4 Some Relations of The Numbers of Generators of I and I-Corner-Elements

If $I = (X^{a_1}Y^{b_1}, X^{a_2}Y^{b_2}, \dots, X^{a_n}Y^{b_n})$ is a monomial ideal in $k[X,Y]$ with $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$, then by $[HRS, (3.13)]$ the I-corner-elements are $X^{a_2-1}Y^{b_1-1}, X^{a_3-1}Y^{b_2-1}, \dots, X^{a_n-1}Y^{b_{n-1}-1}$. Hence if I is a monomial ideal in $k[X,Y]$ with $\upsilon(I) \geq 2$, then I has corner-elements. Thus, it is natural to ask that for $R = k[X_1, \dots, X_d]$ with $d \geq 3$, whether there exists an integer n such that if I is a monomial ideal in R with $\upsilon(I) \geq n$ then I has at least one corner-element. By Remark 2.2.6, we know that $n \geq d$ if such n exists.

4.1 A case where I has no corner-elements

At first, we are interested in finding an integer n such that if I is a monomial ideal in $k[X,Y,Z]$ with $\upsilon(I) \geq n$, then I has at least one corner-element. But unfortunately (or fortunately ?), before we find such an integer n, we have found, for arbitrary $t \geq 3$, an ideal I which has no corner-elements and $\upsilon(I) = t$.

Lemma 4.1.1 *Let $I = (X^{a_1}Y^{b_1}Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \dots, X^{a_t}Y^{b_t}Z^{c_t})$ be a monomial ideal in $k[X,Y,Z]$ with $t \geq 3$, $a_1 \leq a_2 \leq \cdots \leq a_t$, $b_1 \leq b_2 \leq \cdots \leq b_t$, and $c_1 \geq c_2 \geq \cdots \geq c_t$. Then I has no corner-elements. In particular, if $a_1 < a_2 < \cdots < a_t$, $b_1 < b_2 < \cdots < b_t$ and $c_1 > c_2 > \cdots > c_t$, then $\upsilon(I) = t$ and I has no corner-elements.*

Proof. Note that

$$\begin{aligned} I &= (X^{a_1}Y^{b_1}Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \dots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &= (X^{a_1}, X^{a_2}Y^{b_2}Z^{c_2}, \dots, X^{a_t}Y^{b_t}Z^{c_t}) \cap (Y^{b_1}, X^{a_2}Y^{b_2}Z^{c_2}, \dots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &\quad \cap (Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \dots, X^{a_t}Y^{b_t}Z^{c_t}) \end{aligned}$$

For each $i = 2, \dots, t$, since $a_i \geq a_1$, $X^{a_i}Y^{b_i}Z^{c_i} \in (X^{a_1})$. Hence

$$(X^{a_1}, X^{a_2}Y^{b_2}Z^{c_2}, \dots, X^{a_t}Y^{b_t}Z^{c_t}) = (X^{a_1}).$$

Similarly, for each $i = 2, \ldots, t$, since $b_i \geq b_1$, $X^{a_i}Y^{b_i}Z^{c_i} \in (Y^{b_1})$. Hence

$$(Y^{b_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) = (Y^{b_1}).$$

Therefore,

$$\begin{aligned} I &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &\quad \cap (Z^{c_1}, Y^{b_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &\quad \cap (Z^{c_1}, Z^{c_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}). \end{aligned}$$

For each $i = 3, \ldots, t$, since $a_i \geq a_2$, $X^{a_i}Y^{b_i}Z^{c_i} \in (X^{a_2}) \subseteq (Z^{c_1}, X^{a_2})$. Hence

$$(Z^{c_1}, X^{a_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) = (Z^{c_1}, X^{a_2}).$$

And for each $i = 3, \ldots, t$, since $b_i \geq b_2$, $X^{a_i}Y^{b_i}Z^{c_i} \in (Y^{b_2}) \subseteq (Z^{c_1}, Y^{b_2})$. Hence

$$(Z^{c_1}, Y^{b_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) = (Z^{c_1}, Y^{b_2}).$$

Moreover, since $c_1 \geq c_2$, we get

$$I = (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap (Z^{c_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).$$

Similarly as above, we have

$$\begin{aligned} I &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap (Z^{c_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap (Z^{c_2}, X^{a_3}) \cap (Z^{c_2}, Y^{b_3}) \\ &\quad \cap (Z^{c_3}, X^{a_4}Y^{b_4}Z^{c_4}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &= \cdots \\ &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{i-1}}, X^{a_i}) \cap (Z^{c_{i-1}}, Y^{b_i}) \\ &\quad \cap (Z^{c_i}, X^{a_{i+1}}Y^{b_{i+1}}Z^{c_{i+1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\ &= \cdots \\ &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{t-2}}, X^{a_{t-1}}) \\ &\quad \cap (Z^{c_{t-2}}, Y^{b_{t-2}}) \cap (Z^{c_{t-1}}, X^{a_t}Y^{b_t}Z^{c_t}) \\ &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{t-1}}, X^{a_t}) \\ &\quad \cap (Z^{c_{t-1}}, Y^{b_t}) \cap (Z^{c_t}). \end{aligned}$$

If $I = Q_1 \cap Q_2 \cap \cdots \cap Q_l$ is the unique irredundant parametric decomposition of I, by Algorithm 2.1.7, we know that

$$Q_i \in \{(X^{a_1}), (Y^{b_1}), (Z^{c_1}, X^{a_2}), (Z^{c_1}, Y^{b_2}), \ldots, (Z^{c_{t-1}}, X^{a_t}), (Z^{c_{t-1}}, Y^{b_t}), (Z^{c_t})\}$$

for all $i = 1, \ldots, l$. Then $Rad(Q_i) \subsetneq (X, Y, Z)$ for all $i = 1, \ldots, l$. Therefore, by Proposition 2.2.1, I has no corner-elements. □

We can generalize this example which is an ideal in $k[X, Y, Z]$ with no corner-elements and get an example which is an ideal in $R = k[X_1, X_2, \ldots, X_d]$ with no corner-elements.

