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指導教授：陳界山 博士

Lagrangian Globalization method together with new NCP-functions for the Nonlinear Complementarity Problem

研 究 生：黃裕偉

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致　謝

僅將誌謝獻給每一個曾經在我的人生路上給我鼓勵的你們。

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Yu-Wei Huang

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Abstract. In this paper, we look into the detailed properties of six NCP-functions. Based on these NCP functions, we present a Lagrangian globalization (LG) algorithm model for solving the nonlinear complementarity problem. In particular, this algorithm model does not depend on some specific NCP function. Under several theoretical assumptions on NCP functions. We prove that the algorithm model is well-defined. Several NCP functions applicable to the LG-method are analyzed in details and shown to satisfy these assumptions. Furthermore, we identify not only the properties of NCP functions which enable them to be used in the LG method but also their properties which enable the strict complementarity condition to be removed from the convergence conditions of the LG method.

Keywords. NCP-function; Nonlinear complementarity problem; Lagrangian globalization.

1 Introduction

The nonlinear complementarity problem (NCP) [11, 17] is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ \ F(x) \geq 0, \langle x,\ \ F(x)\rangle = 0$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \cdots, F_n)^T$ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The NCP(F) can be equivalently reformulated as a system of nonlinear equations,

$$\Phi(x) = 0 \tag{1}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz but not differentiable and defined by.

$$\Phi(x) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix},$$

where ϕ is NCP-function.

The LG-method associates an objective function f with the system (1) to give an equality constrained nonlinear programming problem of the from

$$\min \quad f(x) \quad s.t. \;\; \Phi(x) = 0, \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is assumed to be continuously differentiable. For example, $f(x) = \alpha e^T x$, where e is the vector of all ones, whereas $\alpha \neq 0$ is a constant. See [16] for specific choices of f.

The Lagrangian and augmented Lagrangian of (2) are respectively

$$L(x, \lambda) := f(x) + \lambda^T \Phi(x), \tag{3}$$

$$P_c(z) := P_c(x, \lambda) := L(x, \lambda) + \frac{1}{2} c ||\Phi(x)||^2, \tag{4}$$

where $\lambda \in \mathbb{R}^n$ is the Lagrange multiplier vector, c is a nonnegative real parameter, $z = (x, \lambda) \in \mathbb{R}^n$. We propose a descent algorithm for solving the unconstrained nonsmooth optimization problem

$$\min_{z \in \mathbb{R}^{2n}} P_c(z). \tag{5}$$

We prove that, the negative of a generalized gradient direction of the objective function P_c is a descent direction of P_c at any noncritical point of P_c.

We will briefly introduce the NCP function used in this paper. A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if it satisfies

$$\phi(a, b) = 0 \iff a \geq 0, \; b \geq 0, \; ab = 0. \tag{6}$$

Many NCP-functions and merit functions have been explored and proposed in many literature, see [10] for a survey. Among them, the Fischer-Burmeister (FB) function and the Natural-Residual (NR) function are two effective NCP-functions. The FB function $\phi_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\rm FB}(a, b) = \sqrt{a^2 + b^2} - (a + b), \tag{7}$$

and the NR function $\phi_{\rm NR} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\rm NR}(a, b) = a - (a - b)_+ = \min\{a, b\}, \tag{8}$$

where $(t)_+$ means $\max\{0, t\}$ for any $t \in \mathbb{R}$.

Recently, the generalized Fischer-Burmeister function $\phi^p_{\rm FB}$ which includes the Fischer-Burmeister as a special case was considered in [2, 3, 4, 6, 22]. Indeed, the function $\phi^p_{\rm FB}$ is a natural extension of the $\phi_{_{FB}}$ function, in which the 2-norm in $\phi_{_{FB}}$ is replaced by general p-norm. In other words, $\phi^p_{_{FB}} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\phi^p_{_{FB}}(a,b) = \|(a,b)\|_p - (a+b), \tag{9}$$

where $p > 1$ and $\|(a,b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. The detailed geometric view of $\phi^p_{_{FB}}$ is depicted in [22]. Corresponding to $\phi^p_{_{FB}}$, there is a merit function $\psi^p_{_{FB}} : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$\psi^p_{_{FB}}(a,b) = \frac{1}{2}\left|\phi^p_{_{FB}}(a,b)\right|^2.$$

For any given $p > 1$, the function $\psi^p_{_{FB}}$ is a nonnegative NCP-function and smooth on $\mathbb{R}^2$. Note that $\phi^p_{_{FB}}$ is a natural "continuous" type of generalization of the FB function $\phi_{_{FB}}$.

To the contrast, what does "generalized natural-residual function" look like? In [5], Chen *et al.* give an answer to the long-standing open question. More specifically, the generalized natural-residual function, denoted by $\phi^p_{_{NR}}$, is defined by

$$\phi^p_{_{NR}}(a,b) = a^p - (a-b)^p_+, \tag{10}$$

with $p > 1$ being a positive odd integer. As remarked in [5], the main idea to create it relies on "discrete generalization", not the "continuous generalization". Note that when $p = 1$, $\phi^p_{_{NR}}$ reduces to the natural residual function $\phi_{_{NR}}$.

Unlike the surface of $\phi^p_{_{FB}}$, the surface of $\phi^p_{_{NR}}$ is not symmetric which may cause some difficulties in further analysis in designing solution methods. To this end, Chang *et al.* [1] try to symmetrize the function $\phi^p_{_{NR}}$. The first-type symmetrization of $\phi^p_{_{NR}}$, denoted by $\phi^p_{_{S-NR}}$ is proposed as

$$\phi^p_{_{S-NR}}(a,b) = \begin{cases} a^p - (a-b)^p & ,\text{if} \quad a > b, \\ a^p = b^p & ,\text{if} \quad a = b, \\ b^p - (b-a)^p & ,\text{if} \quad a < b, \end{cases} \tag{11}$$

where $p > 1$ being positive odd integer. It is shown in [1] that $\phi^p_{_{S-NR}}$ is an NCP-function with symmetric surface, but it is not differentiable. Therefore, it is natural to ask whether there exists another symmetrization function that has not only symmetric surface but also is differentiable. Fortunately, Chang *et al.* [1] also figure out the second symmetrization of $\phi^p_{_{NR}}$, denoted by $\psi^p_{_{S-NR}}$, which is proposed as

$$\psi^p_{_{S-NR}}(a,b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ a^p b^p - (b-a)^p a^p & \text{if} \quad a < b, \end{cases} \tag{12}$$

where $p > 1$ being positive odd integer.

The idea of "discrete generalization" looks simple, but it is novel and important. In fact, we also apply such idea to construct more NCP-functions. For example, we apply it to the Fischer-Burmeister function to obtain $\phi^p_{\text{D-FB}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi^p_{\text{D-FB}}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p, \tag{13}$$

where $p > 1$ is a positive odd integer. This function is proved as an NCP-function in [14]. In addition, it can also serve as a complementarity function for second-order cone complementarity problem (SOCCP) [14].

One can see that the second symmetrization of ϕ^p_{NR} is differentiable and symmetric simultaneously. Now, A new idea comes from the other expression of minimum function

$$\min\{a,b\} = a - (a-b)_+ = \frac{(a+b) - |a-b|}{2}.$$

Hence, [21] define a hole new NCP-function $\phi^p_{\text{A-NR}}$ with $p > 1$ being a positive odd integer as follow

$$\phi^p_{\text{A-NR}}(a,b) = \left(\frac{a+b}{2}\right)^p - \left(\frac{|a-b|}{2}\right)^p = \frac{1}{2^p}\left[(a+b)^p - |a-b|^p\right]. \tag{14}$$

Notice that when $p = 1$, $\phi^p_{\text{A-NR}}$ reduces to the natural-residual function.

The aforementioned six types of NCP-functions are discovered. Even though we have the feature of differentiability, we point out that the Newton method may not be applied directly because the Jacobian at a degenerate solution to NCP may be singular (see [12, 13]). Nonetheless, the feature of differentiability may enable that some other methods relying on differentiability (like quasi-Newton methods, neural network methods) can be employed directly for solving NCP. In this paper, we applies of these six NCP-functions by Lagragian globalization method and presents a descent algorithm for the LG phase.

The organization of the paper is as follow. In Section 2, we review briefly some basic properties of semismooth function and present some preliminary results which are related closely to problem (5). In Section 3, we analyzing the properties of these NCP functions. In Section 4, we use the results in Section 3, and prove the descent property of the generalized gradients of P_c.

2 Preliminaries

In this section, we recall some background concepts and materials which will play an important role in the subsequent analysis. We begin with the so-called semismooth functions. Semismooth function, as introduced by Mifflin [15] for functionals and further extended by Qi and Sun [18] for vector-valued functions, is of particular interest due to the central role it plays in the superlinear convergence analysis of certain generalized Newton methods, see [18]. First, we say that $F : \mathbb{R}^n \to \mathbb{R}^m$ is *strictly continuous* (also called locally Lipschitz continuous) at $x \in \mathbb{R}^n$ [20, Chap. 9] if there exist scalars $\kappa > 0$ and $\delta > 0$ such that

$$\|F(y) - F(z)\| \leq \kappa \|y - z\| \quad \forall y, z \in \mathbb{R}^n \ \text{ with } \ \|y - x\| \leq \delta \ \text{ and } \ \|z - x\| \leq \delta.$$

The mapping F is locally Lipschitz continuous if F is locally Lipschitz continuous at every $x \in \mathbb{R}^n$. If δ can be taken to be ∞, then F is Lipschitz continuous with Lipschitz constant κ. We say F is directionally differentiable at $x \in \mathbb{R}^n$ if

$$F'(x; h) := \lim_{t \to 0^+} \frac{F(x + th) - F(x)}{t} \quad \text{exists} \quad \forall h \in \mathbb{R}^n;$$

and F is directionally differentiable if F is directionally differentiable at every $x \in \mathbb{R}^n$. If F is locally Lipschitz continuous, then F is almost everywhere differentiable by Rademacher's Theorem, see [20, Section 9J]. In this case, the generalized Jacobian $\partial F(x)$ of F at x (in the Clarke sense) can be defined as the convex hull of B-subdifferential $\partial_B F(x)$, where

$$\partial_B F(x) := \left\{ \lim_{x^j \to x} \nabla F(x^j) \,\Big|\, F \text{ is differentiable at } x^j \in \mathbb{R}^n \right\}.$$

Assume F is locally Lipschitz continuous. We say F is *semismooth at* $x \in \mathbb{R}^n$ if F is directionally differentiable at $x \in \mathbb{R}^n$ and, for any $V \in \partial F(x + h)$, we have

$$F(x + h) - F(x) - Vh = o(\|h\|). \tag{15}$$

The following results can be found in [9, 18]. For more details about semismooth function, see Qi and Sun [18].

Lemma 2.1. *Suppose that $p : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$ are Lipschitzian near $x \in \mathbb{R}^n$ and near $p(x)$, respectively. Then the following statements hold.*

(a) *The composite function $\vartheta = g \circ p$ is Lipschitzian near x and*

$$\partial \vartheta \subseteq conv\{\partial g(p(x)) \partial p(x)\}.$$

In particular, if g is regular at $p(x)$ and p is continuously differentiable at x, then

$$\partial \vartheta = \partial g(p(x)) \nabla p(x)^T.$$

(b) *If p and g are semismooth at x and $p(x)$. respectively, then the composite function ϑ is also semismooth at x and*

$$\vartheta'(x; d) = g'(p(x); p'(x; d)), \quad d \in \mathbb{R}^n.$$

(c) *If p and g are strongly semismooth at x and $p(x)$, respectively, then the composite function ϑ is also strongly semismooth at x.*

In order to facilitate the differentiation of these NCP functions, we need three technical lemmas.

Lemma 2.2. *Let $p > 1$. Then*

(a) *the function $f(t) = |t|^p$ is differentiable and $f'(t) = p \cdot sgn(t) \cdot |t|^{p-1}$.*

(b) *the function $f(t) = t^p|t|$ is differentiable and $f'(t) = (p+1) \cdot t^{p-1} \cdot |t|$.*

Proof.

(a) By definition of absolute value, we have $f(t) = \begin{cases} t^p & ,\text{if} \quad t \geq 0, \\ (-t)^p & ,\text{if} \quad t < 0. \end{cases}$

If $t \neq 0$, then it is clearly for $f'(t) = \begin{cases} pt^{p-1} & ,\text{if} \quad t > 0, \\ -p(-t)^{p-1} & ,\text{if} \quad t < 0. \end{cases}$

If $t = 0$, then $\begin{cases} \lim\limits_{t \to 0^+} \dfrac{f(t) - 0}{t - 0} = \lim\limits_{t \to 0^+} \dfrac{t^P}{t} = 0, \\ \lim\limits_{t \to 0^-} \dfrac{f(t) - 0}{t - 0} = \lim\limits_{t \to 0^-} \dfrac{(-t)^P}{t} = 0, \end{cases}$ hence we have $f'(0) = 0$.

In summary, we can deduce that $f'(t) = p \cdot sgn(t) \cdot |t|^{p-1}$.

(b) By definition of absolute value, we have $f(t) = \begin{cases} t^{p+1} & ,\text{if} \quad t \geq 0, \\ -t^{p+1} & ,\text{if} \quad t < 0. \end{cases}$

If $t \neq 0$, then it is clearly for $f'(t) = \begin{cases} (p+1)t^p & ,\text{if} \quad t > 0, \\ -(p+1)t^p & ,\text{if} \quad t < 0. \end{cases}$

If $t = 0$, then $\begin{cases} \lim\limits_{t \to 0^+} \dfrac{f(t) - 0}{t - 0} = \lim\limits_{t \to 0^+} \dfrac{t^{P+1}}{t} = 0, \\ \lim\limits_{t \to 0^-} \dfrac{f(t) - 0}{t - 0} = \lim\limits_{t \to 0^-} \dfrac{(-t)^{P+1}}{t} = 0, \end{cases}$ hence we have $f'(0) = 0$.

In summary, we can deduce that $f'(t) = (p+1) \cdot t^{p-1} \cdot |t|$. $\square$

Lemma 2.3. *Let $p > 1$. Then*

(a) *an alternative expression of the function $f(t) = (t)_+$ is $f(t) = \frac{1}{2}(t + |t|)$.*

(b) *if $t \neq 0$, then the function $f(t) = (t)_+$ is differentiable and $f'(t) = \frac{(t)_+}{t}$.*

Proof.

(a) Because we have $t_+ = \max\{t, 0\} = \begin{cases} t & ,\text{if} \quad t \geq 0, \\ 0 & ,\text{if} \quad t < 0, \end{cases}$ and

$$\frac{t + |t|}{2} = \begin{cases} \frac{t+t}{2} & ,\text{if} \quad t \geq 0 \\ \frac{t-t}{2} & ,\text{if} \quad t < 0 \end{cases} = \begin{cases} t & ,\text{if} \quad t \geq 0, \\ 0 & ,\text{if} \quad t < 0, \end{cases}$$

so we obviously have $f(t) = (t)_+ = \dfrac{t+|t|}{2}$.

