A Calculation of SU(3) Clebsch-Gordan Coefficients for Bottom Meson Decays*

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Abstract

The Clebsch-Gordan coefficients for the product of SU(3) representations $3^* \otimes 15$, required for the calculation of the SU(3) amplitudes of bottom meson decays, are obtained by adopting the phase convention of Baird and Biedenharn.

I. INTRODUCTION

The recent discovery[1] of three extremely narrow states Υ (9, 4), Υ'(10.0), Υ'' (10.3) suggests the existence of a new quantum number called "bottom." Consequently, there should also be new flavored particles, called "bottom particles," which carry this new quantum number. That this is indeed the case is evidenced by the discovery[2] of a fourth resonance Υ''' (10.5), which has a broad width of about 20 MeV. The broad width of Υ''' suggests that Υ''' decays directly into $B\overline{B}$ pairs, each carrying a single bottom quark. We are thus provided with a source of B mesons whose decay we can observe.

A study of the decays of bottom mesons may provide us with useful information on the nature of weak interactions. However, the nonleptonic decays are complicated by the interplay of weak and strong interactions. At present, a reliable dynamical calculation of the QCD corrected decay rate is still lacking. On the other hand, the symmetry approach is expected to provide a reliable framework for systematic study on nonleptonic decays[3,4]. Since $m_b >> m_c$ the

SU(3) symmetry relations for B decays will hold much better than charmed meson decays. To obtain the SU(3) amplitudes for the decays of bottom mesons into two pseudoscalars, we need the Clebsch-Gordan(C-G) coefficients for the following products of SU(3) representations: $1 \otimes 8$, $1 \otimes 3$, $1 \otimes 15$, $1 \otimes 6^*$, $1 \otimes 3^*$, $1 \otimes 6$, $3^* \otimes 8$, $3^* \otimes 3$, $3^* \otimes 15$, $3^* \otimes 6^*$, $3^* \otimes 3^*$, $3^* \otimes 6^*$. All the C-G coefficients for the above products, except $3^* \otimes 15$, are given by Rabl, Campbell, Jr., and Wali[5]. The C-G coefficients for $3^* \otimes 15$ was given previously by C. K. Chew[6]. Unfortunately, they adopt the phase convention of de Swart[7]. Although the phase convention of de Swart is widely used in particle physics, it is not the one which can be easily generalized to SU(4). Since $m_b >> m_c - m_q (q = u, d, s)$, the SU(4) symmetry relations have been speculated to be good for B meson decays. The purpose of this paper is to present tables of the SU(3) C-G coefficients of $3^* \otimes 15$ by adopting the phase convention of Baird and Biedenharn[8] such that a generalization to SU(4) may be easily carried out.

II. REPRESENTATION CONTENT OF THE WEAK-INTERACTION HAMILTONIAN

The part of the effective nonleptonic Hamiltonian responsible for bottom meson decays can be written as[4]

$$\begin{aligned} H(\Delta \widetilde{B} = 1) = \frac{G'}{\sqrt{2}} [& V_{13} V_{11}^* \{\bar{u}b, \bar{d}u\} + V_{13} V_{12}^* \{\bar{u}b, \bar{s}u\} \\ & + V_{13} V_{21}^* \{\bar{u}b, \bar{d}c\} + V_{13} V_{22}^* \{\bar{u}b, \bar{s}c\} \\ & + V_{23} V_{11}^* \{\bar{c}b, \bar{d}u\} + V_{23} V_{12}^* \{\bar{c}b, \bar{s}u\} \\ & + V_{23} V_{21}^* \{\bar{c}b, \bar{d}c\} + V_{23} V_{22}^* \{\bar{c}b, \bar{s}c\}] \\ = H_1 + & \ldots\ldots\ldots\ldots + H_8 \end{aligned} \tag{2.1}$$

where the space-time structure and the color index are omitted for brevity. And, V_{ij} (i = 1, 2, 3) are elements of the unitary 3 x 3 mixing matrix V, which is usually parametrized in the special form

$$V = \begin{pmatrix} c_1 & c_3 s_1 & s_1 s_3 \\ -c_2 s_1 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + c_3 s_2 e^{i\delta} \\ s_1 s_2 & -c_1 c_3 s_2 - c_2 s_3 e^{i\delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} \tag{2.2}$$

