# 2 Preliminary

We study the system

$$\begin{cases} v' = \kappa^2 \\ \kappa' = -\kappa v - \omega \end{cases} \tag{2.1}$$

with three different cases $\omega > 0$, $\omega < 0$, and $\omega = 0$.

## 2.1 The case for $\omega > 0$

We define

$$\begin{aligned} \mathcal{D}_1 &= \{(v,\kappa) | \kappa > 0, -\kappa v - \omega > 0\} \\ \mathcal{D}_2 &= \{(v,\kappa) | \kappa > 0, -\kappa v - \omega < 0\} \\ \mathcal{B}_1 &= \{(v,\kappa) | \kappa < 0, -\kappa v - \omega < 0\} \\ \mathcal{B}_2 &= \{(v,\kappa) | \kappa < 0, -\kappa v - \omega > 0\} \end{aligned}$$

Note that there is no equilibrium of the system (2.1).

**Lemma 2.1** *Let $\omega > 0$. The following statements hold.*
*(1) $v' > 0$ for all $(v,\kappa)$.*
*(2) $\kappa' > 0$ for $(v,\kappa) \in \mathcal{D}_1 \cup \mathcal{B}_2$.*
*(3) $\kappa' < 0$ for $(v,\kappa) \in \mathcal{D}_2 \cup \mathcal{B}_1$.*
*(4) The set $\{\kappa \leq 0\}$ and $\overline{\mathcal{B}_2}$ are invariant regions for the system (2.1).*

*Proof.* It is easy to prove the lemma by the phase plane analysis. □

The local existence and uniqueness of solutions of the system (2.1) with a given initial condition is trivial. In the next lemma, we claim that the solutions of the system (2.1) is globally defined.

**Lemma 2.2** *Let $(v,\kappa)$ be the local solution of the system (2.1) with the initial condition $(v(0),\kappa(0)) = (0,\kappa_0)$. Then $(v,\kappa)$ is globally defined such that $\kappa(s)$ is bounded.*

*Proof.* We note that $v(s) := \int_0^s \kappa^2(\xi) d\xi$ is a positive strictly increasing function for $s > 0$, since $\kappa(s)$ can only be zero at most once by Lemma 2.1. First we claim $\kappa(s)$ is bounded above and eventually negative. If $\kappa_0 \leq 0$ then $\kappa(s) \leq 0$ by Lemma 2.1. Suppose that $\kappa_0 > 0$. If $\kappa(s) \geq 0$ then $\kappa' = -\kappa v - \omega \leq -\omega < 0$ and so $\kappa(s) \leq \kappa_0$. Hence $\kappa(s)$ is bounded above. Suppose that $\kappa(s) > 0$ for all $s \geq 0$. Then $\kappa(s) \leq \kappa_0 - \omega s \to -\infty$ as $s \to +\infty$, a contradiction. Therefore there exists $s_0 \in (0,+\infty)$ such that $\kappa(s_0) = 0$ and $\kappa(s) \leq 0$ for all $s \in [s_0,+\infty)$.

Now we claim that $\kappa$ is bounded below. Without loss of generality, we may assume that $\kappa(s) \leq 0$ for all $\kappa(s)$ exists. Note that $\kappa'(0) < 0$. If there exists $s_1 \in (0,+\infty)$ such that $\kappa'(s_1) = 0$, then $\kappa'(s) > 0$ for $s > s_1$, by Lemma 2.1. Hence $\kappa(s) \geq \kappa(s_1)$. Suppose that $\kappa'(s) < 0$ as long as $\kappa(s)$ exists. If $\kappa$ is unbounded then $\kappa(s) \to -\infty$ as $s \to s_2$ for some $s_2 \in (0,+\infty]$. Since $v(s) > 0$ and $v'(s) > 0$, $\kappa'(s) = -\kappa(s)v(s) - \omega \to +\infty$ as $s \to s_2$, a contradiction. Therefore $\kappa$ is bounded and hence $v(s)$ is finite if $s$ is finite. □

**Lemma 2.3** *Let $(v(s), \kappa(s))$ be the solution of (2.1) with the initial value $(0, \kappa_0)$ where $\kappa_0 \leq 0$. Then there exists $s_1 \in (0, +\infty)$ such that $\kappa'(s_1) = 0$.*

