A Study of Lie Group and Lie Algebra in GUT

by

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Abstract

Group theoretical methods needed in the study of grand unified theories are reviewed. We focus our attention on Dynkin's approach since it is applicable to any simple Lie group. Tables of dimensions and indices for the irreducible representations of SU(5), SO(10) and E_6 are presented.

Key words: grand unified theory, E_6model, Dykin's method

I. Introduction

Low energy particle phenomenology is successfully described by the Standard Model in which the weak and electromagnetic interactions are unified[1,2] by a gauge theory based on the group SU(2)xU(1) and the strong interactions are described by Quantum Chromodynamics (QCD) based on the color group SU(3). One of the most probable and interesting extension of Standard Model for energies up to 10^{15} GeV is the grand unified theory (GUT) in which the strong and elecroweak interactions are unified by embedding the SU(3)xSU(2)xU(1) group into a simple group G with a single gauge coupling constant g:

$$G \supset SU(3) \times SU(2) \times U(1).$$

The simple group G used in GUT is usually very large and complicated. The most popular ones are SU(5), SO(10) and E_6.

Among the various models proposed for grand unification, the one based on E_6^4 is currently of great interest. It was originally proposed as a generalization of the SU(5) and SO(10) models. It has many nice features: (1) it is anomaly free; (2) it possesses useful complex representations; (3) the fermions of each generation belong to a single irreducible representation; (4) all symmetry breaking can be realized by using only those Higgs representations which give mass to the fermions.

Recently, developments in superstring [5,6] theories have renewed interest in E_6 as the group for grand unification. Present versions of superstring theories start out being defined in 10 demensions, with enormous internal symmetry group like SO(32) or $E_8 \times E_8$. It seems that the E_8xE8 superstring theory may lead to a successful low energy phenomenology. Compactification of this theory to 4 dimensions may reduce one of the E_8 symmetry group to and E_6 gauge group, which in turn contains SO(10) and SU(5) as subgroup:

$E_8 - E_6 \supset SO(10) \supset SU(5)$.

Thus the superstring-inspired E_6 model is also the simplest generalization of the SU(5) and SO(10) models. Therefore, it is of great interest to study the groups SU(5), SO(10) and E_6 which may be relevant for grand unified theory. In this paper, we shall focus our attention mainly on these groups. The traditional tensor techniques mainly on these groups. The traditional tensor techniques become quite cumbersome for the analyses of these groups. A great simplification can be made by the use of Dynkin's method.[7] Moreover this technique is very general and can be applied to all classical groups $A\ell$, $B\ell$,$C\ell$,$D\ell$, and the exceptional groups F_2,G_4,E_6, E_7,E_8. In this work, Dynkin's method will be used for the analsis of E_6. SO(10) and SU(5).

II. Group Theory

The group theory was introduced into mathematics in the nineteenth century by a young Frenchman called Evariste Galois. He hurriedly wrote down all his ideas the night before be was killed in a duel at the age of 20. Inspired by E. Galois' brilliant work, the Norwegian mathematician Sophus Lie began his research on analytic continuous group (Lie group) and discovered 4 series of simple Lie algebra, i.e. A_ℓ, B_ℓ, C_ℓ, and D_ℓ. Then the rank of a Lie group is defined by Killing, who also established the existence of the exceptional Lie algebra, i.e. G_2, F_4, E_6, E_7 and E8. The French mathematician Cartan sigificantly improved the work of Killing. The irreducible representations of simple Lie group was constructed by Weyl. Then in 1950's. the method of simple roots & Dynkin diiagrams was introduced by the Russian mathematician Dynkin.

We begin by defining the notation and developing the group theory needed for analyzing the grand unified models. We refer the reader to Ref.8-11 for a complete explanation an derivation of the group theory involved. Our emphasis will be on the Dynkin formalism since it is applicable to any simple Lie group.

A. Basic Notions[11]

A set of elements G is called a group if there is defined an operation, called group multiplication, among the elements of G such that 4 conditions are met: (1) the set G is closed under this operation; (2) operation is associative; (3) the identity element exists; (4) the inverse element exists.

A set of element M is called a topological space if there is defined for any subset V of M an operation called closure (denoted by a bar) such that (1) $\overline{m}=m$ if m is a single element of M; (2) $\overline{U \cup V}=\overline{U} \cup \overline{V}$ if U and V are subsets of M; (3) $\overline{\overline{U}}=\overline{U}$ if U is a subset of. M.

A set of elements G is called a topological group if (1) G is an abstract group; (2) G is a topological space; and (3) the group operation is a continuous operation in the topological space G.

A topological space G is called separable if for any 2 elements $x,y \in G$ there exists 2 open sets U and V of G which have no elements in common, i.e.

$$x \in U, \quad Y \in V$$

$$U \cap V = 0$$

A topological space satisfying this condition is called a Hausdorff space.

A Hausdorff space (separable topological space) which is locally euclidean is called a topological manifold.

A space M is called locally euclidean if one is able to introduce cartesian coordinates in any sufficiently small neighborhoods V of any point P of this space.

Consider a topological manifold M. Given any poit $p \in M$, one can introduce cartesian coordinates in a certain neghborhood V of p. If these coordinates introduced in the various points of M can be related analytically to each other in the region where they overlap, then M is called an analytic manifold.

A Lie group is a set G such that (1) G is a group; (2) G is an analytic manifold; and (3) the group product is described by analyic functions. So far as a physicist is concerned, a Lie group is a group of transformation of variables which depend analytically on a finite set of r parameters. A group G is called compact if the domain of parameters of G is bounded and closed.

Lie showed that the elements of a Lie group which can be reached continuously from the identity are determined from the elements lying in the neighborhood of the identity. Thus we can obtain a representation of the group by means of unitary operators

$$g(a_\gamma) \longrightarrow U(a_\gamma)=e^{i\sum_{\nu=1}^{r} a_\nu X_\nu} \qquad (2.1)$$

where $a_1, a_2, \cdots\cdots, a_r$ are real parameters and X_1, $X_2, \cdots\cdots, X_r$ are the hermitian generaors of the group. The number of parameters r, which is also the number of generators, is called the order of the group.

In spite of the continutity of the parameters, the study of the local structure can be reduced to the study of Lie algebra.

The Lie algebras may be defined as finite-dimensional vector spaces equipped with a multiplication law $L \times L \to L$ denoted by $[X,Y]$ which satisfies the conditions

$$[X,Y] = -[Y,X]$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

$$\text{for } X,Y,Z, \in L. \qquad (2.2)$$

In many cases, the prouct $[X,Y]$ may be realized by commutator, i.e.

$$[X,Y] = XY - YX \qquad (2.3)$$

A fundamental theorem established by Lie is that the commutator of the generators $X_\mu (\mu=1,2.....,r)$ of a Lie group are linear combinations of the generators, i,e.

$$[X_\mu, X_\nu] = \sum_{\lambda=1}^{r} C_{\mu\nu}^{\lambda} X_\lambda \qquad (2.4)$$

where $C_{\mu\nu}^{\lambda}$ are called structure constants. They satisfy the following relations:

$$C_{\mu\nu}^{\lambda} = -C_{\nu\mu}^{\lambda}$$

$$C_{\nu\mu}^{\beta} C_{\beta\lambda}^{\delta} + C_{\mu\lambda}^{\beta} C_{\beta\nu}^{\delta} + C_{\lambda\nu}^{\beta} C_{\beta\mu}^{\delta} = 0 \qquad (2.5)$$

A Lie algebra is called Abelian if it consists of r generators, all of which commute. Consider a subalgbra M of a Lie algebra L. If the commutator of any operator $a \in L$ with any operator $b \in M$ also lies in M, i,e.

$$[a , b] \quad M \qquad (2.6)$$

for $a \in L$, $b \in M$
then M is said to be an invariant subalgebra or ideal.

