l.s.c.* multifunction having nonempty closed images, and each $T(x)$ is convex for $x \in X \setminus Z$, then T admits a continuous selection.

Theorem 4.2. Let X be paracompact, Y a normed linear space, and Z be a closed subset of X with $dim_X Z \leq 0$. If $S : X \longrightarrow Y$ is an *a.l.s.c.* multifunction such that $S(x)$ is convex for all $x \in X \setminus Z$, then for any $\epsilon > 0$, S has a continuous ϵ-approximate selection.

Proof : Let $\epsilon > 0$ be arbitrary but fixed, and define $U_y := \{x \in X \mid y \in B_\epsilon(S(x))\}$. By Lemma 1.3, the collection $\{intU_y \mid y \in Y\}$ forms an open cover of X. Since X is paracompact, there exists a locally finite open cover $\{V_y \mid y \in Y\}$ of X with

$$V_y \subset clV_y \subset intU_y \subset U_y, \forall y \in Y.$$

For each $x \in X$, let

$$F_x := \{y \in Y \mid x \in clV_y\}.$$

Then F_x is finite and $F_x \subset B_\epsilon(S(x))$. Let $H = X \setminus Z$ and for each $h \in H$, we define

$$G_h := int\{x \in X \mid coF_h \subset B_\epsilon(S(x))\} \setminus \bigcup_{y \notin F_h} clV_y.$$

<u>Claim 1</u>: For each $h \in H$, $h \in G_h$.

For each $h \in H$, F_h is finite, say $F_h = \{y_1, y_2, \cdots, y_k\}$. For each $i = 1, 2, \cdots, k$,

$$h \in clV_{y_i} \subset intU_{y_i}.$$

So, there exists a neighborhood N_i of h in $X \setminus Z$ such that $h \in N_i \subset U_{y_i}$, which implies

$$y_i \in B_\epsilon(S(p)), \forall p \in N_i.$$

Then $W_h := \bigcap_{i=1}^k N_i$ is a neighborhood of h, and for each $p \in W_h$, $y_i \in B_\epsilon(S(p))$ for each $i = 1, 2, 3, \cdots, k$, and hence $F_h \subset B_\epsilon(S(p))$. Since each $S(p)$ is convex,

$$coF_h \subset B_\epsilon(S(p)),\ \forall p \in W_h.$$

Thus, $h \in W_h \subset \{x \in X \mid coF_h \subset B_\epsilon(S(x))\}$. Moreover, if $h \in \bigcup_{y \notin F_h} clV_y$, there exists $y \notin F_h$ such that $h \in clV_y$, which implies $y \in F_h$. It is impossible. Hence, we have $h \notin \bigcup_{y \notin F_h} clV_y$, and this completes the claim.

<u>Claim 2</u>: $F_x \subset F_h$ for all $x \in G_h$.

For each $x \in G_h$ and any $y \in F_x$, we have $x \in clV_y$ and $x \notin \bigcup_{y \notin F_h} clV_y$, which implies $y \in F_h$. This yields that $F_x \subset F_h$, $\forall x \in G_h$.

Claim 3: G_h is open for each $h \in H$.

Note that G_h can be written as the following intersection

$$\bigcap_{y \notin F_h} (U \setminus clV_y),$$

where $U := int\{x \in X \mid coF_h \subset B_\epsilon(S(x))\}$ is an open set. Since $\{V_y \mid y \in Y\}$ is locally finite, each $x \in G_h$ has a neighborhood $\omega_x \subset U$ such that $\omega_x \cap clV_y \neq \emptyset$ for finitely many $y \in Y$. The finite intersection $\bigcap_{y \notin F_h}[\omega_x \cap (X \setminus clV_y)]$ is an open neighborhood of x contained in G_h. This shows that G_h is open.

Let $G := \bigcup_{h \in H} G_h$, and let $E := X \setminus G$. Then E is closed in X and $E \subset Z$, so $\dim_X E \leq 0$. The relatively open cover $\{V_y \cap E \mid y \in Y\}$ of E has a relatively open disjoint refinement $\{D_y \mid y \in Y\}$.

For each $y \in Y$, we let $W_y := V_y \cap (D_y \cup G)$. Then $\{W_y \mid y \in Y\}$ is a locally finite open cover of X, and thus there is a partition of unity $\{p_y \mid y \in Y\}$ subordinated to $\{W_y \mid y \in Y\}$; that is, each $p_y : X \longrightarrow [0,1]$ is continuous such that

(1) $p_y(z) = 0$, if $z \notin W_y$,

(2) $\sum_{y \in Y}(p_y(z)) = 1,\ \forall z \in X$.

Define $f : X \longrightarrow Y$ by

$$f(x) := \sum_{y \in Y}(p_y(x))y,\ \forall x \in X.$$

Clearly f is continuous, so we need only to check that $f(x) \in B_\epsilon(S(x))$ for all $x \in X$.

If $x \in E$, then $f(x) = y \in B_\epsilon(S(x))$ for the unique $y \in Y$ such that $x \in D_y$. If $x \in G$, then $x \in G_h$ for some $h \in H$, so

$$f(x) \in coF_x \subset coF_h \subset B_\epsilon(S(x)).$$

This shows that f is a continuous ϵ-approximate selection of S. □

Again, using the same idea to prove Corollary 2.2, we have a parallel result as follows.