*Proof.* Suppose that $\kappa'(s) < 0$ for all $s \geq 0$, that is, $(v(s), \kappa(s))$ stays in $\mathcal{B}_1$ for all $s \geq 0$. By Lemma 2.2, $\kappa(s) \to c$ as $s \to +\infty$ for some finite $c$. We claim that $v(s) \to +\infty$ as $s \to +\infty$. Suppose that $v(s) \to d$ as $s \to +\infty$ where $d$ is finite. Then there exists a sequence $\{s_n\}$ such that $s_n \to \infty$, $\kappa'(s_n) \to 0$, and $v'(s_n) \to 0$ as $n \to +\infty$. By (2.1), we have

$$\begin{cases} v'(s_n) = \kappa^2(s_n) \\ \kappa'(s_n) = -\kappa(s_n)v(s_n) - \omega, \end{cases}$$

and, by letting $n \to +\infty$, we obtain that

$$\begin{cases} 0 = c^2 \\ 0 = -cd - \omega. \end{cases}$$

Hence $(d, c)$ is an equilibrium of the system (2.1), a contradiction. Therefore $v(s) \to +\infty$ and so $\kappa(s) \to 0$. But $\kappa(s) \leq 0$ and $\kappa'(s) < 0$ for all $s \in [0, +\infty)$, a contradiction. Therefore, there exists $s_1 \in (0, +\infty)$ such that $\kappa'(s_1) = 0$. This proves the lemma. □

**Lemma 2.4** *Let $(v(s), \kappa(s))$ be the solution of (2.1) with the initial value $(0, \kappa_0)$ where $\kappa_0 > 0$. Then there exist $s_1 \in (0, +\infty)$ such that $\kappa(s_1) = 0$ and $s_2 \in (s_1, +\infty)$ such that $\kappa'(s_2) = 0$.*

*Proof.* By the proof of Lemma 2.2 there exists $s_1 \in (0, +\infty)$ such that $\kappa(s_1) = 0$ and $\kappa(s) \leq 0$ for all $s \in (s_1, +\infty)$. By the proof of Lemma 2.3 there exists $s_2 \in (s_1, +\infty)$ such that $\kappa'(s_2) = 0$. Therefore the lemma holds. □

Then we have the following asymptotically behavior.

**Proposition 2.5** *Let $(v(s), \kappa(s))$ be the solution of (2.1) with the initial value $(0, \kappa_0)$. Then $\lim_{s \to +\infty} v(s) = +\infty$ and $\lim_{s \to +\infty} \kappa(s) = 0$.*

*Proof.* By the previous lemmas, there exists $s_0 > 0$ such that $\kappa'(s_0) = 0$ and $\kappa'(s) > 0$ for $s \in (s_0, +\infty)$. Then $\kappa(s) \to c$ as $s \to +\infty$ for some $c \in (-\infty, 0]$. Since there is no equilibrium of the system (2.1), $v(s) \to +\infty$ as $s \to +\infty$.

We claim that $c = 0$. Since $c$ is finite, we can take a sequence $\{s_n\}$ such that $s_n \to \infty$ and $\kappa'(s_n) \to 0$ as $n \to \infty$. If $c < 0$, then by letting $n \to \infty$ in

$$\frac{\kappa'(s_n)}{\kappa(s_n)} = -v(s_n) - \frac{\omega}{\kappa(s_n)},$$

we reach a contradiction. Thus $c = 0$ and the proposition is proved. □

## 2.2 The case for $\omega < 0$

Note that there is also no equilibrium of the system (2.1). Similarly, we define

$$\begin{aligned}\mathcal{D}_1 &= \{(v,\kappa)|\kappa > 0, -\kappa v - \omega > 0\}\\ \mathcal{D}_2 &= \{(v,\kappa)|\kappa > 0, -\kappa v - \omega < 0\}\\ \mathcal{B}_1 &= \{(v,\kappa)|\kappa < 0, -\kappa v - \omega < 0\}\\ \mathcal{B}_2 &= \{(v,\kappa)|\kappa < 0, -\kappa v - \omega > 0\}\end{aligned}$$

Then, by phase plane analysis, we have the following lemma.

**Lemma 2.6** *The following statements hold.*
*(1) $v' > 0$ for all $(v,\kappa)$.*
*(2) $\kappa' > 0$ for $(v,\kappa) \in \mathcal{D}_1 \cup \mathcal{B}_2$.*
*(3) $\kappa' < 0$ for $(v,\kappa) \in \mathcal{D}_2 \cup \mathcal{B}_1$.*
*(4) The set $\{\kappa \geq 0\}$ and $\overline{\mathcal{D}_2}$ are invariant regions for the system (2.1).*

We state the following lemmas without proof, since their proofs are similar to those of the case $\omega > 0$.