Proposition 4.1.2 *Let* $I = (X_1^{a_{1,1}} X_2^{a_{1,2}} \cdots X_d^{a_{1,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}})$ *be a monomial ideal in* R *with* $a_{1,i} \leq a_{2,i} \leq \ldots \leq a_{t,i}$ *for all* $i = 1, \ldots, d-1$ *and* $a_{1,d} \geq a_{2,d} \geq \ldots \geq a_{t,d}$. *Then*

$$I = \bigcap_{i=1}^{d-1} (X_i^{a_{1,i}}) \cap \left[\bigcap_{j=1}^{t-1} \bigcap_{i=1}^{d-1} (X_d^{a_{j,d}}, X_i^{a_{j+1,i}}) \right] \cap (X_d^{a_{t,d}}).$$

Therefore I *has no corner-elements. In particular, if* $a_{1,i} < a_{2,i} < \ldots < a_{t,i}$ *for all* $i = 1, \ldots, d-1$ *and* $a_{1,d} > a_{2,d} > \ldots > a_{t,d}$, *then* $v(I) = t$ *and*

$$I = \bigcap_{i=1}^{d-1} (X_i^{a_{1,i}}) \cap \left[\bigcap_{j=1}^{t-1} \bigcap_{i=1}^{d-1} (X_d^{a_{j,d}}, X_i^{a_{j+1,i}}) \right] \cap (X_d^{a_{t,d}})$$

is the irredundant parametric decomposition of I.

Proof. Note that

$$\begin{aligned} I &= (X_1^{a_{1,1}} X_2^{a_{1,2}} \cdots X_d^{a_{1,d}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &= (X_1^{a_{1,1}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &\quad \cap \cdots \cap (X_{d-1}^{a_{1,d-1}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &\quad \cap (X_d^{a_{1,d}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}). \end{aligned}$$

For all $j = 2, \ldots, t$, since $a_{j,1} \geq a_{1,1}$, $X_1^{a_{j,1}} X_2^{a_{j,2}} \cdots X_d^{a_{j,d}} \in (X_1^{a_{1,1}})$. Hence

$$(X_1^{a_{1,1}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) = (X_1^{a_{1,1}}).$$

Similarly, for each $i \geq 2$, since $a_{j,i} \geq a_{1,i}$ for all $j = 2, 3, \ldots, t$, $X_1^{a_{j,1}} X_2^{a_{j,2}} \cdots X_d^{a_{j,d}} \in (X_i^{a_{1,i}})$. Hence

$$(X_i^{a_{1,i}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) = (X_i^{a_{1,i}}).$$

Therefore,

$$\begin{aligned} I &= (X_1^{a_{1,1}}) \cap \cdots \cap (X_{d-1}^{a_{1,d-1}}) \\ &\quad \cap (X_d^{a_{1,d}}, X_1^{a_{2,1}} X_2^{a_{2,2}} \cdots X_d^{a_{2,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &= (X_1^{a_{1,1}}) \cap \cdots \cap (X_{d-1}^{a_{1,d-1}}) \\ &\quad \cap (X_d^{a_{1,d}}, X_1^{a_{2,1}}, X_1^{a_{3,1}} X_2^{a_{3,2}} \cdots X_d^{a_{3,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &\quad \cap \cdots \cap (X_d^{a_{1,d}}, X_{d-1}^{a_{2,d-1}}, X_1^{a_{3,1}} X_2^{a_{3,2}} \cdots X_d^{a_{3,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &\quad \cap (X_d^{a_{1,d}}, X_d^{a_{2,d}}, X_1^{a_{3,1}} X_2^{a_{3,2}} \cdots X_d^{a_{3,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}). \end{aligned}$$

Let $i = 1, 2, \ldots, d-1$. Since $a_{j,i} \geq a_{2,i}$ for all $j = 3, 4, \ldots, t$, $X_1^{a_{j,1}} X_2^{a_{j,2}} \cdots X_d^{a_{j,d}} \in (X_i^{a_{2,i}})$. Hence

$$(X_d^{a_{1,d}}, X_i^{a_{2,i}}, X_1^{a_{3,1}} X_2^{a_{3,2}} \cdots X_d^{a_{3,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) = (X_d^{a_{1,d}}, X_i^{a_{1,i}}).$$

Also, since $a_{1,d} \geq a_{2,d}$, we have

$$\begin{aligned} I &= (X_1^{a_{1,1}}) \cap \cdots \cap (X_{d-1}^{a_{1,d-1}}) \cap (X_d^{a_{1,d}}, X_1^{a_{2,1}}) \cap \cdots \cap (X_d^{a_{1,d}}, X_{d-1}^{a_{2,d-1}}) \\ &\quad \cap (X_d^{a_{2,d}}, X_1^{a_{3,1}} X_2^{a_{3,2}} \cdots X_d^{a_{3,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}). \end{aligned}$$

With a similar discussion as above, we have

$$\begin{aligned} I &= (X_1^{a_{1,1}}) \cap \cdots \cap (X_{d-1}^{a_{1,d-1}}) \cap (X_d^{a_{1,d}}, X_1^{a_{2,1}}) \cap \cdots \cap (X_d^{a_{1,d}}, X_{d-1}^{a_{2,d-1}}) \\ &\quad \cap (X_d^{a_{2,d}}, X_1^{a_{3,1}}) \cap \cdots \cap (X_d^{a_{2,d}}, X_{d-1}^{a_{t,d-1}}) \\ &\quad \cap (X_d^{a_{3,d}}, X_1^{a_{4,1}} X_2^{a_{4,2}} \cdots X_d^{a_{4,d}}, \ldots, X_1^{a_{t,1}} X_2^{a_{t,2}} \cdots X_d^{a_{t,d}}) \\ &= \cdots \\ &= (X_1^{a_{1,1}}) \cap \cdots \cap (X_{d-1}^{a_{1,d-1}}) \cap (X_d^{a_{1,d}}, X_1^{a_{2,1}}) \cap \cdots \cap (X_d^{a_{1,d}}, X_{d-1}^{a_{2,d-1}}) \\ &\quad \cap \cdots \cap (X_d^{a_{t-1,d}}, X_1^{a_{t,1}}) \cap \cdots \cap (X_d^{a_{t-1,d}}, X_{d-1}^{a_{t,d-1}}) \cap (X_d^{a_{t,d}}). \end{aligned}$$

Therefore,

$$I = \bigcap_{i=1}^{d-1} (X_i^{a_{1,i}}) \cap \left[\bigcap_{j=1}^{t-1} \bigcap_{i=1}^{d-1} (X_d^{a_{j,d}}, X_i^{a_{j+1,i}}) \right] \cap (X_d^{a_t,d}).$$

Hence, similarly as in the proof of Lemma 4.1.1, by Algorithm 2.1.7 and Proposition 2.2.1, I has no corner-elements.

