3 Existence

In this section, we shall using the shooting method to derive the existence of self-similar solutions. For this, we define

$$I_1 = \{\mu > 0 \mid \exists y_0 \in (0,\infty) \text{ such that } g'(y_0;\mu) = 0\},$$
$$I_2 = \{\mu > 0 \mid \exists R < \infty \text{ such that } g(y;\mu) \to 0 \text{ as } y \to R^- \text{ and } g'(y;\mu) < 0 \text{ in } [0,R)\}.$$

Let us first prove I_1 and I_2 are both nonempty.

Lemma 3.1 *I_1 is nonempty.*

Proof. Let $g(y;\mu) = \mu w(\mu^{(1-m)/(2m)}y;\mu)$ for $y > 0$. Then $g(y;\mu)$ is the solution of (P) if and only if $w(z;\mu)$ satisfies (P1):

$$\begin{cases} w'' + \alpha kzw^{\frac{1}{m}-1}w' + \alpha mw^{\frac{1}{m}} = 0, z > 0, \\ w(0) = 1,\ w'(0) = m\mu^{\frac{-1}{2\alpha m}}. \end{cases}$$

If $\mu = 0$, then (P1) becomes

$$\begin{cases} w'' + \alpha kzw^{\frac{1}{m}-1}w' + \alpha mw^{\frac{1}{m}} = 0, z > 0, \\ w(0;0) = 1,\ w'(0;0) = 0. \end{cases}$$

Note that $w''(0;0) = -\alpha m > 0$. We see that $w'(z;0) > 0$ and $w(z;0) > 1$ for small $z > 0$. Since any critical point of $w(z;0)$ must be a local minimum point, $w'(z;0) > 0$ for any z in $(0,\infty)$. Then $w(z;0) > 1$ for any z in $(0,\infty)$ and $w''(z;0) > 0$ for any z in $[0,\infty)$. Hence $w'(z;0)$ is monotone increasing and so $w'(z;0) > w'(1;0)$ for any z in $(1,\infty)$. Given any ε such that $0 < \varepsilon < \min\{\frac{1}{2}, w'(1;0)/2\}$. By the continuous dependence on parameter, there exists $\mu_* > 0$ such that

$$w(z;\mu) > 0, \forall z \in [0,2] \text{ and } w'(z;\mu) > 0, \forall z \in [1,2],$$

for $\mu \in (0,\mu_*)$. Hence $g(y;\mu) > 0, \forall y \in [0, 2\mu^{(m-1)/(2m)}]$ for any $\mu \in (0,\mu_*)$. Since $w'(0;\mu) = m\mu^{\frac{-1}{2\alpha m}} < 0$, there exists $z_0(\mu) < 1$ such that $w'(z_0;\mu) = 0$. Therefore, $g'(\mu^{(m-1)/(2m)}z_0;\mu) = 0$ for $\mu \in (0,\mu_*)$. This shows that $(0,\mu_*) \subseteq I_1$. The lemma follows. □

Lemma 3.2 *I_2 is nonempty.*

Proof. Consider the energy function

$$\begin{aligned} E(y) = E(y;\mu) &= \frac{[g'(y)]^2}{2} + \frac{\alpha m^2}{1+m}[g(y)]^{\frac{1}{m}+1}, \text{ if } m \neq -1, \\ &= \frac{[g'(y)]^2}{2} - \alpha \ln g(y), \text{ if } m = -1. \end{aligned}$$

From $E'(y) = -\alpha kyg^{\frac{1}{m}-1}(g')^2 \geq 0$, it follows that $E(y)$ is nondecreasing in y. If $E(y_1) = E(y_2)$ for some $y_1 < y_2$, then

$$0 = E(y_2) - E(y_1) = \int_{y_1}^{y_2} E'(y)dy = \int_{y_1}^{y_2} \{-\alpha kyg^{\frac{1}{m}-1}(g')^2\}dy.$$

Thus $g'(y) \equiv 0, \forall y \in (y_1, y_2)$. Then $g(y)$ is a constant on (y_1, y_2), a contradiction. Hence $E(y)$ must be strictly monotone increasing in y.

