3 Minimax theorems under jointly upward property

A lot of minimax theorems require the following changeless proportions between the two functions: $f(x, y) \leq g(x, y), \ \forall\, (x, y) \in X \times Y$. However, the condition is not necessary, in general. For example, let g, f be two real-valued functions defined on $[-1, 1] \times [-1, 1]$ by

$$g(x, y) = \begin{cases} (1-x)(1-y^2) & x \geq 0 \\ (1+x)(1-y^2) & x < 0 \end{cases} \quad and \quad f(x, y) = (1-x^2)(1-y^2).$$

Clearly, $g(x, y) \ngeq f(x, y)$. But we still have (see Example 3.3)

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = 0 \leq 0 = \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

This motivates us to introduce the concept of *jointly upward* below.

We say that f and g are *jointly upward* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that for all $x \in X$,

$$\max\{f(x, y_3), g(x, y_3)\} \leq \max\{g(x, y_1), f(x, y_2)\},$$

and

$$|g(x, y_1) - f(x, y_2)| \geq \epsilon \Longrightarrow \max\{f(x, y_3), g(x, y_3)\} \leq \max\{g(x, y_1), f(x, y_2)\} - \delta.$$

Making use of the *jointly upward* property, we will give a two functions minimax theorem without the changeless proportions.

Theorem 3.1. *Let X be a nonempty compact set of a topological space, and let Y be a nonempty set. Let f, g be two real-valued functions defined on $X \times Y$ satisfying the following properties:*
(0) $\sup_X f(x,y) \le \sup_X g(x,y)$, *$\forall y \in Y$ and $\forall y_1, y_2 \in Y$*

$$\sup_X \min\{g(x,y_1), f(x,y_2)\} \le \sup_X \min\{g(x,y_1), g(x,y_2)\}.$$

(i) *$f(\cdot,y)$ and $g(\cdot,y)$ are upper semicontinuous on X for each $y \in Y$;*
(ii) *For any $y_1, \cdots, y_n \in Y$, any $\lambda \in \mathbb{R}$, the set $\cap_{i=1}^n \{x \in X; g(x,y_i) \ge \lambda\}$ is either connected or empty; and for all $y \in Y$, $\{x \in X; f(x,y) \ge \lambda\}$ is either connected or empty.*
(iii) *f and g are jointly upward.*
Then

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \le \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

We need the following.

Lemma 3.2. *Under the conditions of Theorem3.1, if for any $y_1, y_2 \in Y$ and $\beta \in \mathbb{R}$ with $\sup_X \min\{g(x,y_1), g(x,y_2)\} < \beta$, then there exists $y_0 \in Y$ such that $\sup_X f(x,y_0) < \beta$.*

Proof. *Choose $\epsilon > 0$ such that*

$$\sup_X \min\{g(x,y_1), g(x,y_2)\} < \beta - 2\epsilon < \beta \tag{5}$$

Suppose that $\sup_X f(x,y) \ge \beta$ for all $y \in Y$. Let $I(y) = \sup_{x\in X} f(x,y)$, $J(y) = \sup_{x\in X} g(x,y)$, $y \in Y$. Then $\inf_Y J(y) \ge \inf_Y I(y) \ge \beta$. Let

$$A_f(y) = \{x \in X; f(x,y) \ge \beta - 2\epsilon\},$$

$$A_g(y) = \{x \in X; g(x,y) \ge \beta - 2\epsilon\}.$$

Then $A_f(y)$ and $A_g(y)$ are nonempty closed subsets of X for all y in Y by condition (i).

Choose $\delta > 0$ obtained by condition (iii). Next, we shall find $v_1 \in Y$ satisfying either case(a) or case(b):

(a) *$A_g(v_1) \subset A_g(y_1)$ and for all $y \in Y$, $A_f(y) \subset A_g(v_1)$ implies $I(y) > J(v_1) - \delta$,*
or (b) *$A_f(v_1) \subset A_g(y_1)$ and for all $y \in Y$, $A_g(y) \subset A_f(v_1)$ implies $J(y) > I(v_1) - \delta$.*

To see this, if for all y in Y,

$$A_f(y) \subset A_g(y_1) \Longrightarrow I(y) > J(y_1) - \delta,$$

then we take $v_1 = y_1$; otherwise, there exists $y^1 \in Y$ such that $A_f(y^1) \subset A_g(y_1)$ and $I(y^1) \le J(y_1) - \delta$. If for all y in Y,

$$A_g(y) \subset A_f(y^1) \Longrightarrow J(y) > I(y^1) - \delta,$$

then we take $v_1 = y^1$*; otherwise, there exists* $y^2 \in Y$ *such that* $A_g(y^2) \subset A_f(y^1) \subset A_g(y_1)$ *and* $J(y^2) \leq I(y^1) - \delta \leq J(y_1) - 2\delta$.