A Lie algebra is semisimple if it has no nonzero Abelian ideals. A lie algebra is said to be simple if it contains no prope ideals. A Lie group is called simple if the corresponding Lie algebra is semisimple. According to Levi's theorem, any Lie algebra may be written as a semidirect sum of a solvable Lie algebra and a semisimple Lie algebra. Since a semisimple Lie algera is a direct sum of simple Lie algebra, a study of semisimple Lie algebra reduces to the study of simple Lie algebras.

B. Roots

The maximum number of simultaneously diagonalizable generators of a simple Lie algebra G is called the rank of G. The standard form of Lie algebra is the commutation relations of generators written in the Cartan-Weyl basis. In this basis, the generators are divided into 2 sets:
$H_i (i=1,2,\ldots,\ell)$ and $E_\alpha (\alpha=1,2,\ldots,r-\ell)$.
The standard form of Lie algebra is given by the following commutation relations.

$$[H_i , H_j] = 0$$
$$[H_i , E_\alpha] = \alpha_i \; E_\alpha$$
$$[E_\alpha , E_{-\alpha}] = \alpha^i \; H_i$$
$$[E_\alpha , E_\beta] = N_{\alpha,\beta} \; E_{\alpha+\beta} \quad \text{if } \alpha+\beta \text{ is a root and } \alpha+\beta \neq 0$$
$$=0 \quad \text{if } \alpha+\beta \text{ is not a root.} \qquad (2.7)$$

By comparing Eq(2.7) with Eq(2.4), we see that the numbers α_i in Eq(2.7) are essentially the structure constants of the algebra in the Cartan-Weyl basis. For each generator E_α, there are ℓ numbers $\alpha_1, \alpha_2, \cdots\cdots, \alpha_\ell$. which may be regarded as the corrdinates of a point in an ℓ-dimensional space called root space. The vecotor a with component $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is called a root vector or simply a root. The nonzero roots are nondegenerate while the zero roots are ℓ-fold degenerate. The maximal set of commuting hermitian gererators$\{H_1, H_2, \ldots. H_\ell\}$ is called the Cartan subalgebra of G.

Some important properties of roots are: (1) If α is a root, then-α is also a root. Thus, the nonzero roots come in pairs. (2) If α is a root, then 2α is not a root. (3) For each root α, there is only one linearly independent E_α with that root. (4) If α and β are 2 roots, then $\frac{2\alpha\cdot\beta}{\alpha^2}$=m=integer and $\frac{2\alpha\cdot\beta}{\beta^2}$ =n=integer. Furthermore, $\beta - \frac{2\alpha\cdot\beta}{\alpha^2}\alpha = \beta - m\alpha = \gamma$ is also a root. (5) If and β are roots, consider a string of roots:

$$\beta - q\alpha, \ldots, \beta-\alpha, \beta, \beta+\alpha, \ldots, \beta + p\alpha .$$

Then it follows that

$$\frac{2\alpha\cdot\beta}{\alpha^2} = -(p-q) \qquad (2.8)$$

These properties are very useful for finding roots. Moreover, property (4) immediately gives us bcth the angle θ between any 2 roots and the ratio of lengt squared:

$$\cos^2\theta = \frac{mn}{4} \qquad (2.9)$$

$$\frac{\alpha^2}{\beta^2} = \frac{n}{m} \qquad (2.10)$$

Assume $\alpha > \beta$, we have

mn	θ	α^2 / β^2
0	90°	undetermined
1	60°,120°	1
2	45°,135°	2
3	30°,150°	3
4	0°,180°	1

It is a well-known fact that for ℓ(rank)=2, there are only 3 independent simple Lie algebras A_2, B2~C2 and G2. Note that D2~A_1+A_1 and hence not simple.

If the angle between 2 adjacent roots is 60°, the corresponding Lie algebra is called A_2 in Cartan's notation. There are 6 non-zero roots of A_2. They are all of equal length. A convenient choice of normalization is

$$\alpha^2(1) = \alpha^2(2) = \ldots\ldots = 1$$

If the angle between 2 adjacent roots is 45°, the corresponding Lie algebra is called B_2 The corresponding group is generally known as SO(5). There are 8 non-zero roots of B_2. They are of 2 different lengths. Assume $> \beta^2$, then ratio of length is fixed to be $\alpha^2/\beta^2=2$. A convenient choice of normalization is $\alpha^2=2$ and $\beta^2=1$. With this normalizaion, the coordinates of the 8 non-zero roots are

(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)

It is convenient to introduce orthnormal unit vectors $e_1=(1,0)$ and $e_2=(0,1)$. The non-zero roots of B_2 can be expressed as simple combinations of e_1 and e_2

(1, 0)	e_1
(1, 1)	e_1+e_2

(0, 1)	e_2
(-1, 1)	$-e_1 + e_2$
(-1, 0)	$-e_1$
(-1,-1)	$-e_1 - e_2$
(0,-1)	$-e_2$
(1,-1)	$e_1 - e_2$

The above results can be easily generalized to $B\ell$ where $\ell \geq 2$. The non-zero roots of $B\ell$ can be expressed as simple combinations of orthonormal unit vectors $e_1, e_2, \ldots e_\ell$. They are

$\pm e_i$ ($i=1,\ldots\ldots,\ell$) for shorter roots.

$\pm e_i \pm e_j$ ($i,j=1,\ldots\ldots,\ell$; $i \neq j$) for longer roots.

The number of shorter roots is 2ℓ and the number of longer roots $2\ell(\ell-1)$. Thus, the number of non-zero roots is

$2\ell+2\ell(\ell-1)=2\ell^2$. Hence the order of B_ℓ is given by

$$r = \ell + 2\ell^2 = \ell(2\ell+1).$$

Similar expressions for roots in terms of orthonormal unit vectors can be obtained for each simple Lie algebra. There is, however, one slight modification for $A\ell$. Here one needs to introduce orthonormal unit vectors of $A\ell$ lies in a ℓ-dimensional hyperplane perpendicular to (1,1,.....,1). The nonzero roots of simple Lie algbra, expressed in terms of unit vectors e_λ, are listed in Table 1.

The root diagrams for $\ell=3$ are shown in Fig.1 The corresponding Lie algebra are called A_3, B_3 and C_3, respetively. From these diagrams, one can read the angle between any two roots. Note that for $\ell>3$, the angle 30° is not allowed. Thus G_2 is the only group with this angle. The non-zero roots are most easly obtained from Table 1.