We divide our discussion into three cases:

Case 1: $m < -1$. Taking $\mu^* = (\dfrac{-2\alpha}{1+m})^{-\alpha m}$.

$$E(0;\mu) = \frac{m^2\mu^{2+\frac{2k}{m}}}{2} + \frac{\alpha m^2}{1+m}\mu^{\frac{1}{m}+1} \geq 0, \text{ if } \mu \geq \mu^*.$$

Fix any $\mu \geq \mu^*$. If there exists a critical point $0 < y_0 < \infty$ such that $g'(y_0) = 0$, then

$$E(y_0;\mu) = \frac{\alpha m^2}{1+m}[g(y_0)]^{\frac{1}{m}+1} < 0.$$

This is a contradiction. Hence $g'(y) < 0$ in $[0, R)$, where $0 < R \leq \infty$. If $R = \infty$, then $g(y;\mu) \to 0$ and $\exists\{y_n\} \to \infty$ such that $g'(y_n;\mu) \to 0$ as $n \to \infty$. Therefore, $E(y_n;\mu) \to 0$ as $n \to \infty$, a contradiction. Hence $R < \infty$.

Case 2: $-1 < m < 0$. Define

$$F(y) = F(y;\mu) = -\alpha kyg' - \alpha mg. \tag{10}$$

Then we compute that

$$F'(y) + \alpha kyg^{\frac{1}{m}-1}(y)F(y) = (-\alpha k - \alpha m)g'(y).$$

It follows that

$$(\rho F)'(y) = (-\alpha k - \alpha m)g'(y)\rho(y), \tag{11}$$

where $\rho(y) = \exp\{\int_0^y \alpha ksg^{\frac{1}{m}-1}(s)ds\}$.

We first claim that there exists $y_1 > 0$ such that $F(y_1) < 0$ for some $\mu > 0$.

If there is no critical point, then for any $\mu > 1$ and any $n > 1$ there exists $y_1 > 0$ such that $g(y_1) = 1/n$ and $g'(y) < 0$ for any $y \in [0, y_1]$. Next, we suppose that there is a critical point $y_0 > 0$ such that $g'(y_0) = 0$. From $E(y_0;\mu) > E(0;\mu)$, it follows that

$$\begin{aligned} g(y_0) &< [\frac{1+m}{2\alpha}\mu^{2+\frac{2k}{m}} + \mu^{1+\frac{1}{m}}]^{\frac{m}{m+1}} \\ &< [\frac{1+m}{2\alpha}\mu^{2+\frac{2k}{m}}]^{\frac{m}{m+1}}. \end{aligned}$$

This implies that for any $n > \max\{(\frac{2\alpha}{1+m})^{\frac{m}{m+1}},\ 2+\frac{m}{k}\}$, there exists $\mu_0 = (\frac{2\alpha}{1+m})^{\frac{m}{2m+2k}}(\frac{1}{n})^{\frac{m+1}{2m+2k}} > 1$ such that $g(y_0) < 1/n$ for any $\mu \geq \mu_0$. Hence there exists $y_1 > 0$ such that $g(y_1) = 1/n$ and $g'(y) < 0$ for any $y \in [0, y_1]$, if $\mu \geq \mu_0$.

Now, set $\mu_c = (1+c)^{\frac{m}{2m+2k}}(\frac{2\alpha}{1+m})^{\frac{m}{2m+2k}}(\frac{1}{n})^{\frac{m+1}{2m+2k}} > \mu_0$, where $c > 0$. From $E(y_1;\mu_c) > E(0;\mu_c)$, it follows that

$$\begin{aligned} g'(y_1;\mu_c) &< -\sqrt{m^2\mu_c^{2+\frac{2k}{m}} + \frac{2\alpha m^2}{1+m}\mu_c^{1+\frac{1}{m}} - \frac{2\alpha m^2}{1+m}\left(\frac{1}{n}\right)^{1+\frac{1}{m}}} \\ &< -\sqrt{m^2\mu_c^{2+\frac{2k}{m}} - \frac{2\alpha m^2}{1+m}\left(\frac{1}{n}\right)^{1+\frac{1}{m}}} \\ &= m\sqrt{\frac{2c\alpha}{1+m}\left(\frac{1}{n}\right)^{1+\frac{1}{m}}}. \end{aligned} \tag{12}$$

From (9), it follows that

$$-g'(y;\mu_c) < -m\mu_c^{1+\frac{k}{m}} + \alpha kmyg^{\frac{1}{m}}(y),\ \forall y > 0.$$

Since $\mu_c > 1 > (2+\frac{m}{k})/n$,

$$\begin{aligned} \frac{1+\frac{m}{k}}{n} < \mu_c - \frac{1}{n} &= \int_0^{y_1}[-g'(s;\mu_c)]ds \\ &< -m\mu_c^{1+\frac{k}{m}}y_1 + \alpha km\left(\frac{1}{n}\right)^{\frac{1}{m}}\int_0^{y_1} sds \\ &< -m\mu_c^{1+\frac{k}{m}}y_1 + \frac{\alpha km}{2}\left(\frac{1}{n}\right)^{\frac{1}{m}}y_1^2. \end{aligned}$$