Suppose we have chosen y^{n-1}. *If* n *is odd, and for all* y *in* Y,

$$A_f(y) \subset A_g(y^{n-1}) \Longrightarrow I(y) > J(y^{n-1}) - \delta,$$

then we take $v_1 = y^{n-1}$*; otherwise, there exists* $y^n \in Y$ *such that* $A_f(y^n) \subset A_g(y^{n-1})$ *and* $I(y^n) \leq J(y^{n-1}) - \delta \leq J(y_1) - n\delta$. *On the other hand, if* n *is even, and for all* y *in* Y,

$$A_g(y) \subset A_f(y^{n-1}) \Longrightarrow J(y) > I(y^{n-1}) - \delta,$$

then we take $v_1 = y^{n-1}$*; otherwise, there exists* $y^n \in Y$ *such that* $A_g(y^n) \subset A_f(y^{n-1})$ *and* $J(y^n) \leq I(y^{n-1}) - \delta \leq J(y_1) - n\delta$.

Since $\inf_{y\in Y} J(y) > \inf_{y \in Y} I(y) > -\infty$, *this process must stop at some* m. *Let* $v_1 = y^m$. *Thus, if* m *is even, then we have*

$$A_g(v_1) \subset A_f(y^{m-1}) \subset A_g(y^{m-2}) \subset \cdots \subset A_g(y_1)$$

and for all $y \in Y$,

$$A_f(y) \subset A_g(v_1) \Longrightarrow I(y) > J(v_1) - \delta.$$

If m *is odd, then we have*

$$A_f(v_1) \subset A_g(y^{m-1}) \subset A_f(y^{m-2}) \subset \cdots \subset A_g(y_1)$$

and for all $y \in Y$,

$$A_g(y) \subset A_f(v_1) \Longrightarrow J(y) > I(v_1) - \delta.$$

(I) Under case(a), we first notice that there exists $v_2 \in Y$ *such that* $A_f(v_2) \subset A_f(y_2)$ *and for all* $y \in Y$, $A_f(y) \subset A_f(v_2)$ *implies* $I(y) > I(v_2) - \delta$.

Also, by condition (iii), there exists an element y_3 *in* Y *such that*

$$f(x, y_3) \leq \max\{f(x, y_3), g(x, y_3)\} \leq \max\{g(x, v_1), f(x, v_2)\}, \tag{6}$$

and for all $x \in X$,

$$|g(x, v_1) - f(x, v_2)| \geq \epsilon \Longrightarrow f(x, y_3) \leq \max\{f(x, y_3), g(x, y_3)\} \leq \max\{g(x, v_1), f(x, v_2)\} - \delta. \tag{7}$$

By(6),

$$A_f(y_3) \subset A_g(v_1) \cup A_f(v_2). \tag{8}$$

Next, we want to show that $A_f(y_3) \cap A_f(v_2) \neq \emptyset$ *and* $A_f(y_3) \cap A_g(v_1) \neq \emptyset$.

Choose $x \in X$, *with* $f(x, y_3) > \beta - \epsilon$. *Then* $x \in A_f(y_3)$.

If $f(x, v_2) \geq \beta - 2\epsilon$ *then* $x \in A_f(v_2)$, *so* $A_f(y_3) \cap A_f(v_2) \neq \emptyset$. *If* $f(x, v_2) < \beta - 2\epsilon < \beta - \epsilon$, *we must have* $g(x, v_1) > \beta - \epsilon$ *by* (6). *Then* $|g(x, v_1) - f(x, v_2)| > \epsilon$ *By* (7), $f(x, y_3) \leq$