(b) If $t \neq 0$ and by (a), then $f(t) = t_+ = \dfrac{t+|t|}{2} = \begin{cases} t & ,\text{if} \quad t > 0, \\ 0 & ,\text{if} \quad t < 0, \end{cases}$ we can deduce that

$$f'(t) = \begin{cases} 1 & ,\text{if} \quad t > 0 \\ 0 & ,\text{if} \quad t < 0 \end{cases} = \begin{cases} \frac{t}{t} & ,\text{if} \quad t > 0, \\ \frac{0}{t} & ,\text{if} \quad t < 0, \end{cases} = \frac{(t)_+}{t}.$$

Thus, the desired result follows. □

Lemma 2.4. *Let $p > 1$, and p is a integer. Then*

(a) *an alternative expression of the function $f(t) = [(t)_+]^p$ is $f(t) = \frac{t^{p-1}}{2}(t+|t|) = t^{p-1}(t_+)$.If p is an odd integer, then $f(t) = [(t)_+]^p = (t^p)_+ = (t)_+^p$.*

(b) *the function $f(t) = (t_+)^p$ is differentiable and $f'(t) = p \cdot t^{p-2} \cdot (t_+)$.*

Proof. (a) By Lemma 2.2 (a), we have $t_+ = \frac{1}{2}(t+|t|)$.

$$\begin{aligned}
(t_+)^p &= \left(\frac{1}{2}(t+|t|)\right)^p = \frac{1}{2^p}(t+|t|)^p = \frac{1}{2^p}\sum_{k=0}^{p}\binom{p}{k} \cdot |t|^k \cdot t^{p-k} \\
&= \frac{1}{2^p}\left(\sum_{k=even}^{p}\binom{p}{k} \cdot |t|^k \cdot t^{p-k} + \sum_{k=odd}^{p}\binom{p}{k} \cdot |t|^k \cdot t^{p-k}\right) \\
&= \frac{1}{2^p}\left(\sum_{k=even}^{p}\binom{p}{k} \cdot t^{p-1} \cdot t + \sum_{k=odd}^{p}\binom{p}{k} \cdot t^{p-1} \cdot |t|\right) \\
&= \frac{1}{2^p}(2^{p-1} \cdot t^{p-1}t + 2^{p-1} \cdot t^{p-1}|t|) \\
&= t^{p-1} \cdot \frac{t+|t|}{2} = t^{p-1}(t_+).
\end{aligned}$$

If p is a positive odd integer, then $t^{p-1} = |t|^{p-1}$, we can deduce that

$$(t_+)^p = t^{p-1}(t_+) = t^{p-1} \cdot \frac{t+|t|}{2} = \frac{t^p+|t|^p}{2} = \frac{t^p+|t^p|}{2} = (t^p)_+ = (t)_+^p.$$

(b) By Lemma 2.2 (b), we have $\frac{d}{dt}[(t)_+] = \frac{(t)_+}{t}$.
(i) If $t \neq 0$, then

$$\frac{d}{dt}(t_+)^p = p(t_+)^{p-1} \cdot \frac{t_+}{t} = p \cdot t^{p-2} \cdot (t_+).$$

(ii) If $t = 0$, then

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(h_+)^p}{h} = \lim_{h\to 0}\frac{h^{p-1}(h_+)}{h} = 0 = f'(0).$$

In summary, we conclude that $f'(t) = p \cdot t^{p-2} \cdot (t_+)$. □

3 Properties of several NCP functions

We rearrange the NCP functions used in this paper now.

$$\phi^p_{\rm FB}(a,b) = \|(a,b)\|_p - (a+b),$$

where $p > 1$ is an arbitrary fixed real number.

$$\phi^p_{\rm NR}(a,b) = a^p - (a-b)^p_+,$$

$$\phi^p_{\rm S-NR}(a,b) = \begin{cases} a^p - (a-b)^p & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ b^p - (b-a)^p & \text{if} \quad a < b, \end{cases}$$

$$\psi^p_{\rm S-NR}(a,b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ a^p b^p - (b-a)^p a^p & \text{if} \quad a < b, \end{cases}$$

$$\phi^p_{\rm D-FB}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p,$$

$$\phi^p_{\rm A-NR}(a,b) = \left(\frac{a+b}{2}\right)^p - \left(\frac{|a-b|}{2}\right)^p,$$

where $p > 1$ is a positive odd integer.

To define a descent direction for P_c, we use absolute function. In this section, we will give the detailed materials about the generalized gradients and the directional derivatives of these NCP functions, which are very useful in the analysis of the next section.

We consider the absolute function of these NCP-functions

$$\phi^p_{|\rm FB|}(a,b) := |\phi^p_{\rm FB}(a,b)| = \begin{cases} -\phi^p_{\rm FB}(a,b) & \text{if} \quad a > 0,\ b > 0, \\ \phi^p_{\rm FB}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{|\rm NR|}(a,b) := |\phi^p_{\rm NR}(a,b)| = \begin{cases} \phi^p_{\rm NR}(a,b) & \text{if} \quad a > 0,\ b > 0, \\ -\phi^p_{\rm NR}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{|\rm S-NR|}(a,b) := |\phi^p_{\rm S-NR}(a,b)| = \begin{cases} \phi^p_{\rm S-NR}(a,b) & \text{if} \quad a > 0,\ b > 0, \\ -\phi^p_{\rm S-NR}(a,b) & \text{otherwise}, \end{cases}$$

$$\psi^p_{|\rm S-NR|}(a,b) := |\psi^p_{\rm S-NR}(a,b)| = \psi^p_{\rm S-NR}(a,b),$$

$$\phi^p_{|\rm D-FB|}(a,b) := |\phi^p_{\rm D-FB}(a,b)| = \begin{cases} -\phi^p_{\rm D-FB}(a,b) & \text{if} \quad a > 0,\ b > 0, \\ \phi^p_{\rm D-FB}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{|\rm A-NR|}(a,b) := |\phi^p_{\rm A-NR}(a,b)| = \begin{cases} \phi^p_{\rm A-NR}(a,b) & \text{if} \quad a > 0,\ b > 0, \\ -\phi^p_{\rm A-NR}(a,b) & \text{otherwise}. \end{cases}$$

Proposition 3.1. *Let ϕ^p_{FB} be defined as in (9) with $p > 1$ being an arbitrary fixed real number. Then, the following hold.*

(a) *If $(a, b) \neq (0, 0)$, then ϕ^p_{FB} is continuously differentiable at (a, b) and*

$$\nabla \phi^p_{\text{FB}}(a, b) = \left[\frac{\text{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1, \frac{\text{sgn}(b) \cdot |b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1 \right]^T.$$

If $(a, b) = (0, 0)$, then the generalized gradient of ϕ^p_{FB} at (a, b) is

$$\partial \phi^p_{\text{FB}}(a, b) = \{(\xi - 1, \zeta - 1) | \ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}.$$

(b) *If $\phi^p_{|\text{FB}|}(a, b) \neq 0$, then $\phi^p_{|\text{FB}|}$ is continuously differentiable at (a, b) and*

$$\nabla \phi^p_{|\text{FB}|}(a, b) = \begin{cases} -\nabla \phi^p_{\text{FB}}(a, b) & \text{if} \quad a > 0, \ b > 0, \\ \nabla \phi^p_{\text{FB}}(a, b) & \text{otherwise.} \end{cases}$$

If $\phi^p_{|\text{FB}|}(a, b) = 0$, then the generalized gradient of $\phi^p_{|\text{FB}|}$ at (a, b) is

$$\partial \phi^p_{|\text{FB}|}(a, b) = \begin{cases} \{(\rho, 0) | \rho \in [-1, 1]\} & \text{if} \quad a = 0, \ b > 0, \\ \{(0, \rho) | \rho \in [-1, 1]\} & \text{if} \quad a > 0, \ b = 0, \\ \Xi & \text{if} \quad a = 0, \ b = 0, \end{cases}$$

where $\Xi \subseteq conv\{\Xi_1 \bigcup \Xi_2\}$, here

$$\begin{cases} \Xi_1 = \{(1 - \xi, 1 - \zeta) | \xi \geq 0 \text{ and } \zeta \geq 0, \ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}, \\ \Xi_2 = \{(\xi - 1, \zeta - 1) | \xi \leq 0 \text{ or } \zeta \leq 0, \ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}. \end{cases}$$

(c) *The directional derivative of $\phi^p_{|\text{FB}|}$ at (a, b) in the direction $v = (v_1, v_2)^T$ is*

$${\phi^p_{|\text{FB}|}}'((a, b); v) = \begin{cases} \langle \nabla \phi^p_{|\text{FB}|}(a, b), v \rangle & \text{if} \quad \phi^p_{|\text{FB}|}(a, b) \neq 0, \\ |v_1| & \text{if} \quad a = 0, \ b > 0, \\ |v_2| & \text{if} \quad a > 0, \ b = 0, \\ \phi^p_{|\text{FB}|}(v_1, v_2) & \text{if} \quad a = 0, \ b = 0. \end{cases}$$

Proof.
(a) It is clear from [7, Lemma 2.2].

(b) If $\phi^p_{|\text{FB}|}(a, b) \neq 0$ then it is clearly by Proposition 3.1 (a).
(i) For $a = 0$, $b > 0$, we note that

$$\lim_{h \to 0^+} \frac{\phi^p_{|\text{FB}|}(0 + h, b) - \phi^p_{|\text{FB}|}(0, b)}{h} = \lim_{h \to 0^+} \frac{(h + b) - ||(h, b)||_p}{h} = -\frac{\partial \phi^p_{\text{FB}}(0, b)}{\partial a} = 1,$$

$$\lim_{h\to 0^-}\frac{\phi^p_{|FB|}(0+h,b)-\phi^p_{|FB|}(0,b)}{h}=\lim_{h\to 0^-}\frac{||(h,b)||_p-(h+b)}{h}=\frac{\partial\phi^p_{FB}(0,b)}{\partial a}=-1,$$

$$\frac{\partial\phi^p_{|FB|}}{\partial b}=\lim_{h\to 0}\frac{\phi^p_{|FB|}(0,b+h)-\phi^p_{|FB|}(0,b)}{h}=0.$$

So $\phi^p_{|FB|}$ is not differentiable at $(0,b)$. According to the definition of Clarke's generalized gradient,

$$\partial\phi^p_{|FB|}(0,b)=conv\left\{\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|FB|}(a_i,b_i)\,\middle|\,\phi^p_{|FB|}\text{ is differentiable at }(a_i,b_i)\in\mathbb{R}^2\right\},$$

we discuss three cases as below.
(1) If $a_i>0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|FB|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(0,b)}-\nabla\phi^p_{FB}(a_i,b_i)\\ &= \lim_{(a_i,b_i)\to(0,b)}\left[1-\frac{|a_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}},1-\frac{|b_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}\right]^T=[1,0]^T.\end{aligned}$$

(2) If $a_i<0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|FB|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{FB}(a_i,b_i)\\ &= \lim_{(a_i,b_i)\to(0,b)}\left[\frac{sgn(a_i)|a_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}-1,\frac{sgn(b_i)|b_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}-1\right]^T=[-1,0]^T.\end{aligned}$$

(3) For the remainder case, $\nabla\phi^p_{|FB|}(a_i,b_i)$ has no limit as $(a_i,b_i)\to(0,b)$.
From all the above, we conclude that

$$\partial\phi^p_{|FB|}(0,b)=conv\{(1,0),(-1,0)\}=\{(\rho,0)|\rho\in[-1,1]\}.$$

(ii) For $a>0$, $b=0$, the proof is similar to (i).

(iii) For $a=0$, $b=0$.
(1) Suppose that $\phi_p:\mathbb{R}^2\to\mathbb{R}$, $\psi_p:\mathbb{R}^3\to\mathbb{R}$ are defined as below

$$\phi^p(a,b):=||(a,b)||_p=\sqrt[p]{|a|^p+|b|^p},$$

$$\psi^p(a,b,\varepsilon):=\sqrt[p]{|a|^p+|b|^p+\varepsilon^p}.$$

Using an elementary calculation, we immediately obtain that

$$\left(\frac{\partial\phi^p(a,b)}{\partial a},\frac{\partial\phi^p(a,b)}{\partial b}\right)=\left(\frac{sgn(a)|a|^{p-1}}{\left(\sqrt[p]{|a|^p+|b|^p}\right)^{p-1}},\frac{sgn(b)|a|^{p-1}}{\left(\sqrt[p]{|b|^p+|b|^p}\right)^{p-1}}\right),\tag{16}$$

with $(a,b) \neq (0,0)$.

$$\left(\frac{\partial \psi^p(a,b,\varepsilon)}{\partial a}, \frac{\partial \psi^p(a,b,\varepsilon)}{\partial b}\right) = \left(\frac{sgn(a)|a|^{p-1}}{\left(\sqrt[p]{|a|^p+|b|^p+\varepsilon^p}\right)^{p-1}}, \frac{sgn(b)|a|^{p-1}}{\left(\sqrt[p]{|b|^p+|b|^p+\varepsilon^p}\right)^{p-1}}\right), \quad (17)$$

for any fixed $\varepsilon > 0$, since $\frac{\partial \psi^p}{\partial a}$, $\frac{\partial \psi^p}{\partial b}$ are continuous at all $(a,b) \in \mathbb{R}^2$, it follows that $\phi(a,b,\varepsilon)$ is continuously differentiable at all $(a,b) \in \mathbb{R}^2$.

(2) For $\varepsilon > 0$, an elementary calculation yields that

$$\frac{\partial \psi^p(a,b,\varepsilon)}{\partial \varepsilon} = \frac{\varepsilon^{p-1}}{\left(\sqrt[p]{|a|^p+|b|^p+\varepsilon^p}\right)^{p-1}} \leq 1,$$

for any $(a,b) \in \mathbb{R}^2$, $\psi^p(a,b,\varepsilon)$ is continuously differentiable with $\varepsilon > 0$. By the mean-value theorem for $0 < \varepsilon_1 < \varepsilon_2$, there exists some $\varepsilon_0 \in (\varepsilon_1, \varepsilon_2)$ such that

$$\psi^p(a,b,\varepsilon_2) - \psi^p(a,b,\varepsilon_1) = \frac{\psi_p(a,b,\varepsilon_0)}{\partial \varepsilon}(\varepsilon_2 - \varepsilon_1) \leq (\varepsilon_2 - \varepsilon_1).$$

We can use the result to deduce that

$$|\psi^p(a,b,\varepsilon) - \phi^p(a,b)| \leq \varepsilon \longrightarrow 0,$$

as $\varepsilon_2 \to \varepsilon$, $\varepsilon_1 \to 0$, for all $\varepsilon \geq 0$ and $\varepsilon \to 0$.