C. Simple Roots and Dynkin Diagrams

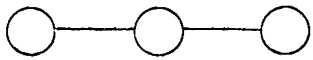
We are now going to define the concept of simple roots which are basic in Dykin's approach. Write the roots in any Cartesian basis. A root is called to be positive if its first non-zero component in that basis is positive. The non-zero roots come in pairs. Consequently, for a group of order r and rank ℓ, there are $(r-\ell)/2$ positive roots.

A simple root is a positive root which cannot be written as the sum of two positive roots. In a simple Lie algebra of rank ℓ there are ℓ simple roots. All the other positive roots are linear combinations of simple roots with integer non-negative coefficients. The set of simple root is usually denoted by Π. The length and angle relation Lieamong the simple roots completely characterize any simple Lie algebra. It can be easily shown that if α, β are simple roots, then $\beta-\alpha$ is not a root. The angle θ between any pair of simple roots satisfies . The allowed angles are 90 , 120 , 135 and 150 . By using the properties of the simple roots and the master formula, we can determine all the roots in terms of the simple roots.

A Dynkin diagram is just a useful diagrammatic notation for writing down the simple roots. The conventions for drawing the Dynkin's diagrams are (1) Each simple root is denoted by a dot; (2) If all simple roots have the same length, each root is designated by an open dot; (3) in case where simple roots come with two different lengths, the longer roots are denoted by open dots and the shorter roots by black dots. (4) The angle between a pair of simple roots, or equivalently the ratio of lengths, is denoted by lines connecting the corresponding dots:

# of lines	Angle	Ratio of lengths
No line	90°	No constraint
One line	120°	1
Two lines	135°	$\sqrt{2}$
Three lines	150°	$\sqrt{3}$

For example, the Dynkin diagram for A =SU(4) is



For the above diagram, one easily obtains that there are 3 simple roots. The angle between α_1 and α_2 is equal to the angle between α_2 and α_3, which is 120° . The angle between α_1 and α_3 is 90° since they are not connected. Note that the following relations hold for the simple roots of SU(4):

$$\alpha_1{}^2 = \alpha_2{}^2 = \alpha_3{}^2 = 1$$
$$\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = -\frac{1}{2} \neq 0$$
$$\alpha_1 \cdot \alpha_3 = 0 \tag{2.11}$$

By generalizing to the case of SU(ℓ+1), we have

$$\alpha_i{}^2 = 1$$
$$\alpha_i \cdot \alpha_{i+1} = -\frac{1}{2}$$
$$\alpha_i \cdot \alpha_j = 0 \qquad \text{if } j \neq i \text{ or } i \pm 1 \tag{2.12}$$

In short,

$$\alpha_i \cdot \alpha_j = \delta_{ij} - \frac{1}{2}\delta_{ij+1} - \frac{1}{2}\delta_{i+1j} \tag{2.13}$$

The Dynkin diagrams for simple Lie algebra are summarized in Fig.2.

From the Dynkin diagrams one can immediately arrive at the following relations: $A_1 \sim B_1 \sim C_1$, $B_2 \sim C_2$, $A_3 \sim D_3$ and $D_2 \sim A_1 \sim + A_1$.

It is conventient to express the simple roots in terms of the orthonormal vectors e_i. Take A_2 for example. There are 6 non-zero roots. The positive roots are e_1-e_2, e_2-e_3, and e_1-e_3. The simple roots are $\alpha_1=e_1-e_2$ and $\alpha_2=e_2-e_3$. In general, the simple roots of A_ℓ are given by e_i-e_{i+1}, $i=1,2,\ldots,\ell$. The simple roots for classical Lie algebras are given in Table 2. Similar expressions can be given for the exceptional Lie algebras.

Given a set of simple roots $\Pi=\{\alpha_1,\alpha_2,\ldots\ldots\alpha_\ell\}$, the Cartan matrix A is defined as the matrix with elements

$$A_{ij}=\frac{2<\alpha_i,\alpha_j>}{<\alpha_j,\alpha_j>} \tag{2.14}$$

Each matrix elemet is an integer. The diagonal elements are all equal to 2. The off-diagonal elements can be only 0, -1, -2, and -3.

The Cartan matrix, together with the master formula for roots, suffices to determine all roots for a given simple Lie algebra. Let β be a postive root of level n:

$$\beta=\sum_i k_i\,\alpha_i \qquad k_i>0\ ,\ \alpha_i\in\Pi \tag{2.15}$$

where $n=\sum_i k_i$ (2.15)

Consider the string of roots for β

$$\beta-q\alpha_i\ ,\cdots\cdots,\ \beta-\alpha_i\ ,\ \beta\ ,\ \beta+\alpha_i\ ,\cdots\cdots,\ \beta+p\alpha_i$$

Since all the roots through the n-th level are known, the number q is known. The number p may be computed by using the master formula

$$-(p-q)=\frac{2<\beta,\alpha_i>}{<\alpha_i,\alpha_i>} \tag{2.16}$$

By using $\beta=\sum k_i\,\alpha_j$, one obtains

$$p=q-\sum_j k_j A_{ji} \tag{2.17}$$

Thus, $\beta+\alpha_i$ is a root if p>0. There is a highest root α_h

such that $\alpha_h+\alpha_i$ cannot be a root for any simple root α_i. Thus, this procedure stops at the highest root.

D.Weights

Let m> be an eigenvector of the diagonal generators $H_i(i=1,2,....\ell)$i.e.

$$H_i|m\rangle = m_i|m\rangle \qquad (2.18)$$

The vector $\vec{m}$ with compnent m_i is called a weight vector. A weight vector is said to be positive if its first non-zero component is positive. Consider two weights $\vec{m}$ and $\vec{m}'$. The weight $\vec{m}$ is said to be larger than the other weight $\vec{m}'$ $(\vec{m}>\vec{m}')$ if $\vec{m}-\vec{m}'$ is positive. The highest weight in an irreducible representation is a weight greater than any of the others. The irreducible representation of a Lie algebra is completely determined by its highest weight.

The Dynkin components of a weight or root Λ is a of integer corrdinates $a_1\ a_2...a_l$) defined by

$$a_i = \frac{2<\Lambda, \alpha_i>}{<\alpha_i,\alpha_i>} \qquad (2.19)$$

By this definition, the Dynkin components of the simple root α_i is simply given by the i-th row of the Cartan matrix A. Because the simple roots are not orthonormal, the scalar product of $\Lambda=(a_1\ a_2...\ a_\ell)$ and $\Lambda'=(a'_1\ a'_2...a'_\ell)$ involves a metric tensor G_{ij}:

$$<\Lambda',\Lambda>= \sum_{i,j}\tilde{a}'_i\ G_{ij}\,a_j \qquad (2.20)$$

where $$G_{ij}=(A^{-1})_{ij}\frac{<\alpha_j,\alpha_j>}{2} \qquad (2.21)$$

The coordinates $[\tilde{a}_1\ \tilde{a}_2...\tilde{a}\ ]$ in dual bais is related to DYnkin coordinates $(a_1\ a_2\ ...a_\ell)$ by

$$\tilde{a}_i = \sum_j G_{ij} a_j \tag{2.22}$$

Then the scalar product is simply

$$< \Lambda', \Lambda > = \sum_i \tilde{a}_i' a_i \tag{2.23}$$

The complete set of weights for each irreducible representation can be derived from the highest weight and the simple roots. Consider a string of weights

$$\Lambda + p_i \alpha_i , \cdots\cdots, \Lambda + \alpha_i , \Lambda , \Lambda - \alpha_i , \cdots , \Lambda - q_i \alpha_i$$

We can obtain the master formula:

$$q_i = p_i + \frac{2 < \Lambda , \alpha_i >}{< \alpha_i , \alpha_i >} = p_i + a_i \tag{2.24}$$

If $q_i > o$, then $\Lambda - \alpha_i$ is a weight. By repeated use of the above master formula, one will obtain all the weights belonging to this representation. One may refer to Ref.10 for a detailed discussion of this procedure.