$\max\{g(x,v_1),f(x,v_2)\}-\delta \le g(x,v_1)-\delta \le J(v_1)-\delta$. *Since this inequality holds for all* $x \in A \equiv \{x; f(x,y_3) > \beta-\epsilon\}$, *we have* $I(y_3) = \sup_{x\in X} f(x,y_3) = \sup_{x\in A} f(x,y_3) \le J(v_1)-\delta$. *Hence* $A_f(y_3) \not\subseteq A_g(v_1)$ *by the choice of* v_1. *By* (8),

$$A_f(y_3) \cap A_f(v_2) \neq \emptyset. \tag{9}$$

Similarly, we can show that

$$A_f(y_3) \cap A_g(v_1) \neq \emptyset. \tag{10}$$

Observe that $A_f(y_3)$ *is nonempty and connected by condition* (*ii*). *Then from* (8), (9) *and* (10), *it follows that* $A_g(v_1) \cap A_f(v_2) \neq \emptyset$, *and hence* $A_g(y_1) \cap A_f(y_2) \neq \emptyset$. *Let* $x_0 \in A_g(y_1) \cap A_f(y_2)$. *Thus,* $\min\{g(x_0,y_1), f(x_0,y_2)\} \ge \beta - 2\epsilon$. *This contradicts* $\sup_X \min\{g(x,y_1),f(x,y_2)\} \le \sup_X \min\{g(x,y_1),g(x,y_2)\} < \beta-2\epsilon$. *Therefore, there exists* y_0 *in* Y *such that* $\sup_X f(x,y_0) < \beta$.

(*II*) *Under case(b),we notice that there exists* $v_2 \in Y$ *such that* $A_g(v_2) \subset A_g(y_2)$ *and for all* $y \in Y$, $A_g(y) \subset A_g(v_2)$ *implies* $J(y) > J(v_2) - \delta$.

Also, by condition (*iii*), *there exists an element* y_3 *in* Y *such that*

$$g(x,y_3) \le \max\{f(x,y_3),g(x,y_3)\} \le \max\{f(x,v_1),g(x,v_2)\}, \tag{11}$$

and for all $x \in X$,

$$|f(x,v_1)-g(x,v_2)| \ge \epsilon \Longrightarrow g(x,y_3) \le \max\{f(x,y_3),g(x,y_3)\} \le \max\{f(x,v_1),g(x,v_2)\}-\delta. \tag{12}$$

By(11),

$$A_g(y_3) \subset A_f(v_1) \cup A_g(v_2). \tag{13}$$

Next, we want to show that $A_g(y_3) \cap A_g(v_2) \neq \emptyset$ *and*$A_g(y_3) \cap A_f(v_1) \neq \emptyset$.

Choose $x \in X$, *with* $g(x,y_3) > \beta-\epsilon$. *Then* $x \in A_g(y_3)$. *If* $g(x,v_2) \ge \beta - 2\epsilon$ *then* $x \in A_g(v_2)$, *so* $A_g(y_3) \cap A_g(v_2) \neq \emptyset$. *If* $g(x,v_2) < \beta-2\epsilon < \beta-\epsilon$, *we must have* $f(x,v_1) > \beta-\epsilon$ *by* (11). *Then* $|f(x,v_1)-g(x,v_2)| > \epsilon$ *By* (12), $g(x,y_3) \le \max\{f(x,v_1),g(x,v_2)\}-\delta \le f(x,v_1)-\delta \le I(v_1)-\delta$. *Since this inequality holds for all* $x \in B \equiv \{x; g(x,y_3) > \beta-\epsilon\}$, *thus* $J(y_3) = \sup_{x\in X} g(x,y_3) = \sup_{x\in B} g(x,y_3) \le I(v_1)-\delta$. *Hence* $A_g(y_3) \not\subseteq A_f(v_1)$ *by the choice of* v_1. *By* (13),

$$A_g(y_3) \cap A_g(v_2) \neq \emptyset. \tag{14}$$

Similarly, we can show that

$$A_g(y_3) \cap A_f(v_1) \neq \emptyset. \tag{15}$$

By condition (*ii*), $A_g(y_3)$ *is nonempty and connected. Then from* (13), (14) *and* (15), *it follows that* $A_f(v_1) \cap A_g(v_2) \neq \emptyset$, *and hence* $A_g(y_1) \cap A_g(y_2) \neq \emptyset$. *Let* $x_0 \in A_g(y_1) \cap A_g(y_2)$. *Thus,* $\min\{g(x_0,y_1), g(x_0,y_2)\} \ge \beta-2\epsilon$. *This contradicts* (5). *Therefore, there exists* y_0 *in* Y *such that* $\sup_x f(x,y_0) < \beta$.