Corollary 4.3. Let X be paracompact, Y a normed linear space, Z a closed subset of X, with $dim_X Z \leq 0$, and $T : X \longrightarrow Y$ be a multifunction. If there exists a multifunction $S : X \longrightarrow Y$ satisfying

(1) S is *a.l.s.c.*, and $S(x)$ is nonempty for each $x \in X$,

(2) $coS(x) \subset T(x)$ for each $x \in X \setminus Z$, and $S(x) \subset T(x)$ for each $x \in Z$,

then for each $\epsilon > 0$, T admits a continuous $\epsilon-$approximate selection.

Now, we redefine S_ϵ as follows:

$$S_\epsilon(x) := \begin{cases} \{f(x) \mid f \in C'_\epsilon(S)\}, & \forall\, x \in X \setminus Z, \\ \{f(x) \mid f \in C_\epsilon(S)\}, & \forall\, x \in Z. \end{cases}$$

Here $C_\epsilon(S)$ is defined as before in Section 2, and $C'_\epsilon(S)$ is defined by

$$C'_\epsilon(S) := \{f : X \longrightarrow Y \mid f \text{ is continuous, and } f(x) \in B_\epsilon(S(x)),\ \forall x \in X \setminus Z\}.$$

Thus, we have some basic facts:

(1) For $\epsilon_1 \leq \epsilon_2$, $C_{\epsilon_1}(S) \subset C_{\epsilon_2}(S)$ and $C'_{\epsilon_1}(S) \subset C'_{\epsilon_2}(S)$, which implies $S_{\epsilon_1}(x) \subset S_{\epsilon_2}(x)$, for each $x \in X$.

(2) $S_\epsilon(x) \subset B_\epsilon(S(x)),\ \forall x \in X$.

(3) For any $\epsilon > 0$, $C_\epsilon(S) \subset C'_\epsilon(S)$.

As in Section 2, we have a series of parallel results. Proposition 2.5 and Proposition 2.7 still hold; however, Proposition 2.3 and Proposition 2.8 should be modified as Proposition 4.4 and Proposition 4.5, respectively.

Proposition 4.4. For each $\epsilon > 0$, S_ϵ is *l.s.c.* and $S_\epsilon(x)$ is convex for each $x \in X \setminus Z$.

Proof : For each $\epsilon > 0$, $x \in X$, and given any open set G with $G \cap S_\epsilon(x) \neq \emptyset$.

(i) If $x \in X \setminus Z$, there exists $f \in C'_\epsilon(S)$ such that $f(x) \in G$, which implies $x \in f^{-1}(G)$. Note that $f^{-1}(G)$ is an open set since f is continuous, there exists a neighborhood $N_x \subset X \setminus Z$ of x such that for any $z \in N_x$, $f(z) \in G$. This yields $S_\epsilon(z) \cap G \neq \emptyset$.

(ii) If $x \in Z$, there exists $f \in C_\epsilon(S)$ such that $f(x) \in G$, and there exists a neighborhood N_x of x such that for any $z \in N_x$, $f(z) \in G$, which implies $S_\epsilon(z) \cap G \neq \emptyset$ since $C_\epsilon(S) \subset C'_\epsilon(S)$.

Now, for each $x \in X \setminus Z$, for any $y_1, y_2 \in S_\epsilon(x)$, there exist $f_1, f_2 \in C'_\epsilon(S)$ such that $f_1(x) = y_1, f_2(x) = y_2$. Given $\lambda \in (0, 1)$, it is clear that $\lambda f_1 + (1-\lambda) f_2$ is continuous, and since $B_\epsilon(S(z))$ is convex for each $x \in X \setminus Z$, we have

$$\lambda f_1(z) + (1-\lambda) f_2(z) \in B_\epsilon(S(z)),\ \forall z \in X \setminus Z.$$

Thus,

$$\lambda f_1 + (1-\lambda) f_2 \in C'_\epsilon(S),$$

and hence

$$\lambda y_1 + (1-\lambda)y_2 = \lambda f_1(x) + (1-\lambda)f_2(x) \in S_\epsilon(x).$$

This completes the proof. $\square$

Proposition 4.5. The multifunction S_0 is *l.s.c.* and $S_0(x)$ is convex for all $x \in X \setminus Z$.

Finally, we can generalize Theorem 2.10 and Theorem 3.10 as follows.

Theorem 4.6. Let X be paracompact, Y a Banach space, and Z be a closed subset of X, with $dim_X Z \leq 0$. If $T : X \longrightarrow Y$ is an *a.l.s.c.* *ECP* multifunction with nonempty closed images satisfying

(1) $T(x)$ is convex for each $x \in X \setminus Z$,

(2) for each $x \in X$, $B_\eta(T(x))$ is compact for some $\eta > 0$,

then T admits a continuous selection.

Proof : We take the special case where $T = S$. It is known that $\overline{S_0}$ is a *l.s.c.* multifunction with closed images and $\overline{S_0}(x)$ is convex for each $x \in X \setminus Z$. By Theorem 4.1, $\overline{S_0}$ admits a continuous selection, and hence T admits a continuous selection. $\square$

Theorem 4.7. Let X be paracompact, Y a complete LC-metric space, and Z be a closed subset of X, with $dim_X Z \leq 0$. If $T : X \longrightarrow Y$ is an *a.l.s.c.* *ECP* multifunction with nonempty closed images satisfying

(1) $T(x)$ is a C-set for each $x \in X \setminus Z$,

(2) for each $x \in X$, $B_\eta(T(x))$ is compact, for some $\eta > 0$,

(3) T has the *one point extension property*,

then T admits a continuous selection.

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