1 Introduction

Let k be a field and let $R = k[X_1, X_2, \ldots, X_d]$ be a polynomial ring with d variables. A monomial in R is a power product $X_1^{a_1} \cdots X_d^{a_d}$ where $a_1, \ldots, a_d$ are nonnegative integers. And a monomial ideal is a proper ideal generated by monomials. It is well-known that every monomial ideal in R can be generated by finitely many monomials. Also, every monomial ideal in R has a unique minimal monomial generating set. If a monomial ideal I is minimally generated by n monomials, we write $v(I) = n$.

Monomial ideals are important in several areas of mathematics. They have been studied in several papers, such as $[EH]$ and $[T]$, and many useful results are known about such ideals. In $[HRS]$ and $[HMRS]$, Heinzer, Mirbagheri, Ratliff, and Shah investigate parametric decompositions of monomial ideals on a regular sequence in a commutative ring with identity 1. In $[HMRS,\ (4.9)]$, they show that the irredundant parametric decomposition of a monomial ideal is unique under certain conditions.

Theorem 1.0.1 $[HMRS,\ (4.9)]$ **(Unique Parametric Decomposition).** *Let R be a commutative ring with identity 1 and let $x_1, \ldots, x_d$ be an R-sequence. Fix a monomial ideal I in R. Let $\mathbf{D}$ be the set of all monomial ideals in R that contain I, and assume that $\mathbf{D}$ is closed under finite intersections. Then every ideal in $\mathbf{D}$ is an irredundant finite intersection of generalized-parameter ideals, and such a representation is unique (up to the order of the factors).*

A generalized-parameter ideal is an ideal of the form $(X_{i_1}^{a_{i_1}}, \ldots, X_{i_h}^{a_{i_h}})$ where $h \leq d$ and $a_{i_1}, \ldots, a_{i_h}$ are positive integers. In $[L]$, Professor Liu gives three algorithms for finding the unique irredundant parametric decomposition of a monomial ideal.

In $[HRS]$, $[HMRS]$ and $[L]$, corner-elements play an important role in finding the unique irredundant parametric decomposition of a monomial ideal. If I is a monomial ideal, an I-corner-element is a monomial z such that $z \notin I$ and $zX_i \in I$ for all $i = 1, \ldots, d$. In this thesis, we study and investigate some properties of corner-elements.

Throughout this thesis, R is the polynomial ring $k[X_1, \dots, X_d]$ where k is a field. Note that the conditions on R in this thesis are stronger than the conditions on R in [HRS], [$HMRS$] and [L]. Hence the results in [HRS], [$HMRS$] and [L] can be applied in this thesis.

In order to discuss corner-elements in monomial ideals, in Chapter 2, we provide an algorithm for finding corner-elements that can be found in [L]. In Section 2.1, we review some important results about monomial ideals and generalized-parameter ideals. In particular, we recall an algorithm [L, (5.1)] for finding the unique irredundant parametric decomposition. And in Section 2.2, we give an algorithm (Proposition 2.2.1) for finding corner-elements using Algorithm 2.1.7.

There are a few algorithms for finding corner-elements provided in several articles, including the algorithm mentioned above, and [HRS, (3.14)] and [L, (3.2)]. However, when we are given a monomial ideal I and try to use these algorithms to find I-corner-elements, we need to compute the irredundant parametric decomposition of I or the monomials not in I first. In Chapter 3, we give two methods to find corner-elements without computing the irredundant parametric decomposition of I nor the monomials not in I. In Section 3.1, we first give an equivalent condition for a monomial to be a corner-element (Proposition 3.1.1 for the case of $d = 3$ and Proposition 3.1.2 for the general case). This result is helpful to understand the features of corner-elements. We analyze Proposition 3.1.2 further and get two methods to find corner-elements. The first method is Algorithm 3.2.2 which is written as a program. But this algorithm is not vary simple nor easy to use. However, we discover that if I is a monomial ideal in $R = k[X_1, X_2, \dots, X_d]$ with $v(I) = d$ then it is easy to see whether I has a corner-element by observing the highest power for each variable among the generators of I (Lemma 3.3.4). Moreover, if I indeed has a corner-element then it is unique and is easy to find (Lemma 3.3.5). Therefore, we get the second method Algorithm 3.3.8 using Lemma 3.3.4.

In Chapter 4, we construct some monomial ideals in $k[X, Y, Z]$ (or in R) having some special property. From Algorithm 3.3.8, we know that if I is a monomial ideal with $\upsilon(I) = t$, then C_d^t is an upper bound of the number of I-corner-elements. Moreover, in [HRS, (3.13)], it is proved that if J is a monomial ideal in $k[X, Y]$ with $\upsilon(J) = t$, then J has precisely $t - 1$ corner-elements. Therefore, J has at least one corner-element if $\upsilon(J) \geq 2$. Hence we are interested in, for each $d \geq 3$, whether there exists an integer n such that I has at least one corner-element if I is a monomial ideal in R with $\upsilon(I) \geq n$. Unfortunately, we have found, for arbitrary $t \geq 3$, a monomial ideal I which has no corner-elements and $\upsilon(I) = t$ (Proposition 4.1.2). Hence for $d \geq 3$, there does not exist an integer n that plays the role of 2 which guarantees the existence of corner-elements in $k[X, Y]$. Finally, if t is an integer with $t \geq 3$ and if m is an integer with $0 \leq m \leq [\frac{t-1}{2}]$, then we can modify the ideal in Lemma 4.1.1 to construct an ideal which is minimally generated by t monomials and has precisely m corner-elements in $k[X, Y, Z]$ (Proposition 4.2.2).