□

Proof of Theorem 3.1. *Let* α *be a real number such that* $\sup_X \inf_Y g(x,y) < \alpha$. *Choose* β *such that*

$$\sup_X \inf_Y g(x,y) < \beta < \alpha. \tag{16}$$

For each $y \in Y$, *let* $L_g(y) = \{x \in X; g(x,y) < \beta\}$. *Then each* $L_g(y)$ *is open, and by* (16),

$$X = \bigcup_{y \in Y} L_g(y).$$

Since X *is compact,* $X = \cup_{i=1}^n L_g(y_i)$ *for some* $y_1, \cdots, y_n \in Y$, *and this implies that*

$$\sup_X \min\{g(x,y_1), \cdots, g(x,y_n)\} < \beta.$$

We want to show, by induction on n, that there exists $y_0 \in Y$ *such that* $\sup_X f(x,y_0) < \alpha$. *For* $n = 1$, *since by the condition* (0), *it follows that*

$$\sup_X f(x,y_0) \leq \sup_X g(x,y_0) = \sup_X g(x,y_1) < \beta < \alpha.$$

For $n = 2$, *since*

$$\sup_X \min\{g(x,y_1), g(x,y_2)\} < \beta,$$

by the above Lemma, there exists $y_0 \in Y$ *such that* $\sup_X f(x,y_0) < \alpha$.

For $n > 2$. *Let* $A = \{x \in X; g(x,y_i) \geq \beta, i = 1, \ldots, n-1\}$. *By upper semicontinuity of* $g(\cdot, y_i)$, *the set* A *is a compact set in* X. *For* $x \in A$, *since* $g(x,y_i) \geq \beta$ *for* $i = 1, \cdots, n-1$ *and* $\min\{g(x,y_1), \cdots, g(x,y_n)\} < \beta$, *we have*

$$\min\{g(x,y_1), \cdots, g(x,y_n)\} = g(x,y_n).$$

Then

$$\begin{aligned} \sup_A g(x,y_n) &= \sup_A \min\{g(x,y_1), \cdots, g(x,y_n)\} \\ &\leq \sup_X \min\{g(x,y_1), \cdots, g(x,y_n)\} < \beta. \end{aligned}$$

Notice that $\sup_X \min\{g(x,y_1), g(x,y_n)\} < \beta$. *Thus, by applying the above Lemma again, we can obtain* $y_0 \in Y$ *such that* $\sup_X f(x,y_0) < \alpha$.

□

Example 3.3. *Let* g, f *be two real-valued functions defined on* $[-1,1] \times [-1,1]$ *by*

$$g(x,y) = \begin{cases} (1-x)(1-y^2) & x \geq 0 \\ (1+x)(1-y^2) & x < 0 \end{cases} \qquad f(x,y) = (1-x^2)(1-y^2),$$

figure 1

We will check the conditions of Theorem3.1

(0)*Since* $\sup_X g(x,y) = 1-y^2 = \sup_X f(x,y)$, $\sup_X \min\{g(x,y_1), f(x,y_2)\} = \min\{1-y_1^2, 1-y_2^2\} = \sup_X \min\{g(x,y_1, g(x,y_2)\}$

(*i*)*Since* $f(\cdot,y)$ *and* $g(\cdot,y)$ *are continuous on* X *for each* $y \in Y$, *we have* $f(\cdot,y)$ *and* $g(\cdot,y)$ *are upper semicontinuous on* X *for each* $y \in Y$.

(*ii*)*for any* $y \in Y$

$$\{x \in X; g(x,y) \geq \lambda\} = \begin{cases} \left[\frac{\lambda}{1-y^2} - 1, 1 - \frac{\lambda}{1-y^2}\right] & if \quad \lambda \in [0, 1-y^2] \\ \emptyset & if \quad \lambda \in (-\infty, 0) \cup (1-y^2, \infty) \end{cases}.$$

Hence, for any $y_1, \cdots, y_n \in Y$, *let* $\hat{y} = \max\{|y_1|, |y_2|, \cdots, |y_n|\}$, *the set*

$$\cap_{i=1}^{n}\{x \in X; g(x,y_i) \geq \lambda\} = \begin{cases} \left[\frac{\lambda}{1-\hat{y}^2} - 1, 1 - \frac{\lambda}{1-\hat{y}^2}\right] & if \quad \lambda \in [0, 1-\hat{y}^2] \\ \emptyset & if \quad \lambda \in (-\infty, 0) \cup (1-\hat{y}^2, \infty) \end{cases}$$

is either connected or empty for any $\lambda \in \mathbb{R}$.