III. Dimensions and Indices

In this section we shall calculate the dimensions and indices for the irreducible representations of the SU(5), SO(10) and E_6. The dimensionality for an irreducible representation Γ of a simple Lie group is given by the famous Weyl-Freduenthal formula:

$$\dim \Gamma = \prod_{\alpha \in \Sigma^+} \frac{< \Lambda + \delta , \alpha >}{< \delta , \alpha >} \tag{3.1}$$

where Σ^+ = the set of positive roots,

Λ = the highest weight of the representation Γ,

$\delta = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$ = half-sum of all positive roots.

It can be proved that the Dynkin coefficients of δ are all unity, i.e. $\delta=(1,1,..,1)$. The positive roots α can be expressed in terms of the simple roots $\alpha_j (j=1,2,...,\ell)$:

$$\alpha = \sum k_j^{\alpha} \ \alpha_j \tag{3.2}$$

Then the dimensionality formula can be put into the following covenient form[10]:

$$\dim \ \Gamma = \prod_{\alpha\in\Sigma^+} \frac{\sum_i (m_i\ k_i^{\alpha} + k_i^{\alpha})<\alpha_i , \alpha_i>}{\sum_i k_i^{\alpha}<\alpha_i , \alpha_i>} \tag{3.3}$$

For the groups A_ℓ, D_ℓ, E_6, E_7 and E_8 which have equal length of roots, this formula further simplifies to

$$\dim \ \Gamma = \prod_{\alpha\in\Sigma^+} \frac{\sum_i (m_i\ k_i^{\alpha} + k_i^{\alpha})}{\sum_i k_i^{\alpha}} \tag{3.4}$$

where $(m_1, m_2, ...m_\ell)$ is the DYnkin coefficeints of the highest weight. Since the algebras of SU(5), SO(10) and E6 have simple roots of equal lengths, Eq.(3.4) will be used in our calculations.

TO calculate the dimensionality of an irreducible representation of a group by using Eq(3.4), we need to know the positive roots for each group. The positive roots can be found by Dynkin's method outlined in Sec II.

The group SO(10) is a rank 5 group with 45 generators. Thus it has 45-5=40 non-zero roots of which 20 are positive. The coefficients k_λ^{α} for the positive roots of SO(10) are listed in Table 3.

The group E_6 is a rank 6 group with 78 generators. Thus is has 78-6=72 non-zero roots of which 36 are positive. The coefficient k_λ^{α} for these 36 positive roots of E_6 are listed in Table 4.

In the case of Su(5), it is a simple matter to show that Eq.(3.4) reproduces the following familiar expression:

$$\dim \Gamma = \frac{1}{2!3!4!}(m_1+1)(m_1+m_2+2)(m_1+m_2+m_3+3)(m_1+m_2+m_3+m_4+4)$$
$$\cdot(m_2+1)(m_2+m_3+2)(m_2+m_3+m_4+3)$$
$$\cdot(m_3+1)(m_3+m_4+2)$$
$$\cdot(m_4+1) \qquad (3.5)$$

The dimensionalities for the irreducible representations of SU(5), SO(10) and E_6 are listed in Table 5-7. Although the tables of dimensions have been presented earlier by W. Mckay and J. Patera in Ref.13, our tables are more complete for SU(5),SO(10) and E_6. The highest dimension listed is 51975 for SU(5), 7399392 for SO(10) and 9808465095 for E6.

The index I_2 of an irreducible representation with the highest weight Λ is defined as[12,13]

$$I_2 = \frac{N(\Lambda)}{N(adj)} C(\Lambda) \qquad (3.6)$$

where $C(\Lambda) = \langle \Lambda, \Lambda + 2\delta \rangle$

is the eigenvalue of the second-order Casimir invarint

$$C = f^i_{jk} f^j_{il} X^k X^l$$

acting on the state with the highest weight Λ, and N(adj) is the dimension of the adjoint representation. The secon d-order indices I_2 introduced above has many applications in physice. For example, it simplifies the decomposition of tensor products of representations. The values of index I_2 are listed in the third column of Table 5-7.

The irreducible representations of a group may fall into several different categories distinguushed by their congruence classes. Each congruence class is characterized by a value of the congruence number c. The group E_6 has three congruence classes of representations. Thus the congruence number of E_6 coincides with triality. It is given by

$$c = a_1 - a_2 + a_4 - a_5 \quad (\text{mod } 3)$$

In addition to the Dynkin label, dimension and index, the triality of E6 irreducible representations are very useful in distinguishindg different representations. For example, the lowest dimensional nontrivial representation of E_6 is 27-dimensional. Howerer, there are two different representations with this dimension: one with highest weight Λ=(100000) and the other with Λ=(000010). The former has triality one and is denoted as 27 while the latter has triality 2 and is denoted as 27*. Similarly the irreducible representaton with Λ=(000100) is denoted as 351, while the representation with =(010000) is denoted as 351*. The properties of triality are also very useful in the decomposition of tensor product of irreducible representations and in the computation of branching rules. A disusion of graund unified theories based on E_6 can be found in Ref.14 and Ref.15.

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Table Captions

Table 1. Roots in terms of orthonormal vectors.

Table 2. Simple roots in terms of orthonormal vectors.

Table 3. The coefficients k_i^α for the positive roots of SO(10).

Table 4. The coefficients k_i^α for the positive roots of E_6.

Table 5. Dimensions and indices for SU(5) irreducible representations.

Table 6. Dimensions and indices for SO(10) irreducible representations.

Table 7. Dimensions and indices for E_6 irreducible representations.

Table 1. Roots in terms of orthonormal vectors

Type	Roots		Order
A_ℓ	$e_i - e_j$	$i,j=1,\ldots,\ell+1$, $i \neq j$	$\ell(\ell+2)$
B_ℓ	$\pm e_i$	$i=1,\ldots,\ell$	$\ell(2\ell+1)$
	$\pm e_i \pm e_j$	$i,j=1,\ldots,\ell$, $i \neq j$	
C_ℓ	$\pm 2e_i$	$i=1,\ldots,\ell$	$\ell(2\ell+1)$
	$\pm e_i \pm e_j$	$i,j=1,\ldots,\ell$, $i \neq j$	
D_ℓ	$\pm e_i \pm e_j$	$i,j=1,\ldots,\ell$, $i \neq j$	$\ell(2\ell-1)$
G_2	$e_i - e_j$	$i,j=1,2,3$, $i \neq j$	14
	$\pm(3e_i - e_1 - e_2 - e_3)$	$i=1,2,3$	
F_4	the roots of B_4 plus $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$		52
E_6	The roots of A_5 plus $\pm\sqrt{2}\, e_7$; $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6) \pm \frac{e_7}{\sqrt{2}}$ with 3 positive signs and 3 negative signs in the first term		78
E_7	The roots of A_7 plus $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$ with 4 positive signs and 4 negative signs		133
E_8	The roots of D_8 plus $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$ with even number of positive signs		248