Now assume $a_{1,i} < a_{2,i} < \ldots < a_{t,i}$ for all $i = 1, \ldots, d-1$ and $a_{1,d} > a_{2,d} > \ldots > a_{t,d}$. We want to show that

$$\begin{aligned} I &= \bigcap_{i=1}^{d-1}(X_i^{a_{1,i}}) \cap \left[\bigcap_{j=1}^{t-1}\bigcap_{i=1}^{d-1}(X_d^{a_{j,d}}, X_i^{a_{j+1,i}}) \right] \cap (X_d^{a_{t,d}}) \\ &= Q_1 \cap Q_2 \cap \cdots \cap Q_s \ \ \textit{where } s = d + (d-1)(t-1) \end{aligned}$$

is irredundant. By Lemma 2.1.6, it suffices to observe that

- for all $i = 1, \ldots, d-1$ and for all $j = 1, \ldots, t-1$, since $a_{1,i} < a_{j+1,i}$, $(X_i^{a_{1,i}}) \nsubseteq (X_d^{a_{j,d}}, X_i^{a_{j+1,i}})$,
- for all $i = 1, \ldots, d-1$ and for all $j = 1, \ldots, t-1$, since $a_{t,d} < a_{j,d}$, $(X_d^{a_{t,d}}) \nsubseteq (X_d^{a_{j,d}}, X_i^{a_{j+1,i}})$,
- for all $i = 1, \ldots, d-1$ and for all $1 \leq j < k \leq t-1$, since $a_{j,d} < a_{k,d}$ and since $a_{j+1,i} < a_{k+1,i}$, $(X_d^{a_{j,d}}, X_i^{a_{j+1,i}}) \nsubseteq (X_d^{a_{k,d}}, X_i^{a_{k+1,i}})$ and $(X_d^{a_{k,d}}, X_i^{a_{k+1,i}}) \nsubseteq (X_d^{a_{j,d}}, X_i^{a_{j+1,i}})$. □

From Proposition 4.1.2, we know that if $R = k[X_1, \ldots, X_d]$ with $d \geq 3$, there does not exist an integer n such that a monomial ideal I in R must have corner-elements if $\upsilon(I) \geq n$.

Now we use the following example to demonstrate our proof.

Example 4.1.3 *Let* $I = (XYZ^4, X^2Y^3Z^3, X^3Y^5Z^2, X^4Y^6Z)$ *in* $k[X, Y, Z]$. *Then* $\upsilon(I) = 4$ *and*

$$\begin{aligned} I &= (X, X^2Y^3Z^3, X^3Y^5Z^2, X^4Y^6Z) \cap (Y, X^2Y^3Z^3, X^3Y^5Z^2, X^4Y^6Z) \\ &\quad \cap (Z^4, X^2Y^3Z^3, X^3Y^5Z^2, X^4Y^6Z) \\ &= (X) \cap (Y) \cap (Z^4, X^2, X^3Y^5Z^2, X^4Y^6Z) \cap (Z^4, Y^3, X^3Y^5Z^2, X^4Y^6Z) \\ &\quad \cap (Z^4, Z^3, X^3Y^5Z^2, X^4Y^6Z) \\ &= (X) \cap (Y) \cap (Z^4, X^2) \cap (Z^4, Y^3) \cap (Z^3, X^3, X^4Y^6Z) \cap (Z^3, Y^5, X^4Y^6Z) \\ &\quad \cap (Z^3, Z^2, X^4Y^6Z). \end{aligned}$$

$$
\begin{aligned}
I &= (X)\cap(Y)\cap(Z^4,X^2)\cap(Z^4,Y^3)\cap(Z^3,X^3)\cap(Z^3,Y^5)\cap(Z^2,X^4)\\
&\quad \cap(Z^2,Y^6)\cap(Z^2,Z)\\
&= (X)\cap(Y)\cap(Z^4,X^2)\cap(Z^4,Y^3)\cap(Z^3,X^3)\cap(Z^3,Y^5)\cap(Z^2,X^4)\\
&\quad \cap(Z^2,Y^6)\cap(Z)\\
&= (X)\cap(Y)\cap(Z)\cap(X^2,Z^4)\cap(X^3,Z^3)\cap(X^4,Z^2)\cap(Y^3,Z^4)\\
&\quad \cap(Y^5,Z^3)\cap(Y^6,Z^2).
\end{aligned}
$$

This is the unique irredundant parametric decomposition of I and hence by Proposition 2.2.1 I has no corner-elements. □

Next, we use another method to prove that the ideal I in Proposition 4.1.2 has no corner-elements.

Remark: Suppose I has an corner-element. Let $z = X_1^{a_1}X_2^{a_2}\cdots X_d^{a_d}$ be an I-corner-element. Then

i) $zX_1 = X_1^{a_1+1}X_2^{a_2}\cdots X_d^{a_d} \in (X_1^{a_{s,1}}X_2^{a_{s,2}}\cdots X_d^{a_{s,d}})$ for some $s = 1,\ldots,t$ and $z = X_1^{a_1}X_2^{a_2}\cdots X_d^{a_d} \notin (X_1^{a_{s,1}}X_2^{a_{s,2}}\cdots X_d^{a_{s,d}})$. Hence $a_{s,2} \le a_2$ and $a_1 < a_{s,1} \le a_1+1$.

ii) $zX_2 = X_1^{a_1}X_2^{a_2+1}X_3^{a_3}\cdots X_d^{a_d} \in (X_1^{a_{k,1}}X_2^{a_{k,2}}X_3^{a_{k,3}}\cdots X_d^{a_{k,d}})$ for some $k = 1,\ldots,t$ and $z = X_1^{a_1}X_2^{a_2}\cdots X_d^{a_d} \notin (X_1^{a_{k,1}}X_2^{a_{k,2}}\cdots X_d^{a_{k,d}})$. Hence $a_{k,1} \le a_1$ and $a_2 < a_{k,2} \le a_2+1$.

Hence $a_{k,1} \le a_1 < a_{s,1}$ and $a_{s,2} \le a_2 < a_{k,2}$. This implies $k < s$ and $s < k$ which is a contradiction. Therefore I has no corner-elements.