It follows that

$$y_1^2 - \frac{2}{\alpha k}\mu_c^{1+\frac{k}{m}}\left(\frac{1}{n}\right)^{-\frac{1}{m}}y_1 - \frac{2(1+\frac{m}{k})}{\alpha km}\left(\frac{1}{n}\right)^{1-\frac{1}{m}} > 0.$$

Hence

$$y_1 > \frac{-1}{\alpha k}(1+c)^{1/2}\left(\frac{2\alpha}{1+m}\right)^{1/2}\left(\frac{1}{n}\right)^{\frac{m-1}{2m}}\left[-1+\sqrt{1+\frac{k(1+\frac{m}{k})(1+m)}{m(1+c)}}\right]. \tag{13}$$

Combining (10), (12) and (13), we get

$$F(y_1;\mu_c) < \frac{\alpha m}{n}\left\{\frac{2\sqrt{c^2+c}}{1+m}\left[-1+\sqrt{1+\frac{k(1+\frac{m}{k})(1+m)}{m(1+c)}}\right]-1\right\}.$$

Note that

$$\lim_{c\to\infty}\frac{2\sqrt{c^2+c}}{1+m}\left[-1+\sqrt{1+\frac{k(1+\frac{m}{k})(1+m)}{m(1+c)}}\right] = \frac{k(1+\frac{m}{k})}{m} > 1.$$

Therefore, there exists $c^* > 0$ such that $F(y_1;\mu_c) < 0$ for any $c \geq c^*$. Taking $\mu^* = \mu_{c^*}$. Then $F(y_1;\mu) < 0$ for any $\mu \geq \mu^*$.

Next, we fix any $\mu \geq \mu^*$. We claim that $g'(y;\mu) < 0$ for any $y > y_1$ as long as $g(y;\mu) > 0$. For contradiction, we suppose that there exists $y_2 > y_1$ such that $g(y;\mu) > 0$ in $[y_1, y_2]$, $g'(y;\mu) < 0$ in $[y_1, y_2)$ and $g'(y_2;\mu) = 0$. Then $F(y_2;\mu) = -\alpha m g(y_2;\mu) > 0$. On the other hand, by (11) we have

$$(\rho F)'(y) < 0 \text{ in } (y_1, y_2),\ \rho(y) > 0, \forall y$$

and $F(y_1;\mu) < 0$. It follows that $F(y_2;\mu) < 0$, a contradiction. Therefore, $g'(y;\mu) < 0$ for any $y > y_1$ as long as $g(y;\mu) > 0$.

Finally, we claim that $g(y;\mu)$ is non-global for any $\mu \geq \mu^*$. Otherwise, if g is positive globally, then by (11) and noting that $\rho < 1$, there is a positive constant λ such that

$$F(y;\mu) < -\lambda, \forall y \geq y_1.$$

We recall (10) to get that

$$-\alpha k y g' - \alpha m g < -\lambda, \forall y \geq y_1.$$

By an integration, we obtain that

$$g(y) < (\frac{y_1}{y})^{m/k} g(y_1) + \frac{\lambda}{\alpha m} - \frac{\lambda}{\alpha m}(\frac{y_1}{y})^{m/k}.$$

By passing y to infinity, we reach a contradiction. Hence $g(y;\mu)$ is non-global for any $\mu \geq \mu^*$.

Case 3: $m = -1$. We first claim that there exists $y_1 > 0$ such that $F(y_1) < 0$ for some $\mu > 0$.

If there is no critical point, then for any $\mu > 1$ and any $n > 1$ there exists $y_1 > 0$ such that $g(y_1) = 1/n$ and $g'(y) < 0$ for any $y \in [0, y_1]$. Next, we suppose that there is a critical point $y_0 > 0$ such that $g'(y_0) = 0$. From $E(y_0;\mu) > E(0;\mu)$, it follows that

$$\begin{aligned}\ln g(y_0) &< \frac{-\mu^{2-2k}}{2\alpha} + \ln \mu \\ &< \frac{-1+2\alpha}{2\alpha}\mu^{2-2k}\end{aligned}$$

for any $\mu > 1$. This implies that for any $n > \max\{\exp[\frac{1-2\alpha}{2\alpha}],\ 2 + \frac{-1}{k\sqrt{1-2\alpha}}\}$, there exists $\mu_0 = (\frac{2\alpha}{-1+2\alpha}\ln\frac{1}{n})^{\frac{1}{2-2k}} > 1$ such that $g(y_0) < 1/n$ for any $\mu \geq \mu_0$. Hence there exists $y_1 > 0$ such that $g(y_1) = 1/n$ and $g'(y) < 0$ for any $y \in [0, y_1]$,if $\mu \geq \mu_0$.