Table 2. Simple roots in terms of orthonormal vectors

Type	Roots	Positive roots	Simple roots
A_ℓ	$e_i - e_j \ (i \neq j)$	$e_i - e_j \ (i < j)$	$e_i - e_{i+1}$
			$i = 1, 2, \ldots, \ell$
B_ℓ	$e_i \pm e_j \ (i \neq j)$	$e_i \pm e_j \ (i < j)$	$e_i - e_{i+1} \ (i = 1, \ldots, \ell-1)$
	$\pm e_i$	e_i	e_ℓ
C_ℓ	$\pm e_i \pm e_j \ (i \neq j)$	$e_i \pm e_j \ (i < j)$	$e_i - e_{i+1} \ (i = 1, \ldots, \ell-1)$
	$\pm 2e_i$	$2e_i$	$2e_\ell$
D_ℓ	$\pm e_i \pm e_j \ (i \neq j)$	$e_i \pm e_j \ (i < j)$	$e_i - e_{i+1} \ (1 = 1, \ldots, \ell-1)$
			$e_{\ell-1} + e_\ell$

Table 3.

	k_1^α	k_2^α	k_3^α	k_4^α	k_5^α
$\alpha=1$	1	0	0	0	0
$\alpha=2$	0	1	0	0	0
$\alpha=3$	0	0	1	0	0
$\alpha=4$	0	0	0	1	0
$\alpha=5$	0	0	0	0	1
$\alpha=6$	1	1	0	0	0
$\alpha=7$	0	1	1	0	0
$\alpha=8$	0	0	1	1	0
$\alpha=9$	0	0	1	0	1
$\alpha=10$	1	1	1	0	0
$\alpha=11$	0	1	1	1	0
$\alpha=12$	0	1	1	0	1
$\alpha=13$	0	0	1	1	1
$\alpha=14$	1	1	1	1	0
$\alpha=15$	0	1	1	1	1
$\alpha=16$	1	1	1	0	1
$\alpha=17$	1	1	1	1	1
$\alpha=18$	0	1	2	1	1
$\alpha=19$	1	1	2	1	1
$\alpha=20$	1	2	2	1	1

Table. 4.

	k_1^α	k_2^α	k_3^α	k_4^α	k_5^α	k_6^α
$\alpha=1$	1	0	0	0	0	0
$\alpha=2$	0	1	0	0	0	0
$\alpha=3$	0	0	1	0	0	0
$\alpha=4$	0	0	0	1	0	0
$\alpha=5$	0	0	0	0	1	0
$\alpha=6$	0	0	0	0	0	1
$\alpha=7$	1	1	0	0	0	0
$\alpha=8$	0	1	1	0	0	0
$\alpha=9$	0	0	1	1	0	0
$\alpha=10$	0	0	0	1	1	0
$\alpha=11$	0	0	1	0	0	1
$\alpha=12$	1	1	1	0	0	0
$\alpha=13$	0	1	1	1	0	0
$\alpha=14$	0	1	1	0	0	1
$\alpha=15$	0	0	1	1	0	1
$\alpha=16$	0	0	1	1	1	0
$\alpha=17$	1	1	1	1	0	0
$\alpha=18$	1	1	1	0	0	1
$\alpha=19$	0	1	1	1	1	0
$\alpha=20$	0	1	1	1	0	1
$\alpha=21$	0	0	1	1	1	1
$\alpha=22$	1	1	1	1	1	0
$\alpha=23$	1	1	1	1	0	1
$\alpha=24$	0	1	1	1	1	1
$\alpha=25$	0	1	2	1	0	1

$\alpha=26$	1	1	1	1	1	1
$\alpha=27$	1	1	2	1	0	1
$\alpha=28$	0	1	2	1	1	1
$\alpha=29$	1	1	2	1	1	1
$\alpha=30$	1	2	2	1	0	1
$\alpha=31$	0	1	2	2	1	1
$\alpha=32$	1	2	2	1	1	1
$\alpha=33$	1	1	2	2	1	1
$\alpha=34$	1	2	2	2	1	1
$\alpha=35$	1	2	3	2	1	1
$\alpha=36$	1	2	3	2	1	2