The SU(3) representation content of the effective nonleptonic Hamiltonian can be easily obtained by noting that c, b, $\bar{c}$, $\bar{b}$ in Eq. (2.1) are singlet operators, while the quark operators ($\bar{u}$, $\bar{d}$, $\bar{s}$) and (u, d, s) transform as 3 and 3*, respectively. Thus, H ($\Delta \tilde{B}$ = 1) transform under SU(3) as 8, 3, 15, 6*, 3* and 6. For more details of the representation content, see Ref. (3) and (4). It is well known that the bottom mesons transform[9] as 1 and 3* under SU(3). Therefore, to compute the SU(3) decay amplitudes of bottom mesons into two pseudoscalars we need the C–G coefficients for the following products of representations: $1 \otimes 8$, $1 \otimes 3$, $1 \otimes 15$, $1 \otimes 6^*$, $1 \otimes 3^*$, $1 \otimes 6$, $3 \otimes 8$, $3 \otimes 3$, $3 \otimes 15$, $3 \otimes 6^*$, $3 \otimes 3^*$, $3 \otimes 6$. As discussed in Sec. 1, all the C-G coefficients of these products, except $3^* \otimes 15$, have been obtained previously. The purpose of this paper is to obtain the C-G coefficients for $3^* \otimes 15$ by adopting the phase convention of Baird and Biedenharn[8].

III. NOTATIONS AND PHASE CONVENTIONS

The newly discovered resonances are all bound states of (b$\bar{b}$), where b is the bottom quark and $\bar{b}$ is the antiquark. According to the quark model, the bottom quark b, together with the u, d, s, and c, forms a fundamental representation of SU(5). In SU(5) we have four additive quantum numbers. They are I_3 (third component of isospin), Y (hypercharge), C (charm) and B(bottom). The quantum numbers of quarks are given[9] in Table 1, where the hypercharge is defined by

$$Y = \frac{1}{\sqrt{3}} \lambda_8 \tag{3.1}$$

or

$$Y = B + S - \frac{1}{3} C + \frac{1}{3} \tilde{B} \tag{3.2}$$

In the above equation, B and $\tilde{B}$ denote the baryon and bottom number respectively.

The SU(5) symmetry is badly broken due to heavy mass of b-quark, i.e. $m_b \approx 4.8$ GeV. However, the SU(3) symmetry relations should hold well for bottom meson decays. The main purpose for this paper is to calculate the SU(3) C-G coefficients for the product $3^* \otimes 15 = 8 + 10 + 27$. Under SU(3), u, d, and s quarks form a fundamental representation 3 while both c and b quarks transform as SU(3) singlet. A state in a given irreducible representation of SU(3) is completely specified by[7]

$$\Phi^{(\mu)}_{(I, I_z, Y)}$$

where μ labels the SU(3) representation; I, I_z are the isospin quantum numbers; and Y stands for the hypercharge. The weight diagrams of the SU(3) representations occured in the product 3* ⊗ 15 = 8 + 10 + 27 are given in Fig. 1. For simplicity, we also denote I, I_z and Y collectively by the symbol υ . The SU(3) C-G coefficients

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \upsilon_1 & \upsilon_2 & \upsilon \end{pmatrix}$$

can be written as

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \upsilon_1 & \upsilon_2 & \upsilon \end{pmatrix} = \begin{pmatrix} \mu_1 & \mu_2 & \mu \\ I_1 Y_1 & I_2 Y_2 & I\, Y \end{pmatrix} C \begin{matrix} I_1 & I_2 & I \\ I_{1z} & I_{2z} & I_z \end{matrix} \qquad (3.3)$$

where $\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ I_1 Y_1 & I_2 Y_2 & I\, Y \end{pmatrix}$ are the SU(2) singlet

or isoscalar factors and $C \begin{matrix} I_1 & I_2 & I \\ I_{1z} & I_{2z} & I_z \end{matrix}$ are the usual SU(2) C-G coefficients.

Our tables give the isoscalar factors.

In computing the isoscalar factors, as well as fixing the phases, it is useful to consider the various subgroups of SU(3). For this purpose it is convenient to write the infinitesimal generators in terms of the non-Hermition matrices E_{ij} , which satisfy the commutation relations.

$$[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj} \qquad (3.4)$$

for i, j, k = 1, 2, 3 and $E^+_{ij} = E_{ji}$

In the fundamental representation the E matrices are very simple. They are the so-called matrix units, e_{ij} , where e_{ij} is the matrix in which the element belonging to the i th row and the j th column is unity and all the other elements are zero. We can define the raising and lowering operators of the different subgroups as follows:

$E_{12} = I_+$, $E_{13} = V_+$, $E_{23} = U_+$

and $I_- = E_{12}^+ = E_{21}$, etc.