(3) Using the formula (17), it is easy to calculate that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\partial \psi^p(a,b,\varepsilon)}{\partial a} &= \begin{cases} \frac{sgn(a)|a|^{p-1}}{\left(\sqrt[p]{|a|^p+|b|^p}\right)^{p-1}} & \text{if} \quad (a,b) \neq (0,0), \\ 0 & \text{if} \quad (a,b) = (0,0), \end{cases} \\ \lim_{\varepsilon \downarrow 0} \frac{\partial \psi^p(a,b,\varepsilon)}{\partial b} &= \begin{cases} \frac{sgn(b)|b|^{p-1}}{\left(\sqrt[p]{|a|^p+|b|^p}\right)^{p-1}} & \text{if} \quad (a,b) \neq (0,0), \\ 0 & \text{if} \quad (a,b) = (0,0), \end{cases} \end{aligned} \quad (18)$$

let $\psi_0^p(a,b) := \left(\lim_{\varepsilon \downarrow 0} \frac{\partial \psi^p}{\partial a}, \lim_{\varepsilon \downarrow 0} \frac{\partial \psi^p}{\partial b}\right)$. By the formula (16) and (18), we have

$$\psi_0^p(a,b) = \begin{cases} \left(\frac{\partial \phi^p(a,b)}{\partial a}, \frac{\partial \phi^p(a,b)}{\partial b}\right) & \text{if} \quad (a,b) \neq (0,0), \\ (0,0) & \text{if} \quad (a,b) = (0,0). \end{cases}$$

The desired result follows by

$$\begin{aligned} \phi^p(h_1,h_2) - \phi^p(0,0) - \psi_0^p(h_1,h_2)^T h &= \sqrt[p]{|h_1|^p+|h_2|^p} - \frac{|h_1|^p+|h_2|^p}{\left(\sqrt[p]{|h_1|^p+|h_2|^p}\right)^{p-1}} \\ &= \sqrt[p]{|h_1|^p+|h_2|^p} - \sqrt[p]{|h_1|^p+|h_2|^p} \\ &= 0, \end{aligned}$$

where $h = (h_1, h_2) \in \mathbb{R}^2$, hence we have

$$\lim_{||h|| \to 0} \frac{\phi^p(a+h_1, b+h_2) - \phi^p(a,b) - \psi_0^p(a+h_1, b+h_2)^T h}{||h||} = 0,$$

with $(a,b) = (0,0)$, hence $|\frac{\partial \psi^p(0,0,\varepsilon)}{\partial a} - \frac{\text{sgn}(a)\cdot|a|^{p-1}}{\|(a,b)\|_p^{p-1}}| \to 0$, and $|\frac{\partial \psi^p(0,0,\varepsilon)}{\partial a} - \frac{\text{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}| \to 0$, as $(a,b) \to (0,0)$ and $\varepsilon \to 0$.

(4) Let $\xi_i := \frac{\partial \phi^p(a_i,b_i)}{\partial a} = \frac{\text{sgn}(a_i)\cdot|a_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}$ and $\zeta_i := \frac{\partial \phi^p(a_i,b_i)}{\partial b} = \frac{\text{sgn}(b_i)\cdot|b_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}$, and they satisfy $|\xi_i|^{\frac{p}{p-1}} + |\zeta_i|^{\frac{p}{p-1}} = 1$. By (1)-(3), hence we have ξ, ζ with $\xi_i \longrightarrow \xi$, $\zeta_i \longrightarrow \zeta$, (ξ, ζ) is satisfying $|\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} = 1$, so we can deduce that

$$\lim_{(a_i,b_i)\to(0,0)} \nabla \phi^p_{|\text{FB}|}(a_i,b_i) = \begin{cases} \lim\limits_{(a_i,b_i)\to(0,0)} -\nabla\phi^p_{\text{FB}}(a_i,b_i) & \text{if} \quad a_i > 0, b_i > 0, \\ \lim\limits_{(a_i,b_i)\to(0,0)} \nabla\phi^p_{\text{FB}}(a_i,b_i) & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \lim\limits_{(a_i,b_i)\to(0,0)} \left[1 - \frac{|a_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}, 1 - \frac{|b_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}}\right]^T & \text{if} \quad a_i > 0, b_i > 0, \\ \lim\limits_{(a_i,b_i)\to(0,0)} \left[\frac{sgn(a_i)|a_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}} - 1, \frac{sgn(b_i)|b_i|^{p-1}}{\|(a_i,b_i)\|_p^{p-1}} - 1\right]^T & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \lim\limits_{i\to\infty} [1-\xi_i, 1-\zeta_i]^T & \text{if} \quad \xi_i \geq 0, \zeta_i \geq 0, \\ \lim\limits_{i\to\infty} [\xi_i - 1, \zeta_i - 1]^T & \text{otherwise,} \end{cases}$$

$$= \begin{cases} [1-\xi, 1-\zeta]^T & \text{if} \quad \xi \geq 0, \zeta \geq 0, \\ [\xi - 1, \zeta - 1]^T & \text{otherwise,} \end{cases}$$

we conclude that

$$\partial \phi^p_{|\text{FB}|}(0,0) = \Xi \subseteq conv\{(1-\xi, 1-\zeta), (\xi - 1, \zeta - 1)\}.$$

(c) (i) If $a = 0$, $b > 0$, then given any $v = (v_1, v_2)^T$, we define

$$g(t) := \phi^p_{\text{FB}}(tv_1, b + tv_2) = ||(tv_1, b+tv_2)||_p - (b + tv_1 + tv_2)$$

If $tv_1 \neq 0$, or $b + tv_2 \neq 0$, then $g(t)$ is differentiable. So we have

$$\begin{aligned} \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} &= \frac{dg(t)}{dt} \\ &= \frac{v_1 \cdot sgn(tv_1)\cdot|tv_1|^{p-1} + v_2 \cdot sgn(b+tv_2)\cdot|b+tv_2|^{p-1}}{||(tv_1, b+tv_2)||_p^{p-1}} - (v_1 + v_2) \end{aligned}$$

and

$$\begin{cases} g'(0) = \frac{v_2|b|^{p-1}}{|b|^{p-1}} - (v_1 + v_2) = -v_1, \\ g'(0) = \lim\limits_{t\to 0} \frac{g(t) - g(0)}{t - 0} = \lim\limits_{t\to 0} \frac{g(t)}{t}. \end{cases}$$

So, we have $\lim_{t\to 0} \frac{g(t)}{t} = -v_1$. We can use the result to deduce that

$$
\begin{aligned}
\phi^p_{|\mathrm{FB}|}{}'((0,b);v) &= \lim_{t\to 0^+} \frac{\phi^p_{|\mathrm{FB}|}(tv_1, b+tv_2) - \phi^p_{|\mathrm{FB}|}(0,b)}{t} = \lim_{t\to 0^+} \frac{\phi^p_{|\mathrm{FB}|}(tv_1, b+tv_2)}{t} \\
&= \begin{cases} \lim_{t\to 0^+} \frac{-\phi^p_{\mathrm{FB}}(tv_1, b+tv_2)}{t} & \text{if} \quad v_1 > 0 \\ \lim_{t\to 0^+} \frac{\phi^p_{\mathrm{FB}}(tv_1, b+tv_2)}{t} & \text{if} \quad v_1 < 0 \end{cases} \\
&= \begin{cases} \lim_{t\to 0^+} \frac{-g(t)}{t} & \text{if} \quad v_1 > 0 \\ \lim_{t\to 0^+} \frac{g(t)}{t} & \text{if} \quad v_1 < 0 \end{cases} \\
&= \begin{cases} v_1 & ,\text{if} \quad v_1 > 0 \\ -v_1 & ,\text{if} \quad v_1 < 0 \end{cases} \\
&= |v_1|.
\end{aligned}
$$

(ii) For $a > 0$, $b = 0$, then the proof is similar to (i) and $\phi^p_{|\mathrm{FB}|}{}'((0,b);v) = |v_2|$.

(iii) If $a = 0$, $b = 0$, then given any $v = (v_1, v_2)^T$. For $t > 0$, we can deduce that

$$\phi^p_{|\mathrm{FB}|}(at, bt) = |\|(at, bt)\|_p - (at + bt)| = t|\|(a,b)\|_p - (a+b)| = t\phi^p_{|\mathrm{FB}|}(a,b).$$

So, we can use the result to deduce that

$$\phi^p_{|\mathrm{FB}|}{}'((0,0);v) = \lim_{t\to 0^+} \frac{\phi^p_{|\mathrm{FB}|}(tv_1, tv_2) - \phi^p_{|\mathrm{FB}|}(0,0)}{t} = \lim_{t\to 0^+} \frac{t\phi^p_{|\mathrm{FB}|}(v_1, v_2)}{t} = \phi^p_{|\mathrm{FB}|}(v_1, v_2).$$

Thus, the desired result follows. □

Remark 3.1. *The generalized gradient of $\phi^p_{|\mathrm{FB}|}$ at $(0,0)$ is Ξ. $\Xi = conv\{\Xi_1 \cup \Xi_2\}$ is not $\Xi \subseteq conv\{\Xi_1 \cup \Xi_2\}$ at $p = 2$, hence we guess that when $p > 1$ is an arbitrary fixed real number, it is " = " to not " ⊆ ", but unfortunately we only complete the included part. We look forward to the completion of the equal part.*

Proposition 3.2. *Let ϕ^p_{NR} be defined as in (10) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of ϕ^p_{NR} is*

$$\phi^p_{\mathrm{NR}}(a,b) = a^p - \frac{1}{2}\left((a-b)^p + (a-b)^{p-1}|a-b|\right).$$

(b) *The function* $\phi^p_{\rm NR}$ *is continuously differentiable with*

$$\nabla\phi^p_{\rm NR}(a,b) = p\begin{bmatrix} a^{p-1}-(a-b)^{p-2}(a-b)_+ \\ (a-b)^{p-2}(a-b)_+ \end{bmatrix}.$$

(c) *If* $(a,b)\in\Omega^C$*, then* $\phi^p_{|\rm NR|}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{|\rm NR|}(a,b) = \begin{cases} \nabla\phi^p_{\rm NR}(a,b) & \text{if} \quad a>0, b>0, \\ -\nabla\phi^p_{\rm NR}(a,b) & \quad \text{otherwise.} \end{cases}$$

If $a>0$*,* $b=0$ *then the generalized gradient of* $\phi^p_{|\rm NR|}$ *at* (a,b) *is*

$$\partial\phi^p_{|\rm NR|}(a,b) = \{(0,\rho\cdot pa^{p-1})|\rho\in[-1,1]\},$$

where $\Omega=\{(a,b)|a\geq 0, b=0\}$*.*

(d) *The function* $\phi^p_{|\rm NR|}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{|\rm NR|}(0,0)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(e) *The directional derivative of* $\phi^p_{|\rm NR|}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{|\rm NR|}{}'((a,b);v) = \begin{cases} \langle\nabla\phi^p_{|\rm NR|}(a,b),v\rangle & \text{if} \quad (a,b)\in\Omega^C, \\ |v_2|pa^{p-1} & \text{if} \quad a>0,\ b=0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}$$

Proof.
(a)-(b) It is clear from [5, Proposition 2.2].

(c) If $\phi^p_{|\rm NR|}(a,b)\neq 0$, then it is clearly by Proposition 3.2 (b).
If $\phi^p_{|\rm NR|}(a,b)=0$, by the definition of $\phi^p_{|\rm NR|}$, then we have

$$\phi^p_{|\rm NR|}(a,b) = \begin{cases} a^p-(a-b)^p & \text{if} \quad a>b>0, \\ a^p=b^p & \text{if} \quad a=b>0, \\ a^p & \text{if} \quad b>a>0, \\ -a^p=-b^p & \text{if} \quad a=b<0, \\ (a-b)^p-a^p & \text{if} \quad a>0>b \ or \ 0>a>b, \\ -a^p & \text{if} \quad 0>b>a \ or \ b>0>a. \end{cases}$$

(i) For $a=0$, $b>0$, we note that

$$\lim_{h\to 0^+}\frac{\phi^p_{|\rm NR|}(0+h,b)-\phi^p_{|\rm NR|}(0,b)}{h} = \lim_{h\to 0^+}\frac{h^p}{h}=0,$$

$$\lim_{h\to 0^-}\frac{\phi^p_{|\mathrm{NR}|}(0+h,b)-\phi^p_{|\mathrm{NR}|}(0,b)}{h}=\lim_{h\to 0^-}\frac{-h^p}{h}=0,$$

$$\frac{\partial\phi^p_{|\mathrm{NR}|}}{\partial b}=\lim_{h\to 0}\frac{\phi^p_{|\mathrm{NR}|}(0,b+h)-\phi^p_{|\mathrm{NR}|}(0,b)}{h}=0,$$

hence the partial derivative of $\phi^p_{|\mathrm{NR}|}$ exists at $(0,b)$, we have

$$\nabla\phi^p_{|\mathrm{NR}|}(0,b)=\left(\frac{\partial\phi^p_{|\mathrm{NR}|}}{\partial a}(0,b),\frac{\partial\phi^p_{|\mathrm{NR}|}}{\partial b}(0,b)\right)=(0,0).$$

Because ϕ^p_{NR} is differentiable at $(0,b)$, so we have

$$\frac{|\ \phi^p_{\mathrm{NR}}(a_i,b_i)-\phi^p_{\mathrm{NR}}(0,b)-\langle(0,0),(a_i,b_i)\rangle\ |}{\sqrt{a_i^2+b_i^2}}=\frac{|\phi^p_{\mathrm{NR}}(a_i,b_i)|}{\sqrt{a_i^2+b_i^2}}\longrightarrow 0,$$

as $(a_i,b_i)\to(0,b)$. We can use the result to deduce that

$$\frac{|\ \phi^p_{|\mathrm{NR}|}(a_i,b_i)-\phi^p_{|\mathrm{NR}|}(0,b)-\langle(0,0),(a_i,b_i)\rangle\ |}{\sqrt{a_i^2+b_i^2}}=\frac{|\phi^p_{|\mathrm{NR}|}(a_i,b_i)|}{\sqrt{a_i^2+b_i^2}}=\frac{|\phi^p_{\mathrm{NR}}(a_i,b_i)|}{\sqrt{a_i^2+b_i^2}}\longrightarrow 0,$$

as $(a_i,b_i)\to(0,b)$. So, $\phi^p_{|\mathrm{NR}|}$ is differentiable at $(0,b)$ with $\nabla\phi^p_{|\mathrm{NR}|}(0,b)=[0,0]^T$. To deduce $\nabla\phi^p_{|\mathrm{NR}|}(a,b)$ be continuously at $(0,b)$, we change the representation of $\nabla\phi^p_{|\mathrm{NR}|}$ by polar coordinate,

$$\begin{aligned}\nabla\phi^p_{\mathrm{NR}}(a,b) &= p\begin{bmatrix} a^{p-1}-(a-b)^{p-2}(a-b)_+ \\ (a-b)^{p-2}(a-b)_+\end{bmatrix}\\ &= pr^{p-1}\begin{bmatrix}\cos^{p-1}\theta-(\cos\theta-\sin\theta)^{p-2}(\cos\theta-\sin\theta)_+ \\ (\cos\theta-\sin\theta)^{p-2}(\cos\theta-\sin\theta)_+\end{bmatrix},\end{aligned}$$

hence we have $r\to b$, $\theta\to\frac{\pi}{2}$ as $(a_i,b_i)\to(0,b)$. So, we can deduce that

$$\lim_{\theta\to(\frac{\pi}{2})^+}\nabla\phi^p_{|\mathrm{NR}|}(r,\theta)=\lim_{\theta\to(\frac{\pi}{2})^+}\nabla\phi^p_{\mathrm{NR}}(r,\theta)=[0,0]^T,$$

$$\lim_{\theta\to(\frac{\pi}{2})^-}\nabla\phi^p_{|\mathrm{NR}|}(r,\theta)=\lim_{\theta\to(\frac{\pi}{2})^-}-\nabla\phi^p_{\mathrm{NR}}(r,\theta)=[0,0]^T.$$

$\nabla\phi^p_{|\mathrm{NR}|}(0,b)$ is continuously at $(0,b)$.