And for all $y \in Y$,

$$\{x \in X; f(x,y) \geq \lambda\} = \begin{cases} \left[-\sqrt{1 - \frac{\lambda}{1-y^2}}\ ,\ \sqrt{1 - \frac{\lambda}{1-y^2}}\ \right] & if \quad \lambda \in [0, 1-y^2] \\ \emptyset & if \quad \lambda \in (-\infty, 0) \cup (1-y^2, \infty) \end{cases}$$

is either connected or empty for any $\lambda \in \mathbb{R}$.

(*iii*)*Clearly,* $\max\{f(x,y), g(x,y)\} = f(x,y)$, $\forall (x,y) \in X \times Y$.

If $|g(x,y_1) - f(x,y_2)| \geq \epsilon$, *we take* $\delta = \epsilon$, *and let* $k = \min\{1-y_1^2, 1-y_2^2\}$. *Then we have* $(1+x)\frac{k^2}{3} \leq 1$ *and* $(1-x)\frac{k^2}{3} \leq 1$

Let $y_3 = \sqrt{1 - \frac{k^2}{3}}$, *we have* $f(x,y_3) = (1-x^2)\frac{k^2}{3}$ *and*

(a) *if* $g(x,y_1) \geq f(x,y_2)+\epsilon$, *then* $g(x,y_1)-\delta \geq f(x,y_2) = (1-x^2)(1-y_2^2) \geq (1-x^2)k \geq (1-x^2)\frac{k^2}{3} = f(x,y_3)$

(b) *if* $f(x,y_2) \geq g(x,y_1)+\epsilon$, *then*

$$\begin{cases} for \quad x \geq 0 & f(x,y_2)-\delta \geq g(x,y_1) = (1-x)(1-y_1^2) \geq (1-x)k \geq (1-x^2)\frac{k^2}{3} = f(x,y_3); \\ for \quad x < 0 & f(x,y_2)-\delta \geq g(x,y_1) = (1+x)(1-y_1^2) \geq (1+x)k \geq (1-x^2)\frac{k^2}{3} = f(x,y_3). \end{cases}$$

That is, f and g are jointly upward.
Thus, by Theorem 3.1, we have

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

In fact, $\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} g(x,y) = 0$.

Corollary 3.4. (*Lin and Yu [5]*) *Let X be a nonempty compact topological space and let Y be a nonempty set. Let f, g be two real-valued functions defined on $X\times Y$ with the following properties:*
(0) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X\times Y$;
(*i*) $f(\cdot,y)$ *and* $g(\cdot,y)$ *are upper semicontinuous on X for each $y \in Y$;*
(*ii*) *For any* $y_1,\cdots,y_n \in Y$, *any* $\lambda \in \mathbb{R}$, *the set* $\cap_{i=1}^n\{x\in X; g(x,y_i)\geq\lambda\}$ *is either connected or empty;*
(*iii*) *f and g are jointly upward.*
Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

When $f=g$ are upward or t-convex, then f and g are jointly upward. Thus we have

Corollary 3.5. (*Simons [11] and Geraghty-Lin [10]*) *Let X be a nonempty compact topological space and let Y be a nonempty set. Let f be a real-valued function defined on $X\times Y$ such that* $\inf_Y\sup_X f(x,y) > -\infty$ *and which satisfies the following properties:*
(*i*) $f(\cdot,y)$ *is upper semicontinuous on X for each $y\in Y$;*
(*ii*) *For any* $y_1,\cdots,y_n \in Y$, *any* $\lambda\in\mathbb{R}$, *the set* $\cap_{i=1}^n\{x\in X; f(x,y_i)\geq\lambda\}$ *is either connected or empty;*
(*iii*) *f is either upward or t-convex on Y.*
Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$