Table 5 : Dimension and Index for SU(5) irreps

Dynkin label	Dimension	Index	Dynkin label	Dimension	Index	Dynkin label	Dimension	Index
(0 0 0 0)	1	0	(0 0 0 1)	5	1	(1 0 0 0)	5	1
(0 0 1 0)	10	3	(0 1 0 0)	10	3	(0 0 0 2)	15	7
(2 0 0 0)	15	7	(1 0 0 1)	24	10	(0 0 0 3)	35	28
(3 0 0 0)	35	28	(0 0 1 1)	40	22	(1 1 0 0)	40	22
(1 0 1 0)	45	24	(0 1 0 1)	45	24	(0 0 2 0)	50	35
(0 2 0 0)	50	35	(1 0 0 2)	70	49	(0 0 0 4)	70	84
(2 0 0 1)	70	49	(4 0 0 0)	70	84	(0 1 1 0)	75	50
(0 0 1 2)	105	91	(2 1 0 0)	105	91	(0 0 0 5)	126	210
(5 0 0 0)	126	210	(0 1 0 2)	126	105	(2 0 1 0)	126	105
(1 0 0 3)	160	168	(3 0 0 1)	160	168	(0 0 3 0)	175	210
(0 3 0 0)	175	210	(0 0 2 1)	175	175	(1 0 1 1)	175	140
(1 1 0 1)	175	140	(1 2 0 0)	175	175	(2 0 0 2)	200	200
(0 0 0 6)	210	462	(6 0 0 0)	210	462	(0 2 0 1)	210	203
(1 0 2 0)	210	203	(0 0 1 3)	224	280	(3 1 0 0)	224	280
(0 1 1 1)	280	266	(1 1 1 0)	280	266	(0 1 0 3)	280	336
(3 0 1 0)	280	336	(0 1 2 0)	315	357	(0 2 1 0)	315	357
(1 0 0 4)	315	462	(4 0 0 1)	315	462	(0 0 0 7)	330	924
(7 0 0 0)	330	924	(0 0 1 4)	420	714	(4 1 0 0)	420	714
(0 0 2 2)	420	574	(2 2 0 0)	420	574	(1 0 1 2)	450	510
(2 0 0 3)	450	615	(2 1 0 1)	450	510	(3 0 0 2)	450	615
(1 1 0 2)	480	536	(2 0 1 1)	480	536	(0 0 4 0)	490	882
(0 4 0 0)	490	882	(0 0 0 8)	495	1716	(8 0 0 0)	495	1716
(0 1 0 4)	540	882	(4 0 1 0)	540	882	(1 0 0 5)	560	1092
(5 0 0 1)	560	1092	(0 0 3 1)	560	868	(0 2 0 2)	560	728
(1 3 0 0)	560	868	(2 0 2 0)	560	728	(0 1 1 2)	700	910
(0 3 0 1)	700	1050	(1 0 3 0)	700	1050	(2 1 1 0)	700	910
(0 0 0 9)	715	3003	(9 0 0 0)	715	3003	(0 0 1 5)	720	1596
(1 0 2 1)	720	924	(1 2 0 1)	720	924	(5 1 0 0)	720	1596
(0 0 2 3)	840	1512	(3 2 0 0)	840	1512	(2 0 0 4)	875	1575
(4 0 0 2)	875	1575	(1 0 0 6)	924	2310	(6 0 0 1)	924	2310
(0 1 0 5)	945	2016	(5 0 1 0)	945	2016	(1 0 1 3)	945	1449
(3 1 0 1)	945	1449	(0 1 3 0)	980	1666	(0 3 1 0)	980	1666
(3 0 0 3)	1000	1750	(0 0 0 10)	1001	5005	(10 0 0 0)	1001	5005
(1 1 1 1)	1024	1280	(0 1 2 1)	1050	1540	(1 2 1 0)	1050	1540
(1 1 0 3)	1050	1575	(3 0 1 1)	1050	1575	(0 2 1 1)	1120	1624
(1 1 2 0)	1120	1624	(0 0 1 6)	1155	3234	(6 1 0 0)	1155	3234
(0 0 5 0)	1176	2940	(0 5 0 0)	1176	2940	(0 2 2 0)	1176	1960
(0 2 0 3)	1200	2040	(3 0 2 0)	1200	2040	(2 0 1 2)	1215	1782
(2 1 0 2)	1215	1782	(0 0 3 2)	1260	2478	(2 3 0 0)	1260	2478
(0 0 0 11)	1365	8008	(11 0 0 0)	1365	8008	(0 1 1 3)	1440	2472
(1 0 0 7)	1440	4488	(3 1 1 0)	1440	2472	(7 0 0 1)	1440	4488
(0 0 4 1)	1470	3234	(1 4 0 0)	1470	3234	(4 2 0 0)	1500	3450
(0 0 2 4)	1500	3450	(0 1 0 6)	1540	4158	(2 0 0 5)	1540	3542
(5 0 0 2)	1540	3542	(6 0 1 0)	1540	4158	(1 0 2 2)	1701	2835
(2 2 0 1)	1701	2835	(1 0 1 4)	1750	3500	(4 1 0 1)	1750	3500
(0 0 1 7)	1760	6072	(7 1 0 0)	1760	6072	(0 3 0 2)	1800	3360
(2 0 3 0)	1800	3360	(0 0 0 12)	1820	12376	(12 0 0 0)	1820	12376
(0 4 0 1)	1890	4032	(1 0 4 0)	1890	4032	(1 2 0 2)	1890	3087
(2 0 2 1)	1890	3087	(3 0 0 4)	1925	4235	(4 0 0 3)	1925	4235
(1 1 0 4)	2000	3900	(4 0 1 1)	2000	3900	(1 0 0 8)	2145	8151
(8 0 0 1)	2145	8151	(1 0 3 1)	2205	4116	(1 3 0 1)	2205	4116
(0 2 0 4)	2250	4875	(4 0 2 0)	2250	4875	(0 1 0 7)	2376	7920
(7 0 1 0)	2376	7920	(0 0 0 13)	2380	18564	(13 0 0 0)	2380	18564

Table 6 : Dimension and Index for SO(10) irreps

Dynkin label	Dimension	Index I(2)	Dynkin label	Dimension	Index I(2)
(0 0 0 0 0)	1	0	(1 0 0 0 0)	10	2
(0 0 0 0 1)	16	4	(0 0 0 1 0)	16	4
(0 1 0 0 0)	45	16	(2 0 0 0 0)	54	24
(0 0 1 0 0)	120	56	(0 0 0 0 2)	126	70
(0 0 0 2 0)	126	70	(1 0 0 0 1)	144	68
(1 0 C 1 0)	144	68	(3 0 0 0 0)	210	154
(0 0 0 1 1)	210	112	(1 1 0 0 0)	320	192
(0 1 0 0 1)	560	364	(0 1 0 1 0)	560	364
(4 0 0 0 0)	660	704	(0 0 0 0 3)	672	616
(0 0 0 3 0)	672	616	(2 0 0 0 1)	720	532
(2 0 0 1 0)	720	532	(0 2 0 0 0)	770	616
(1 0 1 0 0)	945	672	(1 0 0 0 2)	1050	840
(1 0 0 2 0)	1050	840	(0 0 1 0 1)	1200	940
(0 0 1 1 0)	1200	940	(2 1 0 0 0)	1386	1232
(0 0 0 1 2)	1440	1256	(0 0 0 2 1)	1440	1256
(1 0 0 1 1)	1728	1344	(5 0 0 0 0)	1782	2574
(3 0 0 0 1)	2640	2772	(3 0 0 1 0)	2640	2772
(0 0 0 0 4)	2772	3696	(0 0 0 4 0)	2772	3696
(0 1 1 0 0)	2970	2706	(0 1 0 0 2)	3696	3696
(0 1 0 2 0)	3696	3696	(1 1 0 0 1)	3696	3388
(1 1 0 1 0)	3696	3388	(0 0 2 0 0)	4125	4400
(6 0 0 0 0)	4290	8008	(2 0 1 0 0)	4312	4312
(1 2 0 0 0)	4410	4802	(3 1 0 0 0)	4608	5632
(2 0 0 0 2)	4950	5390	(2 0 0 2 0)	4950	5390
(1 0 0 0 3)	5280	6248	(1 0 0 3 0)	5280	6248
(0 1 0 1 1)	5940	5808	(0 0 0 1 3)	6930	8778
(0 0 0 3 1)	6930	8778	(0 0 1 0 2)	6930	8008
(0 0 1 2 0)	6930	8008	(0 3 0 0 0)	7644	10192
(4 0 0 0 1)	7920	11132	(4 0 0 1 0)	7920	11132
(0 2 0 1 0)	8064	9184	(0 2 0 0 1)	8064	9184
(2 0 0 1 1)	8085	8624	(1 0 1 1 0)	8800	9240
(1 0 1 0 1)	8800	9240	(0 0 0 2 2)	8910	11088
(7 0 0 0 0)	9438	22022	(0 0 0 0 5)	9504	17160
(0 0 0 5 0)	9504	17160	(0 0 1 1 1)	10560	11968
(1 0 0 1 2)	11088	12628	(1 0 0 2 1)	11088	12628
(4 1 0 0 0)	12870	20592	(3 0 1 0 0)	14784	19712
(2 1 0 0 1)	15120	18564	(2 1 0 1 0)	15120	18564
(2 2 0 0 0)	16380	23296	(0 1 0 0 3)	17280	24288
(0 1 0 3 0)	17280	24288	(3 0 0 0 2)	17325	24640
(3 0 0 2 0)	17325	24640	(1 1 1 0 0)	17920	21504
(8 0 0 0 0)	19305	54912	(5 0 0 1 0)	20592	37180
(5 0 0 0 1)	20592	37180	(1 0 0 4 0)	20790	33726
(1 0 0 0 4)	20790	33726	(1 1 0 0 2)	23040	29696
(1 1 0 2 0)	23040	29696	(2 0 0 0 3)	23760	35508
(2 0 0 3 0)	23760	35508	(0 1 1 1 0)	25200	32060
(0 1 1 0 1)	25200	32060	(0 0 0 1 4)	26400	45320
(0 0 0 4 1)	26400	45320	(1 0 2 0 0)	27720	37576
(3 0 0 1 1)	28160	39424	(0 0 0 0 6)	28314	66066
(0 0 0 6 0)	28314	66066	(0 0 1 0 3)	29568	46816
(0 0 1 3 0)	29568	46816	(0 0 2 1 0)	30800	44660
(0 0 2 0 1)	30800	44660	(5 1 0 0 0)	31680	64064
(0 2 1 0 0)	34398	49686	(0 1 0 1 2)	34992	47628
(0 1 0 2 1)	34992	47628	(1 1 0 1 1)	36750	46550