In summary, the function $\phi^p_{|\mathrm{NR}|}$ is continuously differentiable at $(0,b)$.

(ii) For $a>0$, $b=0$, we note that

$$\lim_{h\to 0^+}\frac{\phi^p_{|\mathrm{NR}|}(a,0+h)-\phi^p_{|\mathrm{NR}|}(a,0)}{h}=\lim_{h\to 0^+}\frac{a^p-(a-h)^p}{h}=\frac{\partial\phi^p_{\mathrm{NR}}(a,0)}{\partial a}=pa^{p-1},$$

$$\lim_{h\to 0^-}\frac{\phi^p_{|\mathrm{NR}|}(a,0+h)-\phi^p_{|\mathrm{NR}|}(a,0)}{h}=\lim_{h\to 0^-}\frac{(a-h)^p-a^p}{h}=-\frac{\partial\phi^p_{\mathrm{NR}}(a,0)}{\partial b}=-pa^{p-1},$$

$$\frac{\partial\phi^p_{|\mathrm{NR}|}}{\partial a}=\lim_{h\to 0}\frac{\phi^p_{|\mathrm{NR}|}(a+h,0)-\phi^p_{|\mathrm{NR}|}(a,0)}{h}=0.$$

So $\phi^p_{|\mathrm{NR}|}$ is not differentiable at $(a,0)$. According to the definition of Clarke's generalized gradient,

$$\partial\phi^p_{|\mathrm{NR}|}(a,0)=conv\left\{\lim_{(a_i,b_i)\to(a,0)}\nabla\phi^p_{|\mathrm{NR}|}(a_i,b_i)\,\middle|\,\phi^p_{|\mathrm{NR}|}\text{ is differentiable at }(a_i,b_i)\in\mathbb{R}^2\right\},$$

we discuss three cases as below.

(1) If $a_i>0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}\lim_{(a_i,b_i)\to(a,0)}\nabla\phi^p_{|\mathrm{NR}|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(a,0)}\nabla\phi^p_{\mathrm{NR}}(a_i,b_i)\\ &= \lim_{(a_i,b_i)\to(a,0)} p\begin{bmatrix} a_i^{p-1}-(a_i-b_i)^{p-2}(a_i-b_i)_+\\ (a_i-b_i)^{p-2}(a_i-b_i)_+\end{bmatrix}=\begin{bmatrix}0\\ pa^{p-1}\end{bmatrix}.\end{aligned}$$

(2) If $a_i>0$, $b_i<0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}\lim_{(a_i,b_i)\to(a,0)}\nabla\phi^p_{|\mathrm{NR}|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(a,0)}-\nabla\phi^p_{\mathrm{NR}}(a_i,b_i)\\ &= \lim_{(a_i,b_i)\to(a,0)} -p\begin{bmatrix} a_i^{p-1}-(a_i-b_i)^{p-2}(a_i-b_i)_+\\ (a_i-b_i)^{p-2}(a_i-b_i)_+\end{bmatrix}=\begin{bmatrix}0\\ -pa^{p-1}\end{bmatrix}.\end{aligned}$$

(3) For the remainder case, $\nabla\phi^p_{|\mathrm{NR}|}(a_i,b_i)$ has no limit as $(a_i,b_i)\to(a,0)$.
From all the above, we conclude that

$$\partial\phi^p_{|\mathrm{NR}|}(a,0)=conv\{(0,pa^{p-1}),(0,-pa^{p-1})\}=\{(0,\rho\cdot pa^{p-1})|\rho\in[-1,1]\}.$$

(d) For $a=0$, $b=0$. Because ϕ^p_{NR} is differentiable at $(0,0)$, so we have

$$\frac{|\ \phi^p_{\mathrm{NR}}(a,b)-\phi^p_{\mathrm{NR}}(0,0)-\langle(0,0),(a,b)\rangle\ |}{\sqrt{a^2+b^2}}=\frac{|\phi^p_{\mathrm{NR}}(a,b)|}{\sqrt{a^2+b^2}}\longrightarrow 0,$$

as $(a,b)\to(0,0)$. We can use the result to deduce that

$$\frac{|\ \phi^p_{|\mathrm{NR}|}(a,b)-\phi^p_{|\mathrm{NR}|}(0,0)-\langle(0,0),(a,b)\rangle\ |}{\sqrt{a^2+b^2}}=\frac{|\phi^p_{|\mathrm{NR}|}(a,b)|}{\sqrt{a^2+b^2}}=\frac{|\phi^p_{\mathrm{NR}}(a,b)|}{\sqrt{a^2+b^2}}\longrightarrow 0,$$

as $(a,b)\to(0,0)$. So, $\phi^p_{|\mathrm{NR}|}$ is differentiable at $(0,0)$ with $\nabla\phi^p_{|\mathrm{NR}|}(0,0)=[0,0]^T$.

(e) (i) If $a > 0$, $b = 0$, then given any $v = (v_1, v_2)^T$, we can deduce that

$$\begin{aligned} {\phi^p_{_{|\mathrm{NR}|}}}'((a,0);v) &= \lim_{t\to 0^+} \frac{\phi^p_{_{|\mathrm{NR}|}}(a+tv_1, tv_2) - \phi^p_{_{|\mathrm{NR}|}}(a,0)}{t} \\ &= \begin{cases} \displaystyle\lim_{t\to 0^+} \frac{(a+tv_1)^p - (a+tv_1-tv_2)^p}{t} & \text{if} \quad v_2 > 0, \\ \displaystyle\lim_{t\to 0^+} \frac{(a+tv_1-tv_2)^p - (a+tv_1)^p}{t} & \text{if} \quad v_2 < 0, \end{cases} \\ &= \begin{cases} v_2 p a^{p-1} & \text{if} \quad v_2 > 0, \\ -v_2 p a^{p-1} & \text{if} \quad v_2 < 0, \end{cases} \\ &= |v_2| p a^{p-1}. \end{aligned}$$

(ii) If $a = 0$, $b = 0$, then given any $v = (v_1, v_2)^T$. For $t > 0$, we can deduce that

$$\phi^p_{_{|\mathrm{NR}|}}(at, bt) = |(at)^p - (at - bt)^p_+| = t^p |a^p - (a-b)^p_+| = t^p \phi^p_{_{|\mathrm{NR}|}}(a,b).$$

So, we can use the result to deduce that

$${\phi^p_{_{|\mathrm{NR}|}}}'((0,0);v) = \lim_{t\to 0^+} \frac{\phi^p_{_{|\mathrm{NR}|}}(tv_1, tv_2) - \phi^p_{_{|\mathrm{NR}|}}(0,0)}{t} = \lim_{t\to 0^+} \frac{t^p \phi^p_{_{|\mathrm{NR}|}}(v_1, v_2)}{t} = 0.$$

Thus, the desired result follows. □

Proposition 3.3. *Let $\phi^p_{_{\mathrm{S-NR}}}$ be defined as in (11) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{_{\mathrm{S-NR}}}$ is*

$$\phi^p_{_{\mathrm{S-NR}}}(a,b) = \begin{cases} \phi^p_{_{\mathrm{NR}}}(a,b) & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ \phi^p_{_{\mathrm{NR}}}(b,a) & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\phi^p_{_{\mathrm{S-NR}}}$ is not differentiable. However, $\phi^p_{_{\mathrm{S-NR}}}$ is continuously differentiable on the set $\Omega_1 := \{(a,b) \,|\, a \neq b\}$ with*

$$\nabla \phi^p_{_{\mathrm{S-NR}}}(a,b) = \begin{cases} p\,[\,a^{p-1} - (a-b)^{p-1},\ (a-b)^{p-1}\,]^T & \text{if} \quad a > b, \\ p\,[\,(b-a)^{p-1},\ b^{p-1} - (b-a)^{p-1}\,]^T & \text{if} \quad a < b. \end{cases}$$

(c) *The function $\phi^p_{_{\mathrm{S-NR}}}$ is differentiable at $(0,0)$ with $\nabla \phi^p_{_{\mathrm{S-NR}}}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.*

(d) *If* $(a,b) \in \Omega_2 := \{(a,b)\,|\,a = b \neq 0\}$*, the generalized gradient of* $\phi^p_{\text{S-NR}}$ *at* (a,a) *is*

$$\partial\phi^p_{\text{S-NR}}(a,a) = \{(\alpha p a^{p-1}, (1-\alpha)pa^{p-1})|\alpha \in [0,1]\}.$$

(e) *If* $(a,b) \in \Omega^C$*, then* $\phi^p_{|\text{S-NR}|}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{|\text{S-NR}|}(a,b) = \begin{cases} \nabla\phi^p_{\text{S-NR}}(a,b) & \text{if} \quad a > 0,\ b > 0, \\ -\nabla\phi^p_{\text{S-NR}}(a,b) & \quad\text{otherwise.} \end{cases}$$

If $(a,b) \in \Omega$*, then the generalized gradient of* $\phi^p_{|\text{S-NR}|}$ *at* (a,b) *is*

$$\partial\phi^p_{|\text{S-NR}|}(a,b) = \begin{cases} \{(\rho\cdot pb^{p-1}, 0)\,|\,\rho \in [-1,1]\} & \text{if} \quad a = 0,\ b > 0, \\ \{(0, \rho\cdot pa^{p-1})\,|\,\rho \in [-1,1]\} & \text{if} \quad a > 0,\ b = 0, \\ \{(\alpha\cdot pa^{p-1}, (1-\alpha)\cdot pa^{p-1})|\alpha \in [0,1]\} & \text{if} \quad a = b > 0, \\ \{(-\alpha\cdot pa^{p-1}, -(1-\alpha)\cdot pa^{p-1})|\alpha \in [0,1]\} & \text{if} \quad a = b < 0, \end{cases}$$

where $\Omega = \{(a,b)\,|\,a = b \neq 0\}\bigcup\{(a,b)\,|\,a = 0, b > 0\}\bigcup\{(a,b)\,|\,a > 0, b = 0\}$.

(f) *The function* $\phi^p_{|\text{S-NR}|}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{|\text{S-NR}|}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(g) *The directional derivative of* $\phi^p_{|\text{S-NR}|}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{|\text{S-NR}|}{}'((a,b);v) = \begin{cases} \langle\nabla\phi^p_{|\text{S-NR}|}(a,b), v\rangle & \text{if} \quad \phi^p_{|\text{S-NR}|}(a,b) \neq 0, \\ |v_1|pb^{p-1} & \text{if} \quad a = 0,\ b > 0, \\ |v_2|pa^{p-1} & \text{if} \quad a > 0,\ b = 0, \\ 0 & \text{if} \quad a = 0,\ b = 0, \\ v_1pa^{p-1} = v_1pb^{p-1} & \text{if} \quad a = b > 0,\ v_1 > v_2, \\ v_2pa^{p-1} = v_2pb^{p-1} & \text{if} \quad a = b > 0,\ v_1 < v_2, \\ -v_1pa^{p-1} = -v_1pb^{p-1} & \text{if} \quad a = b < 0,\ v_1 > v_2, \\ -v_2pa^{p-1} = -v_2pb^{p-1} & \text{if} \quad a = b < 0,\ v_1 < v_2. \end{cases}$$

Proof.
(a)-(d) It is clear from [1, Proposition 2.2].

(e) If $(a,b) \in \Omega^C$, then it is cleasly by Proposition 3.3 (b) and (c).

If $(a,b)\in\Omega$, by the definition of $\phi^p_{|\mathrm{S-NR}|}$, then we have

$$\phi^p_{|\mathrm{S-NR}|}(a,b)=\begin{cases} a^p-(a-b)^p & \text{if} \quad a>b>0,\\ a^p=b^p & \text{if} \quad a=b>0,\\ b^p-(b-a)^p & \text{if} \quad b>a>0,\\ -a^p=-b^p & \text{if} \quad a=b<0,\\ (a-b)^p-a^p & \text{if} \quad a>0>b \ or \ 0>a>b,\\ (b-a)^p-b^p & \text{if} \quad 0>b>a \ or \ b>0>a.\end{cases}$$

(i) For $a=0$, $b>0$, we note that

$$\lim_{h\to 0^+}\frac{\phi^p_{|\mathrm{S-NR}|}(0+h,b)-\phi^p_{|\mathrm{S-NR}|}(0,b)}{h}=\lim_{h\to 0^+}\frac{b^p-(b-h)^p}{h}=\frac{\partial\phi^p_{\mathrm{S-NR}}(0,b)}{\partial a}=pb^{p-1},$$
$$\lim_{h\to 0^-}\frac{\phi^p_{|\mathrm{S-NR}|}(0+h,b)-\phi^p_{|\mathrm{S-NR}|}(0,b)}{h}=\lim_{h\to 0^-}\frac{(b-h)^p-b^p}{h}=-\frac{\partial\phi^p_{\mathrm{S-NR}}(0,b)}{\partial a}=-pb^{p-1},$$
$$\frac{\partial\phi^p_{|\mathrm{S-NR}|}}{\partial b}=\lim_{h\to 0}\frac{\phi^p_{|\mathrm{S-NR}|}(0,b+h)-\phi^p_{|\mathrm{S-NR}|}(0,b)}{h}=0.$$

So $\phi^p_{|\mathrm{S-NR}|}$ is not differentiable at $(0,b)$. According to the definition of Clarke's generalized gradient,

$$\partial\phi^p_{|\mathrm{S-NR}|}(0,b)=conv\left\{\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|\mathrm{S-NR}|}(a_i,b_i)\,\Big|\,\phi^p_{|\mathrm{S-NR}|}\ \text{is differentiable at}\ (a_i,b_i)\in\mathbb{R}^2\right\},$$

we discuss three cases as below.
(1) If $a_i>0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|\mathrm{S-NR}|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{\mathrm{S-NR}}(a_i,b_i)\\ &= \lim_{(a_i,b_i)\to(0,b)}p\begin{bmatrix}(b_i-a_i)^{p-1}\\ b_i^{p-1}-(b_i-a_i)^{p-1}\end{bmatrix}=\begin{bmatrix}pb^{p-1}\\ 0\end{bmatrix}.\end{aligned}$$

(2) If $a_i<0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|\mathrm{S-NR}|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(0,b)}-\nabla\phi^p_{\mathrm{S-NR}}(a_i,b_i)\\ &= \lim_{(a_i,b_i)\to(0,b)}-p\begin{bmatrix}(b_i-a_i)^{p-1}\\ b_i^{p-1}-(b_i-a_i)^{p-1}\end{bmatrix}=\begin{bmatrix}-pb^{p-1}\\ 0\end{bmatrix}.\end{aligned}$$

(3) For the remainder case, $\nabla\phi^p_{|\mathrm{S-NR}|}(a_i,b_i)$ has no limit as $(a_i,b_i)\to(0,b)$.
From all the above, we conclude that

$$\partial\phi^p_{|\mathrm{S-NR}|}(0,b)=conv\{(pb^{p-1},0),(-pb^{p-1},0)\}=\{(\rho\cdot pb^{p-1},0)|\rho\in[-1,1]\}.$$

(ii) For $a > 0$, $b = 0$, the proof is similar to (i).