4.2 Some Special Cases

Let $I = (X^{a_1}Y^{b_1}Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2},\ldots,X^{a_t}Y^{b_t}Z^{c_t})$ be a monomial ideal in $k[X,Y,Z]$. In Section 4.1, we know that if $a_1 \le a_2 \le \cdots \le a_t,\ b_1 \le b_2 \le \cdots \le b_t$, and $c_1 \ge c_2 \ge \cdots \ge c_t$, then I has no corner-elements. Now we are interested in the number of I-corner-elements when we have for some $i = 1,\ldots,t-1,\ b_{i+1} < b_i$ instead of $b_i \le b_{i+1}$ in Lemma 4.1.1. Lemma 4.2.1 is what we get regarding this question.

Lemma 4.2.1 *Let* $I = (X^{a_1}Y^{b_1}Z^{c_1}, \ldots, X^{a_r}Y^{b_r}Z^{c_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t})$ *be a monomial ideal in* $k[X,Y,Z]$ *with* $t \geq 3,\ r \geq 2,\ a_1 \leq \cdots \leq a_{r-1} \leq a_r < a_{r+1} \leq a_{r+2} \leq \cdots \leq a_t,\ b_1 \leq \cdots \leq b_{r-1} \leq b_{r+1} < b_r \leq b_{r+2} \leq \cdots \leq b_t$ *and* $c_1 \geq \cdots \geq c_{r-1} > c_r \geq c_{r+1} \geq c_{r+2} \geq \cdots \geq c_t$. *Then* I *has a unique corner-element* $X^{a_{r+1}-1}Y^{b_r-1}Z^{c_{r-1}-1}$.

Proof. Note that

$$\begin{aligned}
I &= (X^{a_1}Y^{b_1}Z^{c_1}, \ldots, X^{a_{r-1}}Y^{b_{r-1}}Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (X^{a_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_{r-1}}Y^{b_{r-1}}Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Y^{b_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_{r-1}}Y^{b_{r-1}}Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_{r-1}}Y^{b_{r-1}}Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).
\end{aligned}$$

Since $a_2, \ldots, a_t \geq a_1$ and $b_2, \ldots, b_t \geq b_1$, similar as the proof of Lemma 4.1.1, we have

$$\begin{aligned}
I &= (X^{a_1}) \cap (Y^{b_1}) \\
&\quad \cap (Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_{r-1}}Y^{b_{r-1}}Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Z^{c_1}, Y^{b_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Z^{c_1}, Z^{c_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).
\end{aligned}$$

For each $i = 2, \ldots, r-1$, since $a_{i+1}, \ldots, a_t \geq a_i,\ b_{i+1}, \ldots, b_t \geq b_i,$ and $c_{i-1} \geq c_i$, we have

$$\begin{aligned}
I &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \\
&\quad \cap (Z^{c_2}, X^{a_3}Y^{b_3}Z^{c_3}, \ldots, X^{a_{r-1}}Y^{b_{r-1}}Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= \cdots \\
&= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r-2}}, X^{a_{r-1}}) \cap (Z^{c_{r-2}}, Y^{b_{r-1}}) \\
&\quad \cap (Z^{c_{r-1}}, X^{a_r}Y^{b_r}Z^{c_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r-2}}, X^{a_{r-1}}) \cap (Z^{c_{r-2}}, Y^{b_{r-1}}) \\
&\quad \cap (Z^{c_{r-1}}, X^{a_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Z^{c_{r-1}}, Z^{c_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).
\end{aligned}$$

Since $a_{r+1}, \ldots, a_t > a_r$,

$$(Z^{c_{r-1}}, X^{a_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) = (Z^{c_{r-1}}, X^{a_r}).$$

Moreover, we have

$$\begin{aligned}
& (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
= & (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
& \cap(Z^{c_{r-1}}, Y^{b_r}, Y^{b_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
& \cap(Z^{c_{r-1}}, Y^{b_r}, Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).
\end{aligned}$$

Since $a_{r+2}, \ldots, a_t \geq a_{r+1},\ b_{r+2}, \ldots, b_t \geq b_r > b_{r+1}$, and $c_{r-1} > c_{r+1}$, we get

$$\begin{aligned}
& (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
= & (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}) \cap (Z^{c_{r-1}}, Y^{b_{r+1}}) \cap (Y^{b_r}, Z^{c_{r+1}}).
\end{aligned}$$

Hence, since $c_{r-1} > c_r$, we have

$$\begin{aligned}
I = & (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r-2}}, X^{a_{r-1}}) \cap (Z^{c_{r-2}}, Y^{b_{r-1}}) \\
& \cap(Z^{c_{r-1}}, X^{a_r}) \cap (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}) \cap (Z^{c_{r-1}}, Y^{b_{r+1}}) \cap (Y^{b_r}, Z^{c_{r+1}}) \\
& \cap(Z^{c_r}, X^{a_{r+1}}Y^{b_{r+1}}Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).
\end{aligned}$$

For each $i = r+1, \ldots, t$, since $a_{i+1}, \ldots, a_t \geq a_i,\ b_{i+1}, \ldots, b_t \geq b_i$, and $c_r \geq c_{r+1} \geq c_{r+2} \geq \cdots \geq c_t$, we have

$$\begin{aligned}
I = & (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r-2}}, X^{a_{r-1}}) \cap (Z^{c_{r-2}}, Y^{b_{r-1}}) \\
& \cap(Z^{c_{r-1}}, X^{a_r}) \cap (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}) \cap (Z^{c_{r-1}}, Y^{b_{r+1}}) \cap (Y^{b_r}, Z^{c_{r+1}}) \\
& \cap(Z^{c_r}, X^{a_{r+1}}) \cap (Z^{c_r}, Y^{b_{r+1}}) \cap (Z^{c_{r+1}}, X^{a_{r+2}}Y^{b_{r+2}}Z^{c_{r+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
= & (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r-2}}, X^{a_{r-1}}) \cap (Z^{c_{r-2}}, Y^{b_{r-1}}) \\
& \cap(Z^{c_{r-1}}, X^{a_r}) \cap (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}) \cap (Z^{c_{r-1}}, Y^{b_{r+1}}) \cap (Y^{b_r}, Z^{c_{r+1}}) \\
& \cap(Z^{c_r}, X^{a_{r+1}}) \cap (Z^{c_r}, Y^{b_{r+1}}) \cap (Z^{c_{r+1}}, X^{a_{r+2}}) \cap (Z^{c_{r+1}}, Y^{b_{r+2}}) \\
& \cap(Z^{c_{r+2}}, X^{a_{r+3}}Y^{b_{r+3}}Z^{c_{r+3}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
= & \cdots \\
= & (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r-2}}, X^{a_{r-1}}) \\
& \cap(Z^{c_{r-2}}, Y^{b_{r-1}}) \cap (Z^{c_{r-1}}, X^{a_r}) \cap (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}}) \cap (Z^{c_{r-1}}, Y^{b_{r+1}}) \\
& \cap(Y^{b_r}, Z^{c_{r+1}}) \cap (Z^{c_r}, X^{a_{r+1}}) \cap (Z^{c_r}, Y^{b_{r+1}}) \cap (Z^{c_{r+1}}, X^{a_{r+2}}) \\
& \cap(Z^{c_{r+1}}, Y^{b_{r+2}}) \cap \cdots \cap (Z^{c_{t-1}}, X^{a_t}) \cap (Z^{c_{t-1}}, Y^{b_t}) \cap (Z^{c_t}).
\end{aligned}$$