These operators generate the well-known I-spin, U-spin, and V-spin subgroups of SU(3). We recall that in SU(3) the relative phases of the matrix elements of any two out of the three sets $I_\pm$, $U_\pm$, $V_\pm$ can be chosen arbitarily. In addition to the usual Condon-Shortley convention that the matrix elements of $I_\pm$ be positive, deSwart[7] chose to make the matrix elements of $V_\pm$ positive in order to fix the relative phases of the different SU(2) multiplets contained in an irreducible representation of SU(3). However, as discussed by Baird and Biedenharn[8], the most general and convenient phase convention for any SU(n) is to define the matrix elements of $E_{i,i+1}$ $(i = 1, \ldots, n-1)$ to be positive. This leads us to adopt the phase convention of Baird and Biedenharn in our calculation and choose $I_\pm$ and $U_\pm$ to have positive matrix elements. This is because we expect that SU(4) symmetry relations might be useful for B meson decays. Therefore, by adopting this general phase convention, we can easily generalize our SU(3) calculation to SU(4). The matrix elements of $I_\pm$ are well known. The matrix elements of $U_\pm$ can be obtained from

$$U_\pm |I, I_3, Y> = c_\pm |I + 1/2, I_3 \mp 1/2, Y \pm 1> + d_\pm | I - 1/2, I_3 \mp 1/2, Y \pm 1>$$

$$c_+ = \left\{ \frac{(I - I_3 + 1)\,[(p-q)/3 + I + Y/2 + 1]\,[(p+2q)/3 + I + Y/2 + 2]\,[(2p+q)/3 - I - Y/2]}{2(I+1)(2I+1)} \right\}^{1/2}$$

$$d_+ = \left\{ \frac{(I + I_3)\,[(q-p)/3 + I - Y/2]\,[(p+2q)/3 - I + Y/2 + 1]\,[(2p+q)/3 + I - Y/2 + 1]}{2I(2I+1)} \right\}^{1/2} \qquad (3.6)$$

The coefficients c_- and d_- can be obtained by realizing that $(L_+)^+ = L_-$. In (3.6), p and q are the pair of numbers (p, q) which specifies a particular irreducible representation of SU(3), i.e. p denotes the number of columns with one box and q denotes the number of columns with two boxes for the Young tableau of a particular representation of SU(3). Table 2 gives the values of (p, q) corresponding to the 3*, 15, 8, 10 and 27 representations of SU(3).

Within a given multiplet in SU(3) the relative phases are determined by choosing $I_\pm$ and $U_\pm$ to have positive matrix elements. A further condition[7] is required to determine the relative phase between different SU(3) representations in a decomposition of a product. For each representation μ occured in the product, we consider the highest state $\Phi^{(\mu)}_{\upsilon_H}$, i.e. the state with highest I_z in μ, and we choose the coefficient

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \upsilon_1 & \upsilon_2 & \upsilon_H \end{pmatrix}$$

to be positive for the highest I_1 that occurs. If this is not sufficient, we shall in addition require the highest I_2. If this is insufficient, we choose the one with highest I_{1z} to be positive.

The phase factor ξ_1 given in Table 3 contains the symmetry properties of the C-G coefficients:

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \upsilon_1 & \upsilon_2 & \upsilon \end{pmatrix} = \xi_1 \begin{pmatrix} \mu_2 & \mu_1 & \mu \\ \upsilon_2 & \upsilon_1 & \upsilon \end{pmatrix} \tag{3.7}$$

Table 4 give the SU(3) isoscalar factors $\left(\begin{matrix} \mu_1 & \mu_2 \\ \upsilon_1 & \upsilon_2 \end{matrix} \middle| \begin{matrix} \mu \\ \upsilon \end{matrix}\right)$ for the product $3^* \otimes 15 = 8 + 10 + 27$ as a result of our calculation. After this work is finished, I learned that the isoscalar factors for the above product is also given by Anderson and Joshi[10] in a completely different context. They obtained these factors in their studies of the formation of baryonium and exotic baryons. Our results agree with theirs except for a few sign differences due to different choices of the relative phases between SU(3) representations in a decomposition of a product. The isoscalar factors in Table 4 are used in our calculation of the SU(3) amplitudes for the nonleptonic two-body decays of bottom mesons[11].