(iii) For $a = b > 0$, then it is clearly by Proposition 3.3 (d).

(iv) For $a = b < 0$, then $\phi^p_{|\text{S-NR}|}(a, a) = -\phi^p_{\text{S-NR}}(a, a)$ and by Proposition 3.3 (d). We have

$$\partial\phi^p_{|\text{S-NR}|}(a, a) = \partial(-\phi^p_{\text{S-NR}}(a, a)) = \{(-\alpha \cdot pa^{p-1}, -(1-\alpha) \cdot pa^{p-1}) | \alpha \in [0, 1]\}.$$

(f) If $a = 0$, $b = 0$, and $\phi^p_{\text{S-NR}}$ is differentiable at $(0, 0)$, then we have

$$\frac{|\ \phi^p_{\text{S-NR}}(a, b) - \phi^p_{\text{S-NR}}(0, 0) - \langle(0, 0), (a, b)\rangle\ |}{\sqrt{a^2 + b^2}} = \frac{|\phi^p_{\text{S-NR}}(a, b)|}{\sqrt{a^2 + b^2}} \longrightarrow 0,$$

as $(a, b) \to (0, 0)$. We can use the result to deduce that

$$\frac{|\ \phi^p_{|\text{S-NR}|}(a, b) - \phi^p_{|\text{S-NR}|}(0, 0) - \langle(0, 0), (a, b)\rangle\ |}{\sqrt{a^2 + b^2}} = \frac{|\phi^p_{|\text{S-NR}|}(a, b)|}{\sqrt{a^2 + b^2}} = \frac{|\phi^p_{\text{S-NR}}(a, b)|}{\sqrt{a^2 + b^2}} \longrightarrow 0,$$

as $(a, b) \to (0, 0)$. So, $\phi^p_{|\text{S-NR}|}$ is differentiable at $(0, 0)$ with $\nabla\phi^p_{|\text{S-NR}|}(0, 0) = [0, 0]^T$.

(g) (i) If $a = 0$, $b > 0$, then given any $v = (v_1, v_2)^T$, we can deduce that

$$\begin{aligned}
{\phi^p_{|\text{S-NR}|}}'((0, b); v) &= \lim_{t \to 0^+} \frac{\phi^p_{|\text{S-NR}|}(tv_1, b + tv_2) - \phi^p_{|\text{S-NR}|}(0, b)}{t} \\
&= \begin{cases} \lim\limits_{t \to 0^+} \dfrac{(b + tv_2)^p - (b + tv_2 - tv_1)^p}{t} & \text{if} \quad v_1 > 0 \\ \lim\limits_{t \to 0^+} \dfrac{(b + tv_2 - tv_1)^p - (b + tv_2)^p}{t} & \text{if} \quad v_1 < 0 \end{cases} \\
&= \begin{cases} v_1 pb^{p-1} & \text{if} \quad v_1 > 0 \\ -v_1 pb^{p-1} & \text{if} \quad v_1 < 0 \end{cases} \\
&= |v_1| pb^{p-1}.
\end{aligned}$$

(ii) If $a > 0$, $b = 0$, then the proof is similar to (i) and ${\phi^p_{|\text{S-NR}|}}'((a, 0); v) = |v_2| pa^{p-1}$.

(iii) If $a = 0$, $b = 0$, then given any $v = (v_1, v_2)^T$. For $t > 0$, we can deduce that

$$\begin{aligned}
\phi^p_{|\text{S-NR}|}(at, bt) &= \begin{cases} \phi^p_{|\text{NR}|}(at, bt) & \text{if} \quad a > b \\ |(at)^p| = |(bt)^p| & \text{if} \quad a = b \\ \phi^p_{|\text{NR}|}(bt, at) & \text{if} \quad a < b \end{cases} \\
&= \begin{cases} t^p \phi^p_{|\text{NR}|}(a, b) & \text{if} \quad a > b \\ t^p |a^p| = t^p |b^p| & \text{if} \quad a = b \\ t^p \phi^p_{|\text{NR}|}(b, a) & \text{if} \quad a < b \end{cases} \\
&= t^p \phi^p_{|\text{S-NR}|}(a, b).
\end{aligned}$$

So, we can use the result to deduce that

$$\phi^p_{|\mathrm{S-NR}|}{}'((0,0);v) = \lim_{t\to 0^+} \frac{\phi^p_{|\mathrm{S-NR}|}(tv_1,tv_2) - \phi^p_{|\mathrm{S-NR}|}(0,0)}{t} = \lim_{t\to 0^+} \frac{t^p \phi^p_{|\mathrm{S-NR}|}(v_1,v_2)}{t} = 0.$$

(iv) If $a = b > 0$, then given any $v = (v_1, v_2)^T$, we can deduce that

$$\begin{aligned}
\phi^p_{|\mathrm{S-NR}|}{}'((a,a);v) &= \lim_{t\to 0^+} \frac{\phi^p_{|\mathrm{S-NR}|}(a+tv_1, a+tv_2) - \phi^p_{|\mathrm{S-NR}|}(a,a)}{t} \\
&= \lim_{t\to 0^+} \frac{\phi^p_{|\mathrm{S-NR}|}(a+tv_1, a+tv_2) - a^p}{t} \\
&= \begin{cases} \lim\limits_{t\to 0^+} \dfrac{(a+tv_1)^p - (tv_1 - tv_2)^p - a^p}{t} & \text{if} \quad v_1 > v_2 \\ \lim\limits_{t\to 0^+} \dfrac{(a+tv_2)^p - (tv_2 - tv_1)^p - a^p}{t} & \text{if} \quad v_1 < v_2 \end{cases} \\
&= \begin{cases} v_1 p a^{p-1} & \text{if} \quad v_1 > v_2, \\ v_2 p a^{p-1} & \text{if} \quad v_1 < v_2. \end{cases}
\end{aligned}$$

(v) If $a = b < 0$, then the proof is similar to (iv).

Thus, the desired result follows. □

Proposition 3.4. *Let $\psi^p_{\mathrm{S-NR}}$ be defined as in (12) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\psi^p_{\mathrm{S-NR}}$ is*

$$\psi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} \phi^p_{\mathrm{NR}}(a,b) b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ \phi^p_{\mathrm{NR}}(b,a) a^p & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\psi^p_{\mathrm{S-NR}}$ is continuously differentiable with*

$$\nabla \psi^p_{\mathrm{S-NR}}(a,b) = \begin{cases} p \begin{bmatrix} a^{p-1}b^p - (a-b)^{p-1}b^p \\ a^p b^{p-1} - (a-b)^p b^{p-1} + (a-b)^{p-1} b^p \end{bmatrix} & \text{if} \quad a > b, \\ p \begin{bmatrix} a^{p-1}b^p \\ a^p b^{p-1} \end{bmatrix} = p a^{2p-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = p b^{2p-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if} \quad a = b, \\ p \begin{bmatrix} a^{p-1}b^p - (b-a)^p a^{p-1} + (b-a)^{p-1} a^p \\ a^p b^{p-1} - (b-a)^{p-1} a^p \end{bmatrix} & \text{if} \quad a < b. \end{cases}$$

(c) *The function* $\psi^p_{|\text{S-NR}|}$ *is continuously differentiable with*

$$\nabla\psi^p_{|\text{S-NR}|}(a,b) = \nabla\psi^p_{\text{S-NR}}(a,b).$$

(d) *The directional derivative of* $\psi^p_{|\text{S-NR}|}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\psi^p_{|\text{S-NR}|}{}'((a,b);v) = \langle \nabla\psi^p_{|\text{S-NR}|}(a,b), v\rangle.$$

Proof.
(a) and (b) It is clear from [1, Proposition 3.2].
(c) and (d) It is clearly by proposition 3.4 (b). □

Proposition 3.5. *Let* $\phi^p_{\text{D-FB}}$ *be defined as in (13) where* $p > 1$ *is a positive odd integer. Then, the followings hold.*

(a) *The function* $\phi^p_{\text{D-FB}}$ *is continuously differentiable with*

$$\nabla\phi^p_{\text{D-FB}}(a,b) = p\begin{bmatrix} a(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \\ b(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \end{bmatrix}.$$

(b) *If* $\phi^p_{|\text{D-FB}|}(a,b) \neq 0$*, then* $\phi^p_{|\text{D-FB}|}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{|\text{D-FB}|}(a,b) = \begin{cases} -\nabla\phi^p_{\text{D-FB}}(a,b) & \text{if} \quad a>0,\ b>0 \\ \nabla\phi^p_{\text{D-FB}}(a,b) & \text{otherwise.} \end{cases}$$

If $\phi^p_{|\text{D-FB}|}(a,b) = 0$ *with* $(a,b) \neq (0,0)$*, then the generalized gradient of* $\phi^p_{|\text{D-FB}|}$ *at* (a,b) *is*

$$\partial\phi^p_{|\text{D-FB}|}(a,b) = \begin{cases} \{(\rho\cdot pb^{p-1}, 0)\,|\,\rho\in[-1,1]\} & \text{if} \quad a=0,\ b>0, \\ \{(0, \rho\cdot pa^{p-1})\,|\,\rho\in[-1,1]\} & \text{if} \quad a>0,\ b=0. \end{cases}$$

(c) *The function* $\phi^p_{|\text{D-FB}|}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{|\text{D-FB}|}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) *The directional derivative of* $\phi^p_{|\text{D-FB}|}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{|\text{D-FB}|}{}'((a,b);v) = \begin{cases} \langle\nabla\phi^p_{|\text{D-FB}|}(a,b), v\rangle & \text{if} \quad \phi^p_{|\text{D-FB}|}(a,b)\neq 0, \\ |v_1|pb^{p-1} & \text{if} \quad a=0,\ b>0, \\ |v_2|pa^{p-1} & \text{if} \quad a>0,\ b=0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}$$

Proof.

(a) It is clear from [14, Proposition 3.2].

(b) If $\phi^p_{|\mathrm{D-FB}|}(a,b)\neq 0$ then it is clearly by proposition 3.5 (a).

(i) For $a=0$, $b>0$, we note that

$$\lim_{h\to 0^+}\frac{\phi^p_{|\mathrm{D-FB}|}(0+h,b)-\phi^p_{|\mathrm{D-FB}|}(0,b)}{h}=\lim_{h\to 0^+}\frac{(h+b)^p-\left(\sqrt{h^2+b^2}\right)^p}{h}=-\frac{\partial\phi^p_{\mathrm{D-FB}}(0,b)}{\partial a}=pb^{p-1},$$

$$\lim_{h\to 0^-}\frac{\phi^p_{|\mathrm{D-FB}|}(0+h,b)-\phi^p_{|\mathrm{D-FB}|}(0,b)}{h}=\lim_{h\to 0^-}\frac{\left(\sqrt{h^2+b^2}\right)^p-(h+b)^p}{h}=\frac{\partial\phi^p_{\mathrm{D-FB}}(0,b)}{\partial a}=-pb^{p-1},$$

$$\frac{\partial\phi^p_{|\mathrm{D-FB}|}}{\partial b}=\lim_{h\to 0}\frac{\phi^p_{|\mathrm{D-FB}|}(0,b+h)-\phi^p_{|\mathrm{D-FB}|}(0,b)}{h}=0.$$

So $\phi^p_{|\mathrm{D-FB}|}$ is not differentiable at $(0,b)$. According to the definition of Clarke's generalized gradient,

$$\partial\phi^p_{|\mathrm{D-FB}|}(0,b)=conv\left\{\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|\mathrm{D-FB}|}(a_i,b_i)\,\Big|\,\phi^p_{|\mathrm{D-FB}|}\text{ is differentiable at }(a_i,b_i)\in\mathbb{R}^2\right\},$$

we discuss three cases as below.

(1) If $a_i>0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}&\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|\mathrm{D-FB}|}(a_i,b_i)=\lim_{(a_i,b_i)\to(0,b)}-\nabla\phi^p_{\mathrm{D-FB}}(a_i,b_i)\\ =&\lim_{(a_i,b_i)\to(0,b)}-p\begin{bmatrix}a_i(\sqrt{a_i^2+b_i^2})^{p-2}-(a_i+b_i)^{p-1}\\ b_i(\sqrt{a_i^2+b_i^2})^{p-2}-(a_i+b_i)^{p-1}\end{bmatrix}=\begin{bmatrix}pb^{p-1}\\ 0\end{bmatrix}.\end{aligned}$$

(2) If $a_i<0$, $b_i>0$, for any $i\geq n$ and sufficiently large n, then

$$\begin{aligned}&\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{|\mathrm{D-FB}|}(a_i,b_i)=\lim_{(a_i,b_i)\to(0,b)}\nabla\phi^p_{\mathrm{D-FB}}(a_i,b_i)\\ =&\lim_{(a_i,b_i)\to(0,b)}p\begin{bmatrix}a_i(\sqrt{a_i^2+b_i^2})^{p-2}-(a_i+b_i)^{p-1}\\ b_i(\sqrt{a_i^2+b_i^2})^{p-2}-(a_i+b_i)^{p-1}\end{bmatrix}=\begin{bmatrix}-pb^{p-1}\\ 0\end{bmatrix}.\end{aligned}$$

(3) For the remainder case, $\nabla\phi^p_{|\mathrm{D-FB}|}(a_i,b_i)$ has no limit as $(a_i,b_i)\to(0,b)$.
From all the above, we conclude that

$$\partial\phi^p_{|\mathrm{D-FB}|}(0,b)=conv\{(pb^{p-1},0),(-pb^{p-1},0)\}=\{(\rho\cdot pb^{p-1},0)|\rho\in[-1,1]\}.$$

(ii) For $a>0$, $b=0$, then the proof is similar to (i).