Finally, we claim that the only parameter ideal $(Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$ in the above decomposition is not redundant. It suffices to observe that

- since $a_1 < a_{r+1}$, $(X^{a_1}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- since $b_1 < b_r$, $(Y^{b_1}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- for all $i = 2, \ldots, r$, since $a_i < a_{r+1}$, $(Z^{c_{i-1}}, X^{a_i}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- for all $i = 2, \ldots, r-1$, since $b_i < b_r$, $(Z^{c_{i-1}}, Y^{b_i}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- since $b_{r+1} < b_r$, $(Z^{c_{r-1}}, Y^{b_{r+1}}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- since $c_{r+1} < c_{r-1}$, $(Y^{b_r}, Z^{c_{r+1}}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- for all $i = r, \ldots, t-1$, since $c_i < c_{r-1}$, $(Z^{c_i}, X^{a_{i+1}}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$ and $(Z^{c_i}, Y^{a_{i+1}}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$;
- since $c_t < c_{r-1}$, $(Z^{c_t}) \nsubseteq (Z^{c_{r-1}}, Y^{b_r}, X^{a_{r+1}})$.

Hence, by Proposition 2.2.1 and Algorithm 2.1.7, $X^{a_{r+1}-1}Y^{b_r-1}Z^{c_{r-1}-1}$ is the unique I-corner-element. □

In Lemma 4.2.1, we use the method of finding the parameter ideals in the parametric decomposition of I to find the unique I-corner-element. Nevertheless, we can also use the property of corner-elements, i.e., Proposition 3.1.1, to get $c(I) = 1$.

Another Proof of Lemma 4.2.1. First we claim that $X^{a_{r+1}-1}Y^{b_r-1}Z^{c_{r-1}-1}$ is an I-corner-element. By Proposition 3.1.1 it suffices to observe

(i) $a_{r+1} - 1 = a_{r+1} - 1$, $b_r - 1 = b_r - 1$ and $c_{r-1} - 1 = c_{r-1} - 1$.
(ii) $a_{r+1} > a_r$.
(iii) $b_r > b_{r+1}$.
(iv) $c_{r-1} > c_{r+1}$ and $c_{r-1} > c_r$.
(v) $r - 1 \in \Phi_{r+1,r} = \{i \mid a_i < a_{r+1} \text{ and } b_i < b_r\} = \{1, 2, \ldots, r-1\}$ and $c_{r-1} = \min\{c_i \mid i \in \Phi_{r+1,r}\}$.

Hence $X^{a_{r+1}-1}Y^{b_r-1}Z^{c_{r-1}-1}$ is an I-corner-element.

Conversely, let $z = X^aY^bZ^c$ be an I-corner-element. By Proposition 3.1.1 or Proposition 3.1.2, there exist distinct $s, k, l \in \{1, \ldots, t\}$ such that

$$a = a_s - 1,\ b = b_k - 1,\ c = c_l - 1,\ a_s > a_k \text{ and } b_s < b_k.$$

From the setting of generators of I, we have $s = r+1$ and $k = r$. Moreover, by Proposition 3.1.1, $l \in \Phi_{s,k} = \{i \mid a_i < a_s \text{ and } b_i < b_k\} = \{1, 2, \ldots, r-1\}$ and $c_l = min\{c_i \mid i \in \Phi_{s,k}\}$. Hence $c_l = c_{r-1}$. Therefore $X^{a_{r+1}-1}Y^{b_r-1}Z^{c_{r-1}-1}$ is the unique I-corner-element. $\square$

In Lemma 4.2.1, we request $r \geq 2$, i.e., $r \neq 1$, because if $r = 1$ then there does not exist l satisfying Proposition 3.1.1. Now we use Lemma 4.2.1 to find ideals which have precisely m corner-elements.

Proposition 4.2.2 *Let*

$$\begin{aligned} I \;=\; & (X^{a_1}Y^{b_1}Z^{c_1}, X^{a_2}Y^{b_2}Z^{c_2}, \ldots, X^{a_{r_1-1}}Y^{b_{r_1-1}}Z^{c_{r_1-1}}, \\ & X^{a_{r_1}}Y^{b_{r_1}}Z^{c_{r_1}}, X^{a_{r_1+1}}Y^{b_{r_1+1}}Z^{c_{r_1+1}}, \ldots, X^{a_{r_2-1}}Y^{b_{r_2-1}}Z^{c_{r_2-1}}, \\ & X^{a_{r_2}}Y^{b_{r_2}}Z^{c_{r_2}}, X^{a_{r_2+1}}Y^{b_{r_2+1}}Z^{c_{r_2+1}}, \ldots, X^{a_{r_m-1}}Y^{b_{r_m-1}}Z^{c_{r_m-1}}, \\ & X^{a_{r_m-1}}Y^{b_{r_m-1}}Z^{c_{r_m-1}}, X^{a_{r_m}}Y^{b_{r_m}}Z^{c_{r_m}}, X^{a_{r_m+1}}Y^{b_{r_m+1}}Z^{c_{r_m+1}}, \\ & \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \end{aligned}$$

be a monomial ideal in $k[X, Y, Z]$ with $t \geq 3$ and $2m + 1 \leq t$. Suppose

(*i*) $r_1 \geq 2,\ r_m \leq t - 1,$ *and for all* $i = 2, \ldots, m,\ r_i \geq r_{i-1} + 2,$

(*ii*) $a_1 \leq a_2 \leq \cdots \leq a_t$ *and for all* $i = 1, \ldots, m,\ a_{r_i} < a_{r_i+1},$