Table 7 : Dimension and Index for E_6 irreps

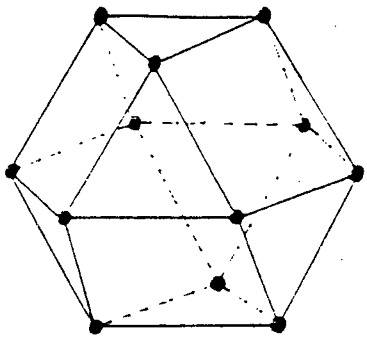
Dynkin label	Dimension	Index I(2)	Triality	Dynkin label	Dimension	Index I(2)	Triality
(0 0 0 0 0 0)	1	0	0	(0 0 0 0 1 0)	27	6	2
(1 0 0 0 0 0)	27	6	1	(0 0 0 0 0 1)	78	24	0
(0 0 0 0 2 0)	351	168	1	(2 0 0 0 0 0)	351	168	2
(0 0 0 1 0 0)	351	150	1	(0 1 0 0 0 0)	351	150	2
(1 0 0 0 1 0)	650	300	0	(1 0 0 0 0 1)	1728	960	1
(0 0 0 0 1 1)	1728	960	2	(0 0 0 0 0 2)	2430	1620	0
(0 0 1 0 0 0)	2925	1800	0	(0 0 0 0 3 0)	3003	2310	0
(3 0 0 0 0 0)	3003	2310	0	(1 1 0 0 0 0)	5824	4032	0
(0 0 0 1 1 0)	5824	4032	0	(1 0 0 1 0 0)	7371	5040	2
(0 1 0 0 1 0)	7371	5040	1	(1 0 0 0 2 0)	7722	5676	2
(2 0 0 0 1 0)	7722	5676	1	(0 0 0 1 0 1)	17550	13800	1
(0 1 0 0 0 1)	17550	13800	2	(0 0 0 0 2 1)	19305	16170	1
(0 0 0 0 4 0)	19305	21120	2	(4 0 0 0 0 0)	19305	21120	1
(2 0 0 0 0 1)	19305	16170	2	(0 0 0 2 0 0)	34398	32340	2
(0 2 0 0 0 0)	34398	32340	1	(1 0 0 0 1 1)	34749	28512	0
(0 0 0 0 0 3)	43758	47124	0	(0 0 0 0 1 2)	46332	43560	2
(1 0 0 0 0 2)	46332	43560	1	(0 0 1 0 1 0)	51975	46200	2
(1 0 1 0 0 0)	51975	46200	1	(0 0 0 1 2 0)	54054	53592	2
(2 1 0 0 0 0)	54054	53592	1	(1 0 0 0 3 0)	61425	64050	1
(3 0 0 0 1 0)	61425	64050	2	(0 1 0 1 0 0)	70070	64680	0
(0 1 0 0 2 0)	78975	76950	0	(2 0 0 1 0 0)	78975	76950	0
(2 0 0 0 2 0)	85293	87480	0	(0 0 0 0 5 0)	100386	145860	1
(5 0 0 0 0 0)	100386	145860	2	(0 0 1 0 0 1)	105600	105600	0
(1 1 0 0 1 0)	112320	108480	2	(1 0 0 1 1 0)	112320	108480	1
(0 0 0 0 3 1)	146432	168960	0	(3 0 0 0 0 1)	146432	168960	0
(0 0 0 1 1 1)	252252	271656	0	(1 1 0 0 0 1)	252252	271656	0
(0 1 0 0 1 1)	314496	336000	1	(1 0 0 1 0 1)	314496	336000	2
(0 0 0 1 3 0)	359424	476160	1	(1 0 0 0 2 1)	359424	402432	2
(2 0 0 0 1 1)	359424	402432	1	(3 1 0 0 0 0)	359424	476160	2
(1 0 0 0 4 0)	371800	514800	0	(4 0 0 0 1 0)	371800	514800	0
(0 0 1 1 0 0)	386100	442200	1	(0 1 1 0 0 0)	386100	442200	2
(0 1 0 0 0 2)	393822	471240	2	(0 0 0 1 0 2)	393822	471240	1
(0 0 0 2 1 0)	412776	515088	1	(1 2 0 0 0 0)	412776	515088	2
(0 0 0 0 6 0)	442442	816816	0	(6 0 0 0 0 0)	442442	816816	0
(0 0 0 0 2 2)	459459	573342	1	(2 0 0 0 0 2)	459459	573342	2
(0 0 1 0 2 0)	494208	591360	1	(2 0 1 0 0 0)	494208	591360	2
(0 0 0 0 0 4)	537966	827640	0	(0 1 0 0 3 0)	579150	752400	2
(3 0 0 1 0 0)	579150	752400	1	(0 2 0 0 1 0)	600600	739200	0
(1 0 0 2 0 0)	600600	739200	0	(2 0 0 0 3 0)	638820	862680	2
(3 0 0 0 2 0)	638820	862680	1	(0 0 0 0 1 3)	741312	1020096	2
(1 0 0 0 0 3)	741312	1020096	1	(1 0 0 0 1 2)	812175	999600	0
(1 0 1 0 1 0)	852930	1006020	0	(4 0 0 0 0 1)	853281	1283568	1
(0 0 0 0 4 1)	853281	1283568	2	(1 1 0 1 0 0)	967680	1182720	1
(0 1 0 1 1 0)	967680	1182720	2	(1 0 0 1 2 0)	972972	1247400	0
(2 1 0 0 1 0)	972972	1247400	0	(2 0 0 1 1 0)	1123200	1430400	2
(1 1 0 0 2 0)	1123200	1430400	1	(0 0 0 2 0 1)	1253070	1692180	2
(0 2 0 0 0 1)	1253070	1692180	1	(0 0 2 0 0 0)	1337050	1851300	0
(0 0 0 3 0 0)	1559376	2399040	0	(0 3 0 0 0 0)	1559376	2399040	0
(0 0 1 0 1 1)	1640925	2131800	2	(1 0 1 0 0 1)	1640925	2131800	1
(0 0 0 0 7 0)	1706562	3879876	2	(7 0 0 0 0 0)	1706562	3879876	1
(1 0 0 0 5 0)	1837836	3235848	2	(5 0 0 0 1 0)	1837836	3235848	1
(0 0 0 1 4 0)	1896180	3208920	0	(4 1 0 0 0 0)	1896180	3208920	0
(0 0 1 0 0 2)	1911195	2744280	0	(0 0 0 1 2 1)	2088450	2927400	2

Figure Captions

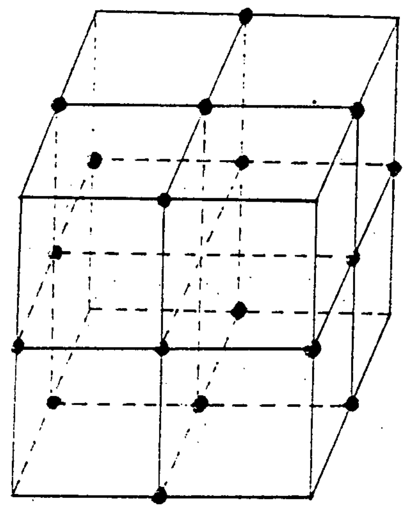
Fig. 1. Root diagrams of $\ell=3$ Lie algebras:

(a)A_3 (b)B_3 and (c)C_3.