(c) If $a=0$, $b=0$, and $\phi^p_{\mathrm{D-FB}}$ is differentiable at $(0,0)$, then we have

$$\frac{|\ \phi^p_{\mathrm{D-FB}}(a,b)-\phi^p_{\mathrm{D-FB}}(0,0)-\langle(0,0),(a,b)\rangle\ |}{\sqrt{a^2+b^2}}=\frac{|\phi^p_{\mathrm{D-FB}}(a,b)|}{\sqrt{a^2+b^2}}\longrightarrow 0,$$

as $(a,b) \to (0,0)$. We can use the result to deduce that

$$\frac{|\ \phi^p_{_{|\mathrm{D-FB}|}}(a,b) - \phi^p_{_{|\mathrm{D-FB}|}}(0,0) - \langle (0,0),(a,b)\rangle\ |}{\sqrt{a^2+b^2}} = \frac{|\phi^p_{_{|\mathrm{D-FB}|}}(a,b)|}{\sqrt{a^2+b^2}} = \frac{|\phi^p_{_{\mathrm{D-FB}}}(a,b)|}{\sqrt{a^2+b^2}} \longrightarrow 0,$$

as $(a,b) \to (0,0)$. So, $\phi^p_{_{|\mathrm{D-FB}|}}$ is differentiable at $(0,0)$ with $\nabla \phi^p_{_{|\mathrm{D-FB}|}}(0,0) = [0,0]^T$.

(d) (i) If $a = 0$, $b > 0$, then given any $v = (v_1, v_2)^T$, we can deduce that

$$\begin{aligned}
\phi^p_{_{|\mathrm{D-FB}|}}{}'((0,b);v) &= \lim_{t\to 0^+} \frac{\phi^p_{_{|\mathrm{D-FB}|}}(tv_1, b+tv_2) - \phi^p_{_{|\mathrm{D-FB}|}}(0,b)}{t} \\
&= \lim_{t\to 0^+} \frac{|\left(\sqrt{(tv_1)^2+(b+tv_2)^2}\right)^p - (b+tv_1+tv_2)|}{t} \\
&= \begin{cases} \lim\limits_{t\to 0^+} \dfrac{\left(\sqrt{(tv_1)^2+(b+tv_2)^2}\right)^p - (b+tv_1+tv_2)^p}{t} & \text{if} \quad v_1 > 0 \\ \lim\limits_{t\to 0^+} \dfrac{(b+tv_1+tv_2)^p - \left(\sqrt{(tv_1)^2+(b+tv_2)^2}\right)^p}{t} & \text{if} \quad v_1 < 0 \end{cases} \\
&= \begin{cases} v_1 p b^{p-1} & \text{if} \quad v_1 > 0 \\ -v_1 p b^{p-1} & \text{if} \quad v_1 < 0 \end{cases} \\
&= |v_1| p b^{p-1}.
\end{aligned}$$

(ii) If $a > 0$, $b = 0$, then the proof is similar to (i) and $\phi^p_{_{|\mathrm{D-FB}|}}{}'((a,0);v) = |v_2| p a^{p-1}$.

(iii) If $a = 0$, $b = 0$, then given any $v = (v_1, v_2)^T$. For $t > 0$, we can deduce that

$$\begin{aligned}
\phi^p_{_{|\mathrm{D-FB}|}}(at, bt) &= |\left(\sqrt{(at)^2+(bt)^2}\right)^p - (at+bt)^p| \\
&= t^p |\left(\sqrt{a^2+b^2}\right)^p - (a+b)^p| \\
&= t^p \phi^p_{_{|\mathrm{D-FB}|}}(a,b).
\end{aligned}$$

So, we can use the result to deduce that

$$\phi^p_{_{|\mathrm{D-FB}|}}{}'((0,0);v) = \lim_{t\to 0^+} \frac{\phi^p_{_{|\mathrm{D-FB}|}}(tv_1, tv_2) - \phi^p_{_{|\mathrm{D-FB}|}}(0,0)}{t} = \lim_{t\to 0^+} \frac{t^p \phi^p_{_{|\mathrm{D-FB}|}}(v_1, v_2)}{t} = 0.$$

Thus, the desired result follows. □

Proposition 3.6. *Let $\phi^p_{_{\mathrm{A-NR}}}$ be defined as in (14) where $p > 1$ is a positive odd integer. Then, the followings hold.*

(a) *The function $\phi^p_{_{\mathrm{A-NR}}}$ is continuously differentiable with*

$$\nabla \phi^p_{_{\mathrm{A-NR}}}(a,b) = \frac{p}{2^p} \left[\begin{array}{c} (a+b)^{p-1} - (a-b)|a-b|^{p-2} \\ (a+b)^{p-1} + (a-b)|a-b|^{p-2} \end{array} \right].$$

(b) *If* $\phi^p_{|\text{A-NR}|}(a,b) \neq 0$, *then* $\phi^p_{|\text{A-NR}|}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{|\text{A-NR}|}(a,b) = \begin{cases} \nabla\phi^p_{\text{A-NR}}(a,b) & \text{if} \quad a>0,\ b>0, \\ -\nabla\phi^p_{\text{A-NR}}(a,b) & \text{otherwise.} \end{cases}$$

If $\phi^p_{|\text{A-NR}|}(a,b) = 0$ *with* $(a,b) \neq (0,0)$, *then the generalized gradient of* $\phi^p_{|\text{A-NR}|}$ *at* (a,b) *is*

$$\partial\phi^p_{|\text{A-NR}|}(a,b) = \begin{cases} \{(\rho \cdot p(\frac{b}{2})^{p-1}, 0) \,|\, \rho \in [-1,1]\} & \text{if} \quad a=0,\ b>0, \\ \{(0, \rho \cdot p(\frac{b}{2})^{p-1}) \,|\, \rho \in [-1,1]\} & \text{if} \quad a>0,\ b=0. \end{cases}$$

(c) *The function* $\phi^p_{|\text{A-NR}|}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{|\text{A-NR}|}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) *The directional derivative of* $\phi^p_{|\text{A-NR}|}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{|\text{A-NR}|}{}'((a,b);v) = \begin{cases} \langle \nabla\phi^p_{|\text{A-NR}|}(a,b), v \rangle & \text{if} \quad \phi^p_{|\text{A-NR}|}(a,b) \neq 0, \\ |v_1| p(\frac{b}{2})^{p-1} & \text{if} \quad a=0,\ b>0, \\ |v_2| p(\frac{a}{2})^{p-1} & \text{if} \quad a>0,\ b=0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}$$

Proof.
(a) It is clear from [21, Proposition 3.4].

(b) If $\phi^p_{|\text{A-NR}|}(a,b) \neq 0$ then it is clearly by proposition 3.6 (a).
(i) For $a=0$, $b>0$, we note that

$$\lim_{h\to 0^+} \frac{\phi^p_{|\text{A-NR}|}(0+h,b) - \phi^p_{|\text{A-NR}|}(0,b)}{h} = \lim_{h\to 0^+} \frac{(h+b)^p - |h-b|^p}{2^p \cdot h} = \frac{\partial\phi^p_{\text{A-NR}}(0,b)}{\partial a} = p\left(\frac{b}{2}\right)^{p-1},$$

$$\lim_{h\to 0^-} \frac{\phi^p_{|\text{A-NR}|}(0+h,b) - \phi^p_{|\text{A-NR}|}(0,b)}{h} = \lim_{h\to 0^-} \frac{|h-b|^p - (h+b)^p}{2^p \cdot h} = -\frac{\partial\phi^p_{\text{A-NR}}(0,b)}{\partial a} = -p\left(\frac{b}{2}\right)^{p-1},$$

$$\frac{\partial\phi^p_{|\text{A-NR}|}}{\partial b} = \lim_{h\to 0} \frac{\phi^p_{|\text{A-NR}|}(0,b+h) - \phi^p_{|\text{A-NR}|}(0,b)}{h} = 0.$$

So $\phi^p_{|\text{A-NR}|}$ is not differentiable at $(0,b)$. According to the definition of Clarke's generalized gradient,

$$\partial\phi^p_{|\text{A-NR}|}(0,b) = conv\left\{ \lim_{(a_i,b_i)\to(0,b)} \nabla\phi^p_{|\text{A-NR}|}(a_i,b_i) \,\middle|\, \phi^p_{|\text{A-NR}|} \text{ is differentiable at }, (a_i,b_i) \in \mathbb{R}^2 \right\},$$

we discuss three cases as below.
(1) If $a_i > 0$, $b_i > 0$, for any $i \geq n$ and sufficiently large n, then

$$\begin{aligned}
\lim_{(a_i,b_i)\to(0,b)} \nabla\phi^p_{|\mathrm{A-NR}|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(0,b)} \nabla\phi^p_{\mathrm{A-NR}}(a_i,b_i) \\
= \lim_{(a_i,b_i)\to(0,b)} \frac{p}{2^p}\begin{bmatrix} (a_i+b_i)^{p-1}-(a_i-b_i)|a_i-b_i|^{p-2} \\ (a_i+b_i)^{p-1}+(a_i-b_i)|a_i-b_i|^{p-2} \end{bmatrix} &= \left[p\left(\frac{b}{2}\right)^{p-1}, 0\right]^T.
\end{aligned}$$

(2) If $a_i < 0$, $b_i > 0$, for any $i \geq n$ and sufficiently large n, then

$$\begin{aligned}
\lim_{(a_i,b_i)\to(0,b)} \nabla\phi^p_{|\mathrm{A-NR}|}(a_i,b_i) &= \lim_{(a_i,b_i)\to(0,b)} -\nabla\phi^p_{\mathrm{A-NR}}(a_i,b_i) \\
= \lim_{(a_i,b_i)\to(0,b)} -\frac{p}{2^p}\begin{bmatrix} (a_i+b_i)^{p-1}-(a_i-b_i)|a_i-b_i|^{p-2} \\ (a_i+b_i)^{p-1}+(a_i-b_i)|a_i-b_i|^{p-2} \end{bmatrix} &= \left[-p\left(\frac{b}{2}\right)^{p-1}, 0\right]^T.
\end{aligned}$$

(3) For the remainder case, $\nabla\phi^p_{|\mathrm{A-NR}|}(a_i,b_i)$ has no limit as $(a_i,b_i)\to(0,b)$.
From all the above, we conclude that

$$\begin{aligned}
\partial\phi^p_{|\mathrm{A-NR}|}(0,b) &= conv\left\{\left(p\left(\frac{b}{2}\right)^{p-1}, 0\right), \left(-p\left(\frac{b}{2}\right)^{p-1}, 0\right)\right\} \\
&= \left\{\left(\rho\cdot p\left(\frac{b}{2}\right)^{p-1}, 0\right) | \rho\in[-1,1]\right\}.
\end{aligned}$$

(ii) For $a > 0$, $b = 0$, then the proof is similar to (i).

(c) If $a = 0$, $b = 0$, and $\phi^p_{\mathrm{A-NR}}$ is differentiable at $(0,0)$, then we have

$$\frac{|\ \phi^p_{\mathrm{A-NR}}(a,b) - \phi^p_{\mathrm{A-NR}}(0,0) - \langle(0,0),(a,b)\rangle\ |}{\sqrt{a^2+b^2}} = \frac{|\phi^p_{\mathrm{A-NR}}(a,b)|}{\sqrt{a^2+b^2}} \longrightarrow 0,$$

as $(a,b)\to(0,0)$. We can use the result to deduce that

$$\frac{|\ \phi^p_{|\mathrm{A-NR}|}(a,b) - \phi^p_{|\mathrm{A-NR}|}(0,0) - \langle(0,0),(a,b)\rangle\ |}{\sqrt{a^2+b^2}} = \frac{|\phi^p_{|\mathrm{A-NR}|}(a,b)|}{\sqrt{a^2+b^2}} = \frac{|\phi^p_{\mathrm{A-NR}}(a,b)|}{\sqrt{a^2+b^2}} \longrightarrow 0,$$

as $(a,b)\to(0,0)$. So, $\phi^p_{|\mathrm{A-NR}|}$ is differentiable at $(0,0)$ with $\nabla\phi^p_{|\mathrm{A-NR}|}(0,0) = [0,0]^T$.

(d) (i) If $a = 0$, $b > 0$, then given any $v = (v_1, v_2)^T$, we can deduce that

$$
\begin{aligned}
{\phi^p_{|\mathrm{A-NR}|}}'((0,b);v) &= \lim_{t \to 0^+} \frac{\phi^p_{|\mathrm{A-NR}|}(tv_1, b+tv_2) - \phi^p_{|\mathrm{A-NR}|}(0,b)}{t} \\
&= \lim_{t \to 0^+} \frac{|(b+tv_1+tv_2)^p - (b-tv_1+tv_2)^p|}{2^p \cdot t} \\
&= \begin{cases} \lim\limits_{t \to 0^+} \dfrac{(b+tv_1+tv_2)^p - (b-tv_1+tv_2)^p}{2^p \cdot t} & \text{if} \quad v_1 > 0 \\ \lim\limits_{t \to 0^+} \dfrac{(b-tv_1+tv_2)^p - (b+tv_1+tv_2)^p}{2^p \cdot t} & \text{if} \quad v_1 < 0 \end{cases} \\
&= \begin{cases} v_1 p \left(\dfrac{b}{2}\right)^{p-1} & \text{if} \quad v_1 > 0 \\ -v_1 p \left(\dfrac{b}{2}\right)^{p-1} & \text{if} \quad v_1 < 0 \end{cases} = |v_1| p \left(\frac{b}{2}\right)^{p-1}.
\end{aligned}
$$

(ii) If $a > 0$, $b = 0$, then the proof is similar to (i) and ${\phi^p_{|\mathrm{A-NR}|}}'((a,0);v) = |v_2| p (\frac{a}{2})^{p-1}$.

(iii) If $a = 0$, $b = 0$, then given any $v = (v_1, v_2)^T$. For $t > 0$, we can deduce that

$$
\begin{aligned}
\phi^p_{|\mathrm{A-NR}|}(at, bt) &= \frac{|(at+bt)^p - |at-bt|^p|}{2^p} \\
&= t^p \cdot \frac{|(a+b)^p - |a-b|^p|}{2^p} \\
&= t^p \phi^p_{|\mathrm{A-NR}|}(a,b).
\end{aligned}
$$

So, we can use the result to deduce that

$$
{\phi^p_{|\mathrm{A-NR}|}}'((0,0);v) = \lim_{t \to 0^+} \frac{\phi^p_{|\mathrm{A-NR}|}(tv_1, tv_2) - \phi^p_{|\mathrm{A-NR}|}(0,0)}{t} = \lim_{t \to 0^+} \frac{t^p \phi^p_{|\mathrm{A-NR}|}(v_1, v_2)}{t} = 0.
$$

Thus, the desired result follows. □

4 Descent Property of Generalized Gradients

In this section, we consider the nonsmooth Eq. (1), we investigate the generaized Jacobians of Φ and generaized gradients of P_c. In particular, we prove that the negative of generalized gradient of P_c at $z = (x, \lambda)$ is a descent direction of P_c if z is not a critical point of P_c, see [8, 19] and references therein.