Now, set $\mu_c = (1+c)^{\frac{1}{2-2k}}(\frac{2\alpha}{-1+2\alpha}\ln\frac{1}{n})^{\frac{1}{2-2k}} > \mu_0$, where $c > 0$. From $E(y_1;\mu_c) > E(0;\mu_c)$, it follows that

$$\begin{aligned}g'(y_1;\mu_c) &< -\sqrt{\mu_c^{2-2k} - 2\alpha\ln\mu_c + 2\alpha\ln\frac{1}{n}} \\ &< -\sqrt{(1-2\alpha)\mu_c^{2-2k} + 2\alpha\ln\frac{1}{n}} \\ &= -\sqrt{-2\alpha c\ln\frac{1}{n}}.\end{aligned} \tag{14}$$

From (9), it follows that

$$-g'(y;\mu_c) < \mu_c^{1-k} - \alpha k y g^{-1}(y),\ \forall y > 0.$$

Since $\mu_c > 1 > (2 + \frac{-1}{k\sqrt{1-2\alpha}})/n$,

$$\begin{aligned} \frac{1+\frac{-1}{k\sqrt{1-2\alpha}}}{n} < \mu_c - \frac{1}{n} &= \int_0^{y_1} [-g'(s;\mu_c)]ds \\ &< \mu_c^{1-k} y_1 - \frac{\alpha k n}{2} y_1^2. \end{aligned}$$

It follows that

$$y_1^2 - \frac{2}{\alpha k n}\mu_c^{1-k} y_1 + \frac{2(1+\frac{-1}{k\sqrt{1-2\alpha}})}{\alpha k}\left(\frac{1}{n}\right)^2 > 0.$$

Hence

$$y_1 > \frac{-1}{\alpha k n}\sqrt{\frac{2\alpha(1+c)}{-1+2\alpha}\ln\frac{1}{n}}\left[-1+\sqrt{1-\frac{k(1+\frac{-1}{k\sqrt{1-2\alpha}})(-1+2\alpha)}{(1+c)\ln\frac{1}{n}}}\right]. \tag{15}$$

Combining (10), (14) and (15), we get

$$F(y_1;\mu_c) < \frac{-\alpha}{n}\left\{\sqrt{\frac{-4(c^2+c)}{-1+2\alpha}\left(\ln\frac{1}{n}\right)^2}\left[-1+\sqrt{1-\frac{k(1+\frac{-1}{k\sqrt{1-2\alpha}})(-1+2\alpha)}{(1+c)\ln\frac{1}{n}}}\right]-1\right\}.$$

Note that

$$\begin{aligned} &\lim_{c\to\infty}\sqrt{\frac{-4(c^2+c)}{-1+2\alpha}\left(\ln\frac{1}{n}\right)^2}\left[-1+\sqrt{1-\frac{k(1+\frac{-1}{k\sqrt{1-2\alpha}})(-1+2\alpha)}{(1+c)\ln\frac{1}{n}}}\right] \\ = &\ -k\sqrt{1-2\alpha}(1+\frac{-1}{k\sqrt{1-2\alpha}})\ > 1. \end{aligned}$$

Therefore, there exists $c^* > 0$ such that $F(y_1;\mu_c) < 0$ for any $c \geq c^*$. Taking $\mu^* = \mu_{c^*}$, then $F(y_1;\mu) < 0$ for any $\mu \geq \mu^*$.

Next, we fix any $\mu \geq \mu^*$. Then, as in Case 2, we have $g'(y;\mu) < 0$ for any $y > y_1$ as long as $g(y;\mu) > 0$.

Finally, we claim that $g(y;\mu)$ is non-global for any $\mu \geq \mu^*$. Otherwise, if g is positive globally, then by (11) and noting that $\rho < 1$, there is a positive constant λ such that

$$F(y;\mu) < -\lambda, \forall y \geq y_1.$$

We recall (10) to get that

$$-\alpha k y g' + \alpha g < -\lambda, \forall y \geq y_1.$$

By an integration, we obtain that

$$g(y) < (\frac{y_1}{y})^{-1/k} g(y_1) - \frac{\lambda}{\alpha} + \frac{\lambda}{\alpha}(\frac{y_1}{y})^{-1/k}.$$

By passing y to infinity, we reach a contradiction. The lemma follows. □

Notice that I_1 and I_2 are disjoint, Lemmas 3.1 and 3.2 imply that $(0, \mu_*) \subseteq I_1$ and $[\mu^*, \infty) \subseteq I_2$.