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REFERENCES

1 S. W. Herb et al., Phys. Rev. Lett. 39,252 (1977)
2 D. Andrews et al., Phys. Rev. Lett. 44, 1108 (1980)
3 D. Zeppenfeld, MPI preprint, MPI-PAE/Pth 34/80 (1980)
4 W. Lin, Z. Phys. C-Particles and Fields 11, 21 (1981)
5 V. Rabl, G. Campbell, Jr., and K. C. Wali, J. Math. Phys. 16, 2494 (1975)
6 C. K. Chew, Tables of the SU(3) Isoscalar Factors, Quebec, Dept. of Physics, Laval University, (1968)
7 J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963)
8 G. E. Baird and L. C. Biedenharn, J. Math. Phys. 5, 1723 (1964)
9 W. Lin, Chin. Jour. Phys. 17, 67 (1979)
10 R. Anderson and G. C. Joshi, J. Math, Phys. 20, 1015 (1979)
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TABLE CAPTIONS

Table 1. Quantum numbers of quarks

Table 2. Value of (p, q) for the representations involved in the product $3^* \otimes 15 = 8 + 10 + 27$

Table 3. Phase factor ξ_1 associated with the C-G coefficients

Table 4. Isoscalar factors (ISF) for the product $3^* \otimes 15 = 8 + 10 + 27$

TABLE 1

Quark	B	Q	I	I_3	Y	C	$\tilde{B}$
u	1/3	2/3	1/2	1/2	1/3	0	0
d	1/3	−1/3	1/2	−1/2	1/3	0	0
s	1/3	−1/3	0	0	−2/3	0	0
c	1/3	2/3	0	0	0	1	0
b	1/3	−1/3	0	0	0	0	− 1

TABLE 2

	(p , q)
3*	(0 , 1)
15	(2 , 1)
27	(2 , 2)
10	(3 , 0)
8	(1 , 1)

TABLE 3

μ_1	μ_2	μ	ξ_1
3*	15	8	−1
		10	−1
		27	1

TABLE 4

I_1	Y_1	I_2	Y_2	μ	I_3	Y_3	ISF
1/2	−1/3	1	4/3	8	1/2	1	$2/\sqrt{5}$
0	2/3	1/2	1/3	8	1/2	1	$-1/\sqrt{5}$
1/2	−1/3	3/2	1/3	8	1	0	$4\sqrt{2}/3\sqrt{5}$
1/2	−1/3	1/2	1/3	8	1	0	$1/3\sqrt{5}$
0	2/3	1	−2/3	8	1	0	$-2/\sqrt{15}$
1/2	−1/3	1/2	1/3	8	0	0	$\sqrt{3/5}$
0	2/3	0	−2/3	8	0	0	$-\sqrt{2/5}$
1/2	−1/3	1	−2/3	8	1/2	−1	$\sqrt{2/5}$
1/2	−1/3	0	−2/3	8	1/2	−1	$1/\sqrt{15}$
0	2/3	1/2	−5/3	8	1/2	−1	$-2\sqrt{2/15}$
0	2/3	3/2	1/3	10	3/2	1	$-1/\sqrt{2}$
1/2	−1/3	1	4/3	10	3/2	1	$1/\sqrt{2}$
1/2	−1/3	3/2	1/3	10	1	0	$-\sqrt{2}/3$
0	2/3	1	−2/3	10	1	0	$-1/\sqrt{3}$
1/2	−1/3	1/2	1/3	10	1	0	2/3
1/2	−1/3	1	−2/3	10	1/2	−1	$-1/\sqrt{2}$
0	2/3	1/2	−5/3	10	1/2	−1	$-1/\sqrt{6}$
1/2	−1/3	0	−2/3	10	1/2	−1	$1/\sqrt{3}$
1/2	−1/3	1/2	−5/3	10	0	−2	−1
0	2/3	1	4/3	27	1	2	1
0	2/3	3/2	1/3	27	3/2	1	$1/\sqrt{2}$
1/2	−1/3	1	4/3	27	3/2	1	$1/\sqrt{2}$
0	2/3	1/2	1/3	27	1/2	1	$2/\sqrt{5}$
1/2	−1/3	1	4/3	27	1/2	1	$1/\sqrt{5}$
1/2	−1/3	3/2	1/3	27	2	0	1
1/2	−1/3	3/2	1/3	27	1	0	$1/\sqrt{15}$
0	2/3	1	−2/3	27	1	0	$\sqrt{2/5}$

I_1	Y_1	I_2	Y_2	μ	I_3	Y_3	ISF
1/2	−1/3	1/2	1/3	27	1	0	$2\sqrt{2/15}$
1/2	−1/3	1/2	1/3	27	0	0	$\sqrt{2/5}$
0	2/3	0	−2/3	27	0	0	$\sqrt{3/5}$
1/2	−1/3	1	−2/3	27	3/2	−1	1
1/2	−1/3	1	−2/3	27	1/2	−1	$1/\sqrt{10}$
0	2/3	1/2	−5/3	27	1/2	−1	$\sqrt{3/10}$
1/2	−1/3	0	−2/3	27	1/2	−1	$\sqrt{3/5}$
1/2	−1/3	1/2	−5/3	27	1	−2	1