For any $i \in \{1, 2, \cdots, n\}$, each element $H_i \in \partial\Phi(x)^T$ and let $V \in \partial_x P_c(x, \lambda)$ be written as follow: $V = \nabla f(x) + H[\lambda + c\Phi(x)]$, where $H = (H_1, \cdots, H_n)$. The directional derivative of $P_c(z)$ at $z=(x,\lambda)\in\mathbb{R}^{2n}$ in the direction $q=\binom{d}{h}$, with $d,h\in\mathbb{R}^n$. From (3) and (4), we can deduce that

$$P_c'(z;q)=\Phi(x)^Th+\nabla f(x)^Td+[\lambda+c\Phi(x)]^T\Phi'(x;d), \tag{19}$$

$$V^Td=\nabla f(x)^Td+[\lambda+c\Phi(x)]^TH^Td. \tag{20}$$

Thus, by (19) and (20), we have

$$P_c'(z;q)-V^Td-\Phi(x)^Th=[\lambda+c\Phi(x)]^T[\Phi'(x;d)-H^Td]. \tag{21}$$

Proposition 4.1. *The function* $\phi^p_{\rm FB}$

(a) *Let* $\Phi^p_{|\rm FB|}(x)=\begin{bmatrix}\phi^p_{|\rm FB|}(x_1,F_1(x))\\ \vdots\\ \phi^p_{|\rm FB|}(x_n,F_n(x))\end{bmatrix}$ *and* $\begin{cases}\overline{I}(x)=I_1(x)\bigcup I_2(x),\\ I(x)=I_{0+}(x)\bigcup I_{+0}(x)\bigcup I_{00}(x),\end{cases}$

where $\begin{cases}I_1(x)=\{i|x_i>0,\ F_i(x)>0\},\\ I_2(x)=\{i|x_i<0 \text{ or } F_i(x)<0\},\end{cases}$ *and* $\begin{cases}I_{0+}(x)=\{i|x_i=0,\ F_i(x)>0\},\\ I_{+0}(x)=\{i|x_i>0,\ F_i(x)=0\},\\ I_{00}(x)=\{i|x_i=0,\ F_i(x)=0\}.\end{cases}$

(b) *For any* $i\in\{1,2,\cdots,n\}$*, each element* $H_i\in\partial\Phi^p_{|\rm FB|}(x)^T$ *and* $V\in\partial_xP_c(x,\lambda)$ *can be written as follow:* $H_i=a_i(x)E_i+b_i(x)\nabla F_i(x)$ *and* $V=\nabla f(x)+H[\lambda+c\Phi^p_{|\rm FB|}(x)]$ *where* E_i *is the ith column of the* $n\times n$ *unit matrix,* $H=(H_1,H_2,\cdots,H_n)$

$$(a_i(x),b_i(x))=\begin{cases}\nabla\phi^p_{|\rm FB|}(x_i,F_i(x)) & \text{if } \ i\in\overline{I}(x),\\ (\rho_i,0) & \text{if } \ i\in I_{0+}(x),\\ (0,\rho_i) & \text{if } \ i\in I_{+0}(x),\\ (\xi_i,\eta_i) & \text{if } \ i\in I_{00}(x),\end{cases}$$

with $\rho_i\in[-1,1]$ *and* $(\xi_i,\eta_i)\in\Xi$*, where* $\Xi\subseteq conv\{\Xi_1\bigcup\Xi_2\}$*, here*

$$\begin{cases}\Xi_1=\{(1-\xi,1-\zeta)|\xi\geq 0 \text{ and } \zeta\geq 0,\ |\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1\},\\ \Xi_2=\{(\xi-1,\zeta-1)|\xi\leq 0 \text{ or } \zeta\leq 0,\ |\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1\}.\end{cases}$$

(c) *The directional derivative of* $\Phi^p_{|\rm FB|-i}(x)=\phi^p_{|\rm FB|}(x_i,F_i(x))$ *at* x *in the direction* d *is given by*

$$\Phi^p_{|\rm FB|-i}{}'(x;d)=\begin{cases}H_i^Td & \text{if } \ i\in\overline{I}(x),\\ |d_i| & \text{if } \ i\in I_{0+}(x),\\ |\nabla F_i(x)^Td| & \text{if } \ i\in I_{+0}(x),\\ \phi^p_{|\rm FB|}(d_i,\nabla F_i(x)^Td) & \text{if } \ i\in I_{00}(x).\end{cases}$$

(d) *For any* $(s,t) \in \Omega'$

$$\phi^p_{|\mathrm{FB}|}(a,b) - (as+bt) \geqslant 0,$$

where $\Omega' = \{(s,t) \mid -1 \leqslant s \leqslant 0, -1 \leqslant t \leqslant 0\} \bigcup \{(s,t) \mid ||(s+1,t+1)||_q \leq 1\}$, $p,q \in [1,\infty]$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof.
(b) and (c) The conclusion follows directly from Lemma 2.1 and Proposition 3.1 (b) and (c).

(d) (i) For $a>0$, $b>0$ and $(s,t)\in\Omega'$, we can deduce that

$$\begin{aligned}\phi^p_{|\mathrm{FB}|}(a,b) - as - bt &= (a+b) - \|(a,b)\|_p - as - bt \\ &= (1-s)a + (1-t)b - \|(a,b)\|_p \geq (a+b) - \|(a,b)\|_p > 0.\end{aligned}$$

(ii) For $a<0$, $b<0$ and $(s,t)\in\Omega'$, then
Let $u=s+1$, $w=t+1$, we have

$$\phi^p_{|\mathrm{FB}|}(a,b) - as - bt = \|(a,b)\|_p - a - b - as - bt = \|(a,b)\|_p - (au+bw).$$

We discuss four cases as below.
(1) If $u\geq 0$, $w \geq 0$, then $\|(a,b)\|_p - (au+bw) \geq 0$.
(2) If $u \geq 0$, $w \leq 0$, then $\|(a,b)\|_p - (au+bw) \geq \|(a,b)\|_p - bw \geq \|(a,b)\|_p - |b| > 0$.
(3) If $u \leq 0$, $w \geq 0$, then $\|(a,b)\|_p - (au+bw) \geq \|(a,b)\|_p - aw \geq \|(a,b)\|_p - |a| > 0$.
(4) If $u \leq 0$, $w \leq 0$, then we use Hölder's inequality

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n}|y_k|^q\right)^{\frac{1}{q}}.$$

We substitute $(x_1,x_2)=(a,b)$, $(y_1,y_2)=(u,w)$ into Hölder's inequality, we can get

$$\begin{aligned}|au|+|bw| &\leq (|a|^p+|b|^p)^{\frac{1}{p}} (|u|^q+|w|^q)^{\frac{1}{q}}, \\ \Rightarrow au+bw \leq |au|+|bw| &\leq ||(a,b)||_p \cdot ||(u,w)||_q \leq ||(a,b)||_p.\end{aligned}$$

So, we can deduce that $||(a,b)||_p - (au+bw) \geq 0$.
By (1)-(4), we can also deduce that $\phi^p_{|\mathrm{FB}|}(a,b) - (as+bt) \geqslant 0$.

(iii) For $a>0$, $b<0$ or $a<0$, $b>0$, and $(s,t)\in\Omega'$. The proof is similar to (ii).

In summary, this proves the assertion $\phi^p_{|\mathrm{FB}|}(a,b) - (as+bt) \geqslant 0$. □

Theorem 4.1. *Assume that $z = (x, \lambda)$ is not a critical point of $P_c(z)$ and that λ is nonpositive. Let*

$$q = \begin{bmatrix} -V \\ -\Phi^p_{|\mathrm{FB}|}(x) \end{bmatrix},$$

where $V \in \partial_x P_c(x, \lambda)$, with $\rho_i \in [-1, 1]$ and $(\xi_i, \eta_i) \in \Omega'$. Then, $P_c'(z; q) < 0$. That is, q is a descent direction of P_c at z.

Proof. By the formula (21) and Proposition 4.1 (a).

Note that, $\begin{cases} \Phi^p_{|\mathrm{FB}|-\mathrm{i}}(x) = 0 & \text{if} \quad i \in I(x), \\ \Phi^p_{|\mathrm{FB}|-\mathrm{i}}{}'(x; d) = H_i^T d & \text{if} \quad i \in \overline{I}(x), \end{cases}$ then

$$\begin{aligned} & P'_c(z; q) - V^T d - \Phi^p_{|\mathrm{FB}|}(x)^T h \\ = \; & [\lambda + c\Phi^p_{|\mathrm{FB}|}(x)]^T [\Phi^p_{|\mathrm{FB}|}{}'(x; d) - H^T d] \\ = \; & \sum_{i \in I(x) \bigcup \overline{I}(x)} [\lambda_i + c\Phi^p_{|\mathrm{FB}|-\mathrm{i}}(x)][\Phi^p_{|\mathrm{FB}|-\mathrm{i}}{}'(x; d) - H_i^T d] \\ = \; & \sum_{i \in I(x)} [\lambda_i + c\Phi^p_{|\mathrm{FB}|-\mathrm{i}}(x)][\Phi^p_{|\mathrm{FB}|-\mathrm{i}}{}'(x; d) - H_i^T d] \\ + \; & \sum_{i \in \overline{I}(x)} [\lambda_i + c\Phi^p_{|\mathrm{FB}|-\mathrm{i}}(x)][\Phi^p_{|\mathrm{FB}|-\mathrm{i}}{}'(x; d) - H_i^T d], \end{aligned}$$

hence we have

$$P'_c(z; q) - V^T d - \Phi^p_{|\mathrm{FB}|}(x)^T h = \sum_{i \in I(x)} \lambda_i [\Phi^p_{|\mathrm{FB}|-\mathrm{i}}{}'(x; d) - H_i^T d]. \tag{22}$$

From Proposition 4.1 (b) and (c), we can deduce that

$$\Phi^p_{|\mathrm{FB}|-\mathrm{i}}{}'(x; d) - H_i^T d = \begin{cases} 0 & \text{if} \quad i \in \overline{I}(x), \\ |d_i| - \rho_i d_i & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| - \rho_i \nabla F_i(x)^T d & \text{if} \quad i \in I_{+0}(x), \\ \phi^p_{|\mathrm{FB}|}(d, \nabla F_i(x)^T d) - \xi_i d_i - \eta_i \nabla F_i(x)^T d & \text{if} \quad i \in I_{00}(x). \end{cases}$$

(i) If $i \in I_{0+}(x)$ and $\rho_i \in [-1, 1]$, then

$$|d_i| - \rho_i d_i \geq |d_i| - d_i \geq 0.$$

(ii) If $i \in I_{+0}(x)$ and $\rho_i \in [-1, 1]$, then

$$|\nabla F_i(x)^T d| - \rho_i \nabla F_i(x)^T d \geq |\nabla F_i(x)^T d| - \nabla F_i(x)^T d \geq 0.$$

(iii) If $i \in I_{00}(x)$ and by Proposition 4.1 (d), then

$$\phi^p_{|\mathrm{FB}|}(d, \nabla F_i(x)^T d) - \xi_i d_i - \eta_i \nabla F_i(x)^T d \geq 0.$$

Because λ is nonpositive and by (i)-(iii), so $\sum_{i\in I(x)} \lambda_i[{\Phi^p_{_{|\mathrm{FB}|-\mathrm{i}}}}'(x;d) - H_i^T d] \leq 0.$

And we use the formula (22) and $q = [d,h]^T = [-V, -\Phi^p_{_{|\mathrm{FB}|}}(x)]^T$, then

$$\begin{aligned}
& P'_c(z;q) - V^T d - \Phi^p_{_{|\mathrm{FB}|}}(x)^T h \leq 0 \\
\Rightarrow\ & P'_c(z;q) - V^T(-V) - \Phi^p_{_{|\mathrm{FB}|}}(x)^T(-\Phi^p_{_{|\mathrm{FB}|}}(x)) \leq 0 \\
\Rightarrow\ & {P'}'_c(z;q) + ||V||^2 + ||\Phi^p_{_{|\mathrm{FB}|}}(x)||^2 \leq 0 \\
\Rightarrow\ & {P'}'_c(z;q) \leq -||V||^2 - ||\Phi^p_{_{|\mathrm{FB}|}}(x)||^2 = -||q||^2 < 0.
\end{aligned}$$

Thus, the desired result follows. □

Proposition 4.2. *The function* $\phi^p_{_{\mathrm{NR}}}$

(a) *Let* $\Phi^p_{_{|\mathrm{NR}|}}(x) = \begin{bmatrix} \phi^p_{_{|\mathrm{NR}|}}(x_1, F_1(x)) \\ \vdots \\ \phi^p_{_{|\mathrm{NR}|}}(x_n, F_n(x)) \end{bmatrix}$ *and* $\begin{cases} \overline{I}(x) = I(x)^C, \\ I(x) = I_{+0}(x) \bigcup I_{00}(x), \end{cases}$

where $\begin{cases} I_{+0}(x) = \{i | x_i > 0,\ F_i(x) = 0\}, \\ I_{00}(x) = \{i | x_i = 0,\ F_i(x) = 0\}. \end{cases}$

(b) *For any* $i \in \{1, 2, \cdots, n\}$, *each element* $H_i \in \partial\Phi^p_{_{|\mathrm{NR}|}}(x)^T$ *and* $V \in \partial_x P_c(x, \lambda)$ *can be written as follow:* $H_i = a_i(x)E_i + b_i(x)\nabla F_i(x)$ *and* $V = \nabla f(x) + H[\lambda + c\Phi^p_{_{|\mathrm{NR}|}}(x)]$ *where* E_i *is the ith column of the* $n \times n$ *unit matrix,* $H = (H_1, H_2, \cdots, H_n)$

$$(a_i(x), b_i(x)) = \begin{cases} \nabla\phi^p_{_{|\mathrm{NR}|}}(x_i, F_i(x)) & \text{if} \quad i \in \overline{I}(x), \\ (0, \rho_i \cdot p x_i^{p-1}) & \text{if} \quad i \in I_{+0}(x), \\ (0, 0) & \text{if} \quad i \in I_{00}(x), \end{cases}$$

with $\rho_i \in [-1, 1]$.