(*iii*) $b_k \leq b_{k+1}$ *for all* $k = 1, \ldots, t-1$ *except* $k = r_1, \ldots, r_m,$ *and for each* $i = 1, \ldots, m,\ b_j \leq b_{r_i+1} < b_{r_i}$ *for all* $j \leq r_i - 1$ *and* $b_{r_i+1} < b_{r_i} \leq b_k$ *for all* $k \geq r_i + 2,$

(*iv*) $c_1 \geq c_2 \geq \cdots \geq c_t$ *and for all* $i = 1, \ldots, m,\ c_{r_i-1} > c_{r_i}.$

Then I has precisely m corner-elements

$$X^{a_{r_1+1}-1}Y^{b_{r_1}-1}Z^{c_{r_1-1}-1}, \ldots, X^{a_{r_m+1}-1}Y^{b_{r_m}-1}Z^{c_{r_m-1}-1}.$$

Proof. Note that from (*i*) and (*iii*), we have $b_1 \leq b_{r_1+1} < b_{r_1} \leq b_{r_2+1} < b_{r_2} \leq \cdots \leq b_{r_i+1} < b_{r_i} \leq \cdots \leq b_{r_m+1} < b_{r_m}$. Similarly as the proof of

Lemma 4.2.1, we have

$$
\begin{aligned}
I &= (X^{a_1}Y^{b_1}Z^{c_1}, \ldots, X^{a_{r_1-1}}Y^{b_{r_1-1}}Z^{c_{r_1-1}}, X^{a_{r_1}}Y^{b_{r_1}}Z^{c_{r_1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r_1-2}}, X^{a_{r_1-1}}) \cap (Z^{c_{r_1-2}}, Y^{b_{r_1-1}}) \\
&\quad \cap (Z^{c_{r_1-1}}, X^{a_{r_1}}Y^{b_{r_1}}Z^{c_{r_1}}, X^{a_{r_1+1}}Y^{b_{r_1+1}}Z^{c_{r_1+1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \\
&\quad \cap (Z^{c_{r_1-2}}, X^{a_{r_1-1}}) \cap (Z^{c_{r_1-2}}, Y^{b_{r_1-1}}) \cap (Z^{c_{r_1-1}}, X^{a_{r_1}}) \\
&\quad \cap (Z^{c_{r_1-1}}, Y^{b_{r_1}}, X^{a_{r_1+1}}Y^{b_{r_1+1}}Z^{c_{r_1+1}}, X^{a_{r_1+2}}Y^{b_{r_1+2}}Z^{c_{r_1+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&\quad \cap (Z^{c_{r_1}}, X^{a_{r_1+1}}Y^{b_{r_1+1}}Z^{c_{r_1+1}}, X^{a_{r_1+2}}Y^{b_{r_1+2}}Z^{c_{r_1+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r_1-2}}, X^{a_{r_1-1}}) \cap (Z^{c_{r_1-2}}, Y^{b_{r_1-1}}) \\
&\quad \cap (Z^{c_{r_1-1}}, X^{a_{r_1}}) \cap (Z^{c_{r_1-1}}, Y^{b_{r_1}}, X^{a_{r_1+1}}) \cap (Z^{c_{r_1-1}}, Y^{b_{r_1+1}}) \cap (Y^{b_{r_1}}, Z^{c_{r_1+1}}) \cap I_1
\end{aligned}
$$

where

$$I_1 = (Z^{c_{r_1}}, X^{a_{r_1+1}}Y^{b_{r_1+1}}Z^{c_{r_1+1}}, X^{a_{r_1+2}}Y^{b_{r_1+2}}Z^{c_{r_1+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).$$

Similarly as above, then we have

$$
\begin{aligned}
I_1 &= (Z^{c_{r_1}}, X^{a_{r_1+1}}Y^{b_{r_1+1}}Z^{c_{r_1+1}}, X^{a_{r_1+2}}Y^{b_{r_1+2}}Z^{c_{r_1+2}}, \ldots, X^{a_{r_2-1}}Y^{b_{r_2-1}}Z^{c_{r_2-1}}, \\
&\quad X^{a_{r_2}}Y^{b_{r_2}}Z^{c_{r_2}}, X^{a_{r_2+1}}Y^{b_{r_2+1}}Z^{c_{r_2+1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (Z^{c_{r_1}}, X^{a_{r_1+1}}) \cap (Z^{c_{r_1}}, Y^{b_{r_1+1}}) \cap (Z^{c_{r_1+1}}, X^{a_{r_1+2}}) \cap (Z^{c_{r_1+1}}, Y^{b_{r_1+2}}) \\
&\quad \cap (Z^{c_{r_1+2}}, X^{a_{r_1+3}}) \cap (Z^{c_{r_1+2}}, Y^{b_{r_1+3}}) \cap \cdots \\
&\quad \cap (Z^{c_{r_2-2}}, X^{a_{r_2-1}}) \cap (Z^{c_{r_2-2}}, Y^{b_{r_2-1}}) \cap (Z^{c_{r_2-1}}, X^{a_{r_2}}) \\
&\quad \cap (Z^{c_{r_2-1}}, Y^{b_{r_2}}, X^{a_{r_2+1}}) \cap (Z^{c_{r_2-1}}, Y^{b_{r_2+1}}) \cap (Y^{b_{r_2}}, Z^{c_{r_2+1}}) \cap I_2
\end{aligned}
$$

where

$$I_2 = (Z^{c_{r_2}}, X^{a_{r_2+1}}Y^{b_{r_2+1}}Z^{c_{r_2+1}}, X^{a_{r_2+2}}Y^{b_{r_2+2}}Z^{c_{r_2+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).$$

For each $i = 3, \ldots, m$, we let

$$I_i = (Z^{c_{r_i}}, X^{a_{r_i+1}}Y^{b_{r_i+1}}Z^{c_{r_i+1}}, X^{a_{r_i+2}}Y^{b_{r_i+2}}Z^{c_{r_i+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}).$$