**Lemma 2.7** *Let $(v,\kappa)$ be the local solution of the system (2.1) with the initial condition $(v(0),\kappa(0)) = (0,\kappa_0)$. Then $(v,\kappa)$ is globally defined such that $\kappa(s)$ is bounded.*

**Lemma 2.8** *Let $(v(s),\kappa(s))$ be the solution of (2.1) with the initial value $(0,\kappa_0)$ where $\kappa_0 > 0$. Then there exists $s_1 \in (0,+\infty)$ such that $\kappa'(s_1) = 0$.*

**Lemma 2.9** *Let $(v(s),\kappa(s))$ be the solution of (2.1) with the initial value $(0,\kappa_0)$ where $\kappa_0 < 0$. Then there exists $s_1 \in (0,+\infty)$ such that $\kappa(s_1) = 0$ and there exists $s_2 \in (s_1,+\infty)$ such that $\kappa'(s_2) = 0$.*

**Proposition 2.10** *Let $(v(s),\kappa(s))$ be the solution of (2.1) with the initial value $(0,\kappa_0)$. Then $\lim_{s\to+\infty} v(s) = +\infty$ and $\lim_{s\to+\infty} \kappa(s) = 0$.*

## 2.3 The case for $\omega = 0$

In this case, the system (2.1) reads as

$$\begin{cases} v' = \kappa^2 \\ \kappa' = -\kappa v. \end{cases} \tag{2.2}$$

We define the function

$$E[s] := \frac{1}{2}v^2(s) + \frac{1}{2}\kappa^2(s). \tag{2.3}$$

First, we note that $(v_0, 0)$ is an equilibrium of the system (2.2) for any $v_0 \in (-\infty,+\infty)$. Note that $\kappa' > 0$ if $(v,\kappa) \in \{v\kappa < 0\}$, $\kappa' < 0$ if $(v,\kappa) \in \{v\kappa > 0\}$, and $v' > 0$ for all $(v,\kappa)$ with $\kappa \neq 0$.

**Proposition 2.11** *Let $(v, \kappa)$ be the local solution of the system (2.2) with the initial condition $(v(0), \kappa(0)) = (0, \kappa_0)$. Then $(v, \kappa)$ is globally defined such that $|v(s)| \leq |\kappa_0|$ and $|\kappa(s)| \leq |\kappa_0|$ for all $s \geq 0$ and $(v, \kappa) \to (|\kappa_0|, 0)$ as $s \to +\infty$.*

*Proof.* By (2.3) and (2.2), $E'[s] = 0$ for all $s \geq 0$, that is, $E[s]$ is a constant function. Since $E[0] = \frac{1}{2}\kappa_0^2$, $E[s] := \frac{1}{2}v^2(s) + \frac{1}{2}\kappa^2(s) = \frac{1}{2}\kappa_0^2$ as long as $(v, \kappa)$ exists. Hence $(v, \kappa)$ is globally defined such that $|v(s)| \leq |\kappa_0|$ and $|\kappa(s)| \leq |\kappa_0|$ for all $s \geq 0$.

Now we claim that $\kappa(s) \to 0$ as $s \to +\infty$ and hence $v(s) \to |\kappa_0|$ as $s \to +\infty$. The case $\kappa_0 = 0$ is trivial. Note that $\kappa(s)$ does not change sign for all $s \geq 0$, since all of the points of $\{\kappa = 0\}$ are fixed. If $\kappa_0 > 0$ then $\kappa'(s) = -\kappa(s)v(s) < 0$ for all $s > 0$. Hence the limit $c := \lim_{s \to +\infty} \kappa(s)$ exists for some $c \geq 0$. Since $c$ is finite, we can find a sequence $s_n \to \infty$ such that $\kappa'(s_n) \to 0$ as $n \to \infty$. If $c > 0$, then by letting $n \to \infty$ in $\frac{\kappa'(s_n)}{\kappa(s_n)} = -v(s_n)$ we obtain that $v(s_n) \to 0$. But, $v$ is a positive monotone increasing function, a contradiction. Therefore, $(v, \kappa) \to (|\kappa_0|, 0)$ as $s \to \infty$ when $\kappa_0 > 0$.

The proof for the case $\kappa_0 < 0$ is similar. This completes the proof. $\square$