Next, we shall prove I_1 and I_2 are both open.

Lemma 3.3 *I_1 is open.*

Proof. Let $\mu_0 \in I_1$. Then there exists $y_0 < \infty$ such that $g'(y_0; \mu_0) = 0$ and $g'(y_0+1; \mu_0) > 0$. Given any ε such that $0 < \varepsilon < \min\{g(y_0; \mu_0)/2,\ g'(y_0 + 1; \mu_0)/2\}$. By the continuous dependence on initial data, there exists $\delta > 0$ such that $|g(y; \mu) - g(y; \mu_0)| < \varepsilon$ and $|g'(y; \mu) - g'(y; \mu_0)| < \varepsilon$ for all $y \in [0, y_0 + 1]$ and for any $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$. This implies that $(\mu_0 - \delta, \mu_0 + \delta) \subset I_1$ and so I_1 is open. □

Lemma 3.4 *I_2 is open.*

Proof. Let $\mu_0 \in I_2$. Then there exists $R_0 < \infty$ such that $g(y; \mu_0) \to 0$ as $y \to R_0^-$ and $g'(y; \mu_0) < 0$ in $[0, R_0)$. First, we consider the function

$$F_0(y) = F(y; \mu_0) = -\alpha k y g'(y; \mu_0) - \alpha m g(y; \mu_0),$$

then $F_0(0) > 0$. From Lemma 2.3, we get that $\lim_{y \to R_0^-} F_0(y) = -\infty$. Hence there is a unique $y_0 \in (0, R_0)$ such that $F_0(y_0) = 0$ and $F_0(y) < 0$ in (y_0, R_0).

Next, we fix any $y_1 \in (y_0, R_0)$. By the standard continuous dependence property, there is a $\delta > 0$ small enough such that any solution $g(y; \mu)$ of (P) with $|\mu - \mu_0| < \delta$ satisfies:

$$g(y; \mu) > 0,\ g'(y; \mu) < 0 \text{ in } [0, y_1] \text{ and } F(y_1; \mu) < 0.$$

Now, we fix any $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$. We claim that $g'(y; \mu) < 0$ for any $y > y_1$ as long as $g(y; \mu) > 0$. For contradiction, we suppose that there exists $y_2 > y_1$ such that $g(y; \mu) > 0$ in $[y_1, y_2]$, $g'(y; \mu) < 0$ in $[y_1, y_2)$ and $g'(y_2; \mu) = 0$. Then $F(y_2; \mu) = -\alpha m g(y_2; \mu) > 0$. On the other hand, by (11) we have

$$(\rho F)'(y) < 0 \text{ in } (y_1, y_2),\ \rho(y) > 0, \forall y$$

and $F(y_1; \mu) < 0$. It follows that $F(y_2; \mu) < 0$, a contradiction. Therefore, $g'(y; \mu) < 0$ for any $y > y_1$ as long as $g(y; \mu) > 0$.

Finally, we claim that $g(y; \mu)$ is non-global for any $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$. Otherwise, if g is positive globally, then by (11) and noting that $\rho < 1$, there is a positive constant λ such that

$$F(y; \mu) < -\lambda, \forall y \geq y_1.$$

We recall (10) to get that

$$-\alpha k y g' - \alpha m g < -\lambda, \forall y \geq y_1.$$

By an integration, we obtain that

$$g(y) < (\frac{y_1}{y})^{m/k} g(y_1) + \frac{\lambda}{\alpha m} - \frac{\lambda}{\alpha m}(\frac{y_1}{y})^{m/k}.$$

By passing y to infinity, we reach a contradiction. The lemma follows. □

Now, we can prove the existence of self-similar solutions.

Theorem 2 *There exists $\mu > 0$ such that $g(y;\mu)$ is globally monotone decreasing to zero.*

Proof. Set $\overline{\mu} = \sup I_1$. Let the corresponding solution $\overline{g}(y) := g(y;\overline{\mu})$. Then $\overline{g}' < 0$ in $[0, R)$, since $\overline{\mu} \notin I_1$. If $\overline{\mu} \in I_2$, then there exists $\delta > 0$ such that $\overline{\mu} - a \in I_2$ for all $a \in (0, \delta)$, since I_2 is open. A contradiction with I_1 and I_2 are disjoint. Hence, $R = \infty$ and so $\overline{g}(y)$ is globally monotone decreasing to zero. □