Then for each $i = 2, \ldots, m-1$, we have

$$
\begin{aligned}
I_i &= (Z^{c_{r_i}}, X^{a_{r_i+1}}Y^{b_{r_i+1}}Z^{c_{r_i+1}}, X^{a_{r_i+2}}Y^{b_{r_i+2}}Z^{c_{r_i+2}}, \ldots, X^{a_{r_{i+1}-1}}Y^{b_{r_{i+1}-1}}Z^{c_{r_{i+1}-1}}, \\
&\quad X^{a_{r_{i+1}}}Y^{b_{r_{i+1}}}Z^{c_{r_{i+1}}}, X^{a_{r_{i+1}+1}}Y^{b_{r_{i+1}+1}}Z^{c_{r_{i+1}+1}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (Z^{c_{r_i}}, X^{a_{r_i+1}}) \cap (Z^{c_{r_i}}, Y^{b_{r_i+1}}) \cap (Z^{c_{r_i+1}}, X^{a_{r_i+2}}) \cap (Z^{c_{r_i+1}}, Y^{b_{r_i+2}}) \\
&\quad \cap (Z^{c_{r_i+2}}, X^{a_{r_i+3}}) \cap (Z^{c_{r_i+2}}, Y^{b_{r_i+3}}) \cap \cdots \cap (Z^{c_{r_{i+1}-2}}, X^{a_{r_{i+1}-1}}) \\
&\quad \cap (Z^{c_{r_{i+1}-2}}, Y^{b_{r_{i+1}-1}}) \cap (Z^{c_{r_{i+1}-1}}, X^{a_{r_{i+1}}}) \cap (Z^{c_{r_{i+1}-1}}, Y^{b_{r_{i+1}}}, X^{a_{r_{i+1}+1}}) \\
&\quad \cap (Z^{c_{r_{i+1}-1}}, Y^{b_{r_{i+1}+1}}) \cap (Y^{b_{r_{i+1}}}, Z^{c_{r_{i+1}+1}}) \cap I_{i+1}
\end{aligned}
$$

and

$$
\begin{aligned}
I_m &= (Z^{c_{r_m}}, X^{a_{r_m+1}}Y^{b_{r_m+1}}Z^{c_{r_m+1}}, X^{a_{r_m+2}}Y^{b_{r_m+2}}Z^{c_{r_m+2}}, \ldots, X^{a_t}Y^{b_t}Z^{c_t}) \\
&= (Z^{c_{r_m}}, X^{a_{r_m+1}}) \cap (Z^{c_{r_m}}, Y^{b_{r_m+1}}) \cap (Z^{c_{r_m+1}}, X^{a_{r_m+2}}) \cap (Z^{c_{r_m+1}}, Y^{b_{r_m+2}}) \\
&\quad \cap (Z^{c_{r_m+2}}, X^{a_{r_m+3}}) \cap (Z^{c_{r_m+2}}, Y^{b_{r_m+3}}) \cap \cdots \cap (Z^{c_t-1}, X^{a_t}) \cap (Z^{c_t-1}, Y^{b_t}) \cap (Z^{c_t}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
I &= (X^{a_1}) \cap (Y^{b_1}) \cap (Z^{c_1}, X^{a_2}) \cap (Z^{c_1}, Y^{b_2}) \cap \cdots \cap (Z^{c_{r_1-2}}, X^{a_{r_1-1}}) \cap (Z^{c_{r_1-2}}, Y^{b_{r_1-1}}) \\
&\quad \cap (Z^{c_{r_1-1}}, X^{a_{r_1}}) \cap (Z^{c_{r_1-1}}, Y^{b_{r_1}}, X^{a_{r_1+1}}) \cap (Z^{c_{r_1-1}}, Y^{b_{r_1+1}}) \cap (Y^{b_{r_1}}, Z^{c_{r_1+1}}) \\
&\quad \cap \bigcap_{i=1}^{m-1} \big[(Z^{c_{r_i}}, X^{a_{r_i+1}}) \cap (Z^{c_{r_i}}, Y^{b_{r_i+1}}) \cap (Z^{c_{r_i+1}}, X^{a_{r_i+2}}) \cap (Z^{c_{r_i+1}}, Y^{b_{r_i+2}}) \\
&\qquad \cap (Z^{c_{r_{i+1}+2}}, X^{a_{r_i+3}}) \cap (Z^{c_{r_i+2}}, Y^{b_{r_i+3}}) \cap \cdots \cap (Z^{c_{r_{i+1}-2}}, X^{a_{ri+1-1}}) \\
&\qquad \cap (Z^{c_{r_{i+1}-2}}, Y^{b_{r_{i+1}-1}}) \cap (Z^{c_{r_{i+1}-1}}, X^{a_{r_{i+1}}}) \cap (Z^{c_{r_{i+1}-1}}, Y^{b_{r_{i+1}}}, X^{a_{r_{i+1}+1}}) \\
&\qquad \cap (Z^{c_{r_{i+1}-1}}, Y^{b_{r_{i+1}+1}}) \cap (Y^{b_{r_{i+1}}}, Z^{c_{r_{i+1}+1}}) \big] \cap (Z^{c_{r_m}}, X^{a_{r_m+1}}) \\
&\quad \cap (Z^{c_{r_m}}, Y^{b_{r_m+1}}) \cap (Z^{c_{r_m+1}}, X^{a_{r_m+2}}) \cap (Z^{c_{r_m+1}}, Y^{b_{r_m+2}}) \cap (Z^{c_{r_m+2}}, X^{a_{r_m+3}}) \\
&\quad \cap (Z^{c_{r_m+2}}, Y^{b_{r_m+3}}) \cap \cdots \cap (Z^{c_t-1}, X^{a_t}) \cap (Z^{c_t-1}, Y^{b_t}) \cap (Z^{c_t}).
\end{aligned}
$$

Finally, we claim that the parameter ideals $(Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$ for all $i = 1, \ldots, m$ in the above decomposition are not redundant. It suffices to observe that for each $i = 1, \ldots, m$,

- since $a_1 < a_{r_i+1}$, $(X^{a_1}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;
- since $b_1 < b_{r_i}$, $(Y^{b_1}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;
- for all $j = 2, \ldots, r_i$, since $a_j < a_{r_i+1}$, $(Z^{c_{j-1}}, X^{a_j}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;
- for all $j = 2, \ldots, r_i-1$, since $b_j < b_{r_i}$, $(Z^{c_{j-1}}, Y^{b_j}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;
- for all $j = 1, \ldots, i-1$, since $b_{r_j+1} < b_{r_j} < b_{r_i}$,

  $(Z^{c_{r_j-1}}, Y^{b_{r_j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$ and $(Y^{b_{r_j}}, Z^{c_{r_j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;
- since $b_{r_i+1} < b_{r_i}$, $(Z^{c_{r_i-1}}, Y^{b_{r_i+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;
- since $c_{r_i+1} < c_{r_i-1}$, $(Y^{b_{r_i}}, Z^{c_{r_i+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;

- for all $j = i+1, \ldots, m$, since $c_{r_j+1} < c_{r_j-1} < c_{r_i-1}$,

  $(Z^{c_{r_j-1}}, Y^{b_{r_j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$ and $(Y^{b_{r_j}}, Z^{c_{r_j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;

- for all $j = r_i, \ldots, t-1$, since $c_j < c_{r_i-1}$, $(Z^{c_j}, X^{a_{j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$ and $(Z^{c_j}, Y^{a_{j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$;

- since $c_t < c_{r_i-1}$, $(Z^{c_t}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$.