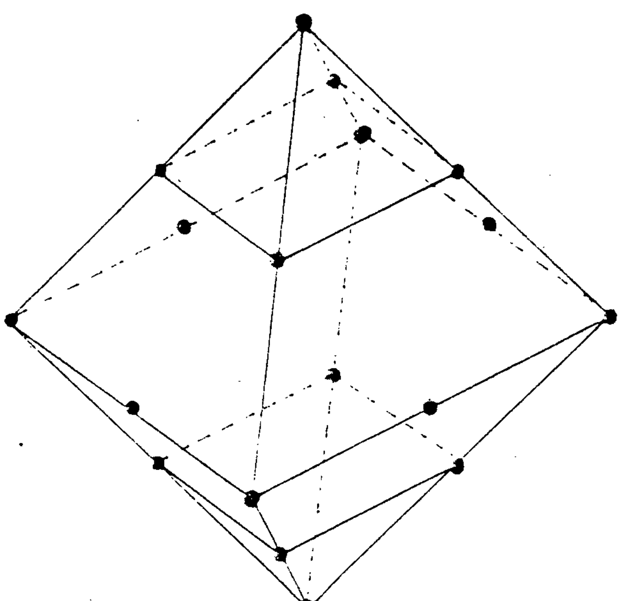
Fig. 2. Dynkin diagrams of simple Lie algebras.



(a)



(b)



(c)

Fig. 1.

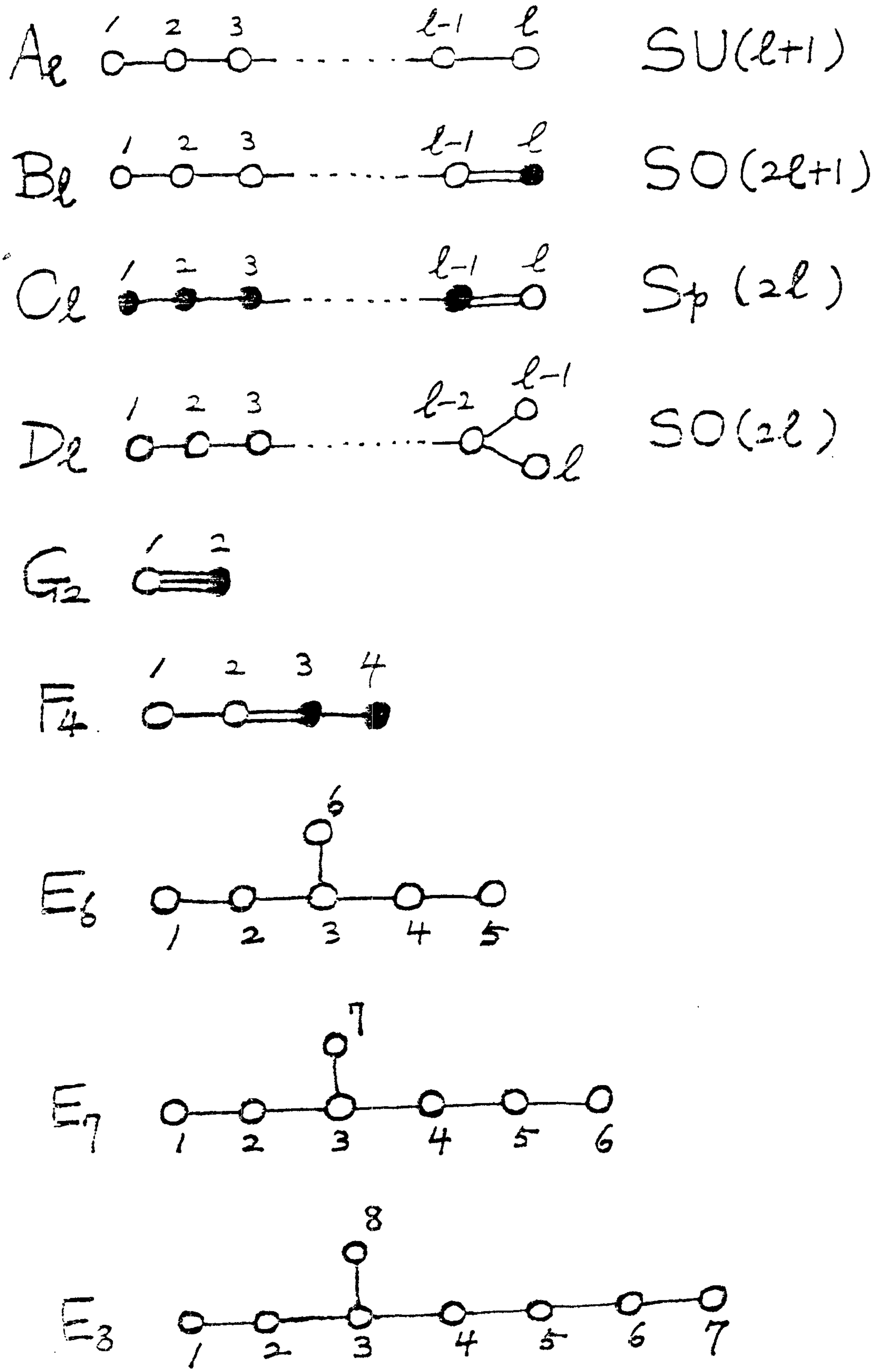

Aℓ
1 2 3 ℓ-1 ℓ
SU(ℓ+1)
Bℓ
1 2 3 ℓ-1 ℓ
SO(2ℓ+1)
Cℓ
1 2 3 ℓ-1 ℓ
Sp(2ℓ)
Dℓ
1 2 3 ℓ-2 ℓ-1 ℓ
SO(2ℓ)
G2
1 2
F4
1 2 3 4
E6
1 2 3 4 5 6
E7
1 2 3 4 5 6 7
E8
1 2 3 4 5 6 7 8

Fig. 2.

李群及李代數在大統一理論中之應用

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摘要：

首先，我們將基本粒子物理中，研究大統一理論所需的群論方法及公式，做一番整理的工作。我們將重點放在應用俄國數學家丁肯(Dynkin)的方法上，因為此方法對於各種簡單李群皆可適用，而不限於某一特定的李群。接著我們計算了SU(5)、SO(10)及E_6等李群各表象之維數和指數，並把結果列表，以供研究大統一理論之參考。