FIGURE CAPTIONS

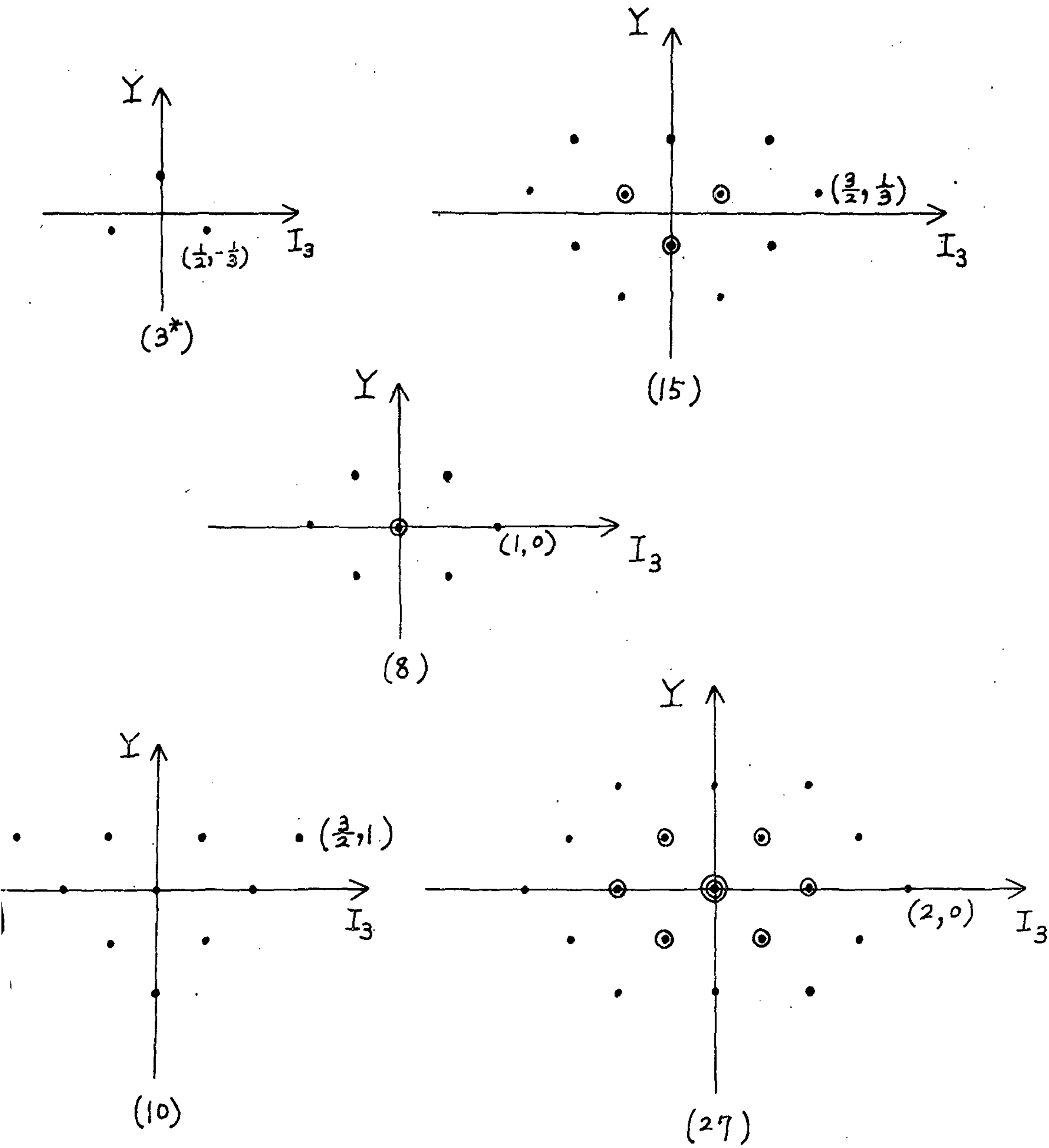

Fig. 1 Weight diagrams for the representations involved in the product 3*
15 = 8 + 10 + 27

下介子弱蛻變之SU(3) C—G係數之計算

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中文摘要

我們根據新的相位之規定，計算了研究下介子弱蛻變之振幅所需之SU(3) C—G係數。