- since $c_{r_1-1} > c_{r_2-1} > \cdots > c_{r_m-1}$ and since $a_{r_1+1} < a_{r_2+1} < \cdots < a_{r_m+1}$, $(Z^{c_{r_j-1}}, Y^{b_{r_j}}, X^{a_{r_j+1}}) \nsubseteq (Z^{c_{r_i-1}}, Y^{b_{r_i}}, X^{a_{r_i+1}})$ for all $j \neq i$.

Hence, by Algorithm 2.1.7 and Proposition 2.2.1, we have that the I-corner-elements are $X^{a_{r_1+1}-1}Y^{b_{r_1}-1}Z^{c_{r_1-1}-1}, \ldots, X^{a_{r_m+1}-1}Y^{b_{r_m}-1}Z^{c_{r_m-1}-1}$. □

Now we prove Proposition 4.2.2 using the property Proposition 3.1.1 of corner-elements.

Another Proof of (4.2.2). First we claim that for each $i = 1, \ldots, m$, $X^{a_{r_i+1}-1}Y^{b_{r_i}-1}Z^{c_{r_i-1}-1}$ is an I-corner-element. By Proposition 3.1.1 it suffices to observe

(*i*) $a_{r_i+1} - 1 = a_{r_i+1} - 1$, $b_{r_i} - 1 = b_{r_i} - 1$ and $c_{r_i-1} - 1 = c_{r_i-1} - 1$.

(*ii*) $a_{r_i+1} > a_{r_i}$.

(*iii*) $b_{r_i} > b_{r_i+1}$.

(*iv*) $c_{r_i-1} > c_{r_i+1}$ and $c_{r_i-1} > c_{r_i}$.

(*v*) $r_i - 1 \in \Phi_{r_i+1,r_i} = \{1, \ldots, r_i - 1\}$ and $c_{r_i-1} = \min\{c_j \mid j \in \Phi_{r_i+1,r_i}\}$.

Thus, $X^{a_{r_i+1}-1}Y^{b_{r_i}-1}Z^{c_{r_i-1}-1}$ is an I-corner-element for all $i = 1, \ldots, m$.

If $z = X^aY^bZ^c$ is an I-corner-element. By Proposition 3.1.1, there exist distinct $s, k, l \in \{1, \ldots, t\}$ such that $a = a_s - 1$, $b = b_k - 1$, $c = c_l - 1$, $a_s > a_k$, and $b_s < b_k$. From the setting of generators of I, we have $s = r_i + 1$ and $k = r_i$ for some $i = 1, \ldots, m$. Moreover, similarly by Proposition 3.1.1, we have $c_l = \min\{c_j \mid j \in \Phi_{s,k} = \{1, 2, \ldots, r_i - 1\}\}$. Hence $c_l = c_{r_i-1}$. Therefore, $X^{a_{r_1+1}-1}Y^{b_{r_1}-1}Z^{c_{r_1-1}-1}, \ldots, X^{a_{r_m+1}-1}Y^{b_{r_m}-1}Z^{c_{r_m-1}-1}$ are the I-corner-elements. □

Let t and m be two integers such that $t \geq 3$ and $0 \leq m \leq [\frac{t-1}{2}]$. From Proposition 4.2.2, we can construct a monomial ideal I which are minimally generated by t monomials and has precisely m corner-elements in $k[X, Y, Z]$.

Example 4.2.3 *Let $t = 7$, $m_1 = 1$ and $m_2 = 3$. Now we construct two monomial ideals I_1 and I_2 with $\upsilon(I_1) = \upsilon(I_2) = 7$ in $k[X, Y, Z]$ such that $c(I_1) = 1$ and $c(I_2) = 3$. Consider the monomial ideal*

$$\begin{aligned} I &= (XYZ^7, X^2Y^2Z^6, X^3Y^3Z^5, X^4Y^4Z^4, X^5Y^5Z^3, X^6Y^6Z^2, X^7Y^7Z) \\ &= (X^{a_1}Y^{b_1}Z^{c_1}, \ldots, X^{a_7}Y^{b_7}Z^{c_7}). \end{aligned}$$

Then $\upsilon(I) = 7$ and by Lemma 4.1.1, we know that I has no corner-elements.

First, we want to modify the ideal I and construct the monomial ideal I_1. By Lemma 4.2.1, we only need to have, for some $2 \leq i \leq 6$, $b_{i+1} < b_i$ instead of $b_i \leq b_{i+1}$. Thus we take $i = 2$ and let

$$I_1 = (XYZ^7, X^2Y^3Z^6, X^3Y^2Z^5, X^4Y^4Z^4, X^5Y^5Z^3, X^6Y^6Z^2, X^7Y^7Z).$$

It is clear that $\upsilon(I_1) = 7$. By Lemma 4.2.1, $X^2Y^2Z^6$ is the unique I_1-corner-element, i.e., $c(I_1) = 1$.

Next, we modify the ideal I and construct the monomial ideal I_2. By Proposition 4.2.2, we only need to have, for some $2 \leq i < j < k \leq 6$, $j-i \geq 2$ and $k - j \geq 2$, $b_{i+1} < b_i$, $b_{j+1} < b_j$, and $b_{k+1} < b_k$, instead of $b_i \leq b_{i+1}$, $b_j \leq b_{j+1}$, and $b_k \leq b_{k+1}$, respectively. Thus we take $i = 2$, $j = 4$, and $k = 6$, and let

$$I_2 = (XYZ^7, X^2Y^3Z^6, X^3Y^2Z^5, X^4Y^5Z^4, X^5Y^4Z^3, X^6Y^7Z^2, X^7Y^6Z).$$

Clearly, $\upsilon(I_2) = 7$ and by Lemma 4.2.2, $X^2Y^2Z^6$, $X^4Y^4Z^4$ and $X^6Y^6Z^2$ are the I_2-corner-elements, i.e., $c(I_2) = 3$.