(c) *The directional derivative of* $\Phi^p_{_{|\mathrm{NR}|-\mathrm{i}}}(x) = \phi^p_{_{|\mathrm{NR}|}}(x_i, F_i(x))$ *at* x *in the direction* d *is given by*

$${\Phi^p_{_{|\mathrm{NR}|-\mathrm{i}}}}'(x;d) = \begin{cases} H_i^T d & \text{if} \quad i \in \overline{I}(x), \\ |\nabla F_i(x)^T d| \cdot p x_i^{p-1} & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x). \end{cases}$$

Proof. (b) and (c) The conclusion follows directly from Lemma 2.1 and Proposition 3.2 (c)-(e). □

Theorem 4.2. *Assume that $z = (x, \lambda)$ is not a critical point of $P_c(z)$ and that λ is nonpositive. Let*

$$q = \begin{bmatrix} -V \\ -\Phi^p_{|\mathrm{NR}|}(x) \end{bmatrix},$$

where $V \in \partial_x P_c(x, \lambda)$, with $\rho_i \in [-1, 1]$. Then, $P_c'(z; q) < 0$. That is, q is a descent direction of P_c at z.

Proof. By the formula (21) and Proposition 4.2 (a).

Note that, $\begin{cases} \Phi^p_{|\mathrm{NR}|-\mathrm{i}}(x) = 0 & \text{if} \quad i \in I(x), \\ \Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) = H_i^T d & \text{if} \quad i \in \overline{I}(x), \end{cases}$ then

$$\begin{aligned} & P'_c(z; q) - V^T d - \Phi^p_{|\mathrm{NR}|}(x)^T h \\ = \; & [\lambda + c\Phi^p_{|\mathrm{NR}|}(x)]^T [\Phi^p_{|\mathrm{NR}|}{}'(x; d) - H^T d] \\ = \; & \sum_{i \in I(x) \bigcup \overline{I}(x)} [\lambda_i + c\Phi^p_{|\mathrm{NR}|-\mathrm{i}}(x)][\Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) - H_i^T d] \\ = \; & \sum_{i \in I(x)} [\lambda_i + c\Phi^p_{|\mathrm{NR}|-\mathrm{i}}(x)][\Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) - H_i^T d] \\ + \; & \sum_{i \in \overline{I}(x)} [\lambda_i + c\Phi^p_{|\mathrm{NR}|-\mathrm{i}}(x)][\Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) - H_i^T d], \end{aligned}$$

hence we have

$$P'_c(z; q) - V^T d - \Phi^p_{|\mathrm{NR}|}(x)^T h = \sum_{i \in I(x)} \lambda_i [\Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) - H_i^T d]. \tag{23}$$

From proposition 4.2 (b) and (c), we deduce that

$$\Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) - H_i^T d = \begin{cases} 0 & \text{if} \quad i \in \overline{I}(x), \\ |\nabla F_i(x)^T d| \cdot p x_i^{p-1} - \rho_i \cdot p x_i^{p-1} \nabla F_i(x)^T d & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x). \end{cases}$$

If $i \in I_{+0}(x)$ and $\rho_i \in [-1, 1]$, then

$$|\nabla F_i(x)^T d| \cdot p x_i^{p-1} - \rho_i \cdot p x_i^{p-1} \nabla F_i(x)^T d = (|\nabla F_i(x)^T d| - \rho_i \nabla F_i(x)^T d) \cdot p x_i^{p-1} \geq 0.$$

Because λ is nonpositive, So $\sum_{i \in I(x)} \lambda_i [\Phi^p_{|\mathrm{NR}|-\mathrm{i}}{}'(x; d) - H_i^T d] \leq 0$.

And we use the formula (23) and $q = [d, h]^T = [-V, -\Phi^p_{|\mathrm{NR}|}(x)]^T$, then

$$\begin{aligned} & P'_c(z; q) - V^T d - \Phi^p_{|\mathrm{NR}|}(x)^T h \leq 0 \\ \Rightarrow \; & P'_c(z; q) - V^T(-V) - \Phi^p_{|\mathrm{NR}|}(x)^T(-\Phi^p_{|\mathrm{NR}|}(x)) \leq 0 \\ \Rightarrow \; & P'_c(z; q) + ||V||^2 + ||\Phi^p_{|\mathrm{NR}|}(x)||^2 \leq 0 \\ \Rightarrow \; & P'_c(z; q) \leq -||V||^2 - ||\Phi^p_{|\mathrm{NR}|}(x)||^2 = -||q||^2 < 0. \end{aligned}$$

Thus, the desired result follows. $\square$

Proposition 4.3. *The function* $\phi^p_{\text{S-NR}}$

(a) *Let* $\Phi^p_{|\text{S-NR}|}(x) = \begin{bmatrix} \phi^p_{|\text{S-NR}|}(x_1, F_1(x)) \\ \vdots \\ \phi^p_{|\text{S-NR}|}(x_n, F_n(x)) \end{bmatrix}$ *and* $\begin{cases} \overline{I}(x) = I_1(x) \bigcup I_2(x) \bigcup I_3(x) \;\; I_4(x), \\ I_0(x) = I_{0+}(x) \bigcup I_{+0}(x) \bigcup I_{00}(x), \\ I_e(x) = I_{++}(x) \bigcup I_{--}(x), \end{cases}$

where $\begin{cases} I_1(x) = \{i | x_i > F_i(x) > 0\}, \\ I_2(x) = \{i | F_i(x) > x_i > 0\}, \\ I_3(x) = \{i | x_i > 0,\; F_i(x) \leq 0 \text{ or } 0 > x_i > F_i(x)\}, \\ I_4(x) = \{i | x_i \leq 0,\; F_i(x) > 0 \text{ or } 0 > F_i(x) > x_i\}, \end{cases}$

$\begin{cases} I_{0+}(x) = \{i | x_i = 0,\; F_i(x) > 0\}, \\ I_{+0}(x) = \{i | x_i > 0,\; F_i(x) = 0\}, \\ I_{00}(x) = \{i | x_i = 0,\; F_i(x) = 0\}, \end{cases}$ *and* $\begin{cases} I_{++}(x) = \{i | x_i = F_i(x) > 0\}, \\ I_{--}(x) = \{i | x_i = F_i(x) < 0\}. \end{cases}$

(b) *For any* $i \in \{1, 2, \cdots, n\}$, *each element* $H_i \in \partial \Phi^p_{|\text{S-NR}|}(x)^T$ *and* $V \in \partial_x P_c(x, \lambda)$ *can be written as follow:* $H_i = a_i(x) E_i + b_i(x) \nabla F_i(x)$ *and* $V = \nabla f(x) + H[\lambda + c\Phi^p_{|\text{S-NR}|}(x)]$ *where* E_i *is the ith column of the* $n \times n$ *unit matrix,* $H = (H_1, H_2, \cdots, H_n)$

$$(a_i(x), b_i(x)) = \begin{cases} \nabla \phi^p_{|\text{S-NR}|}(x_i, F_i(x)) & \text{if } \;\; i \in \overline{I}(x), \\ (\rho_i \cdot p(F_i(x))^{p-1}, 0) & \text{if } \;\; i \in I_{0+}(x), \\ (0, \rho_i \cdot p x_i^{p-1}) & \text{if } \;\; i \in I_{+0}(x), \\ (0, 0) & \text{if } \;\; i \in I_{00}(x), \\ (\alpha \cdot p x_i^{p-1}, (1-\alpha) \cdot p x_i^{p-1}) & \text{if } \;\; i \in I_{++}(x), \\ (-\alpha \cdot p x_i^{p-1}, -(1-\alpha) \cdot p x_i^{p-1}) & \text{if } \;\; i \in I_{--}(x), \end{cases}$$

with $\rho_i \in [-1, 1]$ *and* $\alpha \in [0, 1]$.

(c) *The directional derivative of* $\Phi^p_{|\text{S-NR}|-\text{i}}(x) = \phi^p_{|\text{S-NR}|}(x_i, F_i(x))$ *at* x *in the direction* d *is given by*

$$\Phi^p_{|\text{S-NR}|-\text{i}}{}'(x; d) = \begin{cases} H_i^T d & \text{if} \quad i \in \overline{I}(x), \\ |d_i| \cdot p(F_i(x))^{p-1} & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| \cdot p x_i^{p-1} & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x), \\ p x_i^{p-1} \cdot d_i & \text{if} \quad i \in I_{++}(x),\; d_i > \nabla F_i(x)^T d, \\ p x_i^{p-1} \cdot \nabla F_i(x)^T d & \text{if} \quad i \in I_{++}(x),\; d_i < \nabla F_i(x)^T d, \\ -p x_i^{p-1} \cdot d_i & \text{if} \quad i \in I_{--}(x),\; d_i > \nabla F_i(x)^T d, \\ -p x_i^{p-1} \cdot \nabla F_i(x)^T d & \text{if} \quad i \in I_{--}(x),\; d_i < \nabla F_i(x)^T d. \end{cases}$$

Proof. (b) and (c) The conclusion follows directly from Lemma 2.1 and Proposition 3.3 (e)-(g). $\square$

Theorem 4.3. *Assume that $z = (x, \lambda)$ is not a critical point of $P_c(z)$ and that $\lambda = (\lambda_1, \cdots, \lambda_n)$. Let*

$$q = \begin{bmatrix} -V \\ -\Phi^p_{|\mathrm{S-NR}|}(x) \end{bmatrix},$$

where $V \in \partial_x P_c(x, \lambda)$ and $\begin{cases} \lambda_i \le -c(\overline{x}_{ub})^p & \text{if} \quad i \in I_{++}(x) \\ \lambda_i \ge -c(\overline{x}_{lb})^p & \text{if} \quad i \in I_{--}(x) \\ \lambda_i \le 0 & \textit{otherwise} \end{cases}$ *, with $\rho_i \in [-1, 1]$, $\alpha \in [0, 1]$,* $\overline{x}_{ub} = \max\limits_{i \in I_{++}(x)} x_i$, $\overline{x}_{lb} = \min\limits_{i \in I_{--}(x)} x_i$ *Then, $P_c'(z; q) < 0$. That is, q is a descent direction of P_c at z.*

Proof. By the formula (21) and Proposition 4.3 (a).

Note that, $\begin{cases} \Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}(x) = 0 & \text{if} \quad i \in I_0(x), \\ \Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}(x) = x_i^p & \text{if} \quad i \in I_e(x), \\ {\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) = H_i^T d & \text{if} \quad i \in \overline{I}(x), \end{cases}$ then

$$\begin{aligned}
& {P'}_c(z; q) - V^T d - \Phi^p_{|\mathrm{S-NR}|}(x)^T h \\
= \ & [\lambda + c\Phi^p_{|\mathrm{S-NR}|}(x)]^T [{\Phi^p_{|\mathrm{S-NR}|}}'(x; d) - H^T d] \\
= \ & \sum_{i \in I_0(x) \bigcup I_e(x) \bigcup \overline{I}(x)} [\lambda_i + c\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}(x)][{\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d] \\
= \ & \sum_{i \in I_0(x)} [\lambda_i + c\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}(x)][{\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d] \\
+ \ & \sum_{i \in I_e(x)} [\lambda_i + c\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}(x)][{\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d] \\
+ \ & \sum_{i \in \overline{I}(x)} [\lambda_i + c\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}(x)][{\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d],
\end{aligned}$$

hence we have

$$\begin{aligned}
{P'}_c(z; q) - V^T d - \Phi^p_{|\mathrm{S-NR}|}(x)^T h \ = \ & \sum_{i \in I_0(x)} \lambda_i [{\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d] \\
+ \ & \sum_{i \in I_e(x)} [\lambda_i + c x_i^p][{\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d].
\end{aligned}$$

From Proposition 4.3 (b) and (c), we deduce that
${\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}}'(x; d) - H_i^T d$

$$= \begin{cases} 0 & \text{if} \quad i \in \overline{I}(x), \\ |d_i| \cdot p(F_i(x))^{p-1} - \rho_i \cdot p(F_i(x))^{p-1} d_i & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| \cdot p x_i^{p-1} - \rho_i \cdot p x_i^{p-1} \nabla F_i(x)^T d \cdot & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x), \\ p x_i^{p-1} \cdot d_i - \alpha \cdot p x_i^{p-1} d_i - (1-\alpha) \cdot p x_i^{p-1} \cdot \nabla F_i(x)^T d & \text{if} \quad i \in I_{++}(x),\ d_i > \nabla F_i(x)^T d, \\ p x_i^{p-1} \cdot \nabla F_i(x)^T d - \alpha \cdot p x_i^{p-1} d_i - (1-\alpha) \cdot p x_i^{p-1} \cdot \nabla F_i(x)^T d & \text{if} \quad i \in I_{++}(x),\ d_i < \nabla F_i(x)^T d, \\ -p x_i^{p-1} \cdot d_i + \alpha \cdot p x_i^{p-1} d_i + (1-\alpha) \cdot p x_i^{p-1} \cdot \nabla F_i(x)^T d & \text{if} \quad i \in I_{--}(x),\ d_i > \nabla F_i(x)^T d, \\ -p x_i^{p-1} \cdot \nabla F_i(x)^T d + \alpha \cdot p x_i^{p-1} d_i + (1-\alpha) \cdot p x_i^{p-1} \cdot \nabla F_i(x)^T d & \text{if} \quad i \in I_{--}(x),\ d_i < \nabla F_i(x)^T d. \end{cases}$$

(i) If $i \in I_{0+}(x)$ and $\rho_i \in [-1, 1]$, then

$$|d_i| \cdot p(F_i(x))^{p-1} - \rho_i \cdot p(F_i(x))^{p-1} d_i = (|d_i| - \rho_i d_i) \cdot p(F_i(x))^{p-1} \geq 0.$$

(ii) If $i \in I_{+0}(x)$ and $\rho_i \in [-1, 1]$, then

$$|\nabla F_i(x)^T d| \cdot p x_i^{p-1} - \rho_i \cdot p x_i^{p-1} \nabla F_i(x)^T d = (|\nabla F_i(x)^T d| - \rho_i \nabla F_i(x)^T d) \cdot p x_i^{p-1} \geq 0.$$

By (i), (ii), and λ is nonpositive, so we deduce that

$$\sum_{i \in I_0(x)} \lambda_i [\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}{}'(x; d) - H_i^T d] \leq 0.$$

(iii) If $i \in I_{++}(x)$, $\alpha \in [0, 1]$, and $d_i > \nabla F_i(x)^T d$, then

$$\begin{aligned} & p x_i^{p-1} \cdot d_i - \alpha \cdot p x_i^{p-1} d_i - (1-\alpha) \cdot p x_i^{p-1} \cdot \nabla F_i(x)^T d \\ = \ & p x_i^{p-1} \cdot (d_i - \alpha \cdot d_i - (1-\alpha) \cdot \nabla F_i(x)^T d) \\ = \ & p x_i^{p-1} \cdot (1-\alpha)(d_i - \nabla F_i(x)^T d) \geq 0. \end{aligned}$$

(iv) If $i \in I_{++}(x)$, $\alpha \in [0, 1]$, and $d_i < \nabla F_i(x)^T d$, then

$$\begin{aligned} & p x_i^{p-1} \cdot \nabla F_i(x)^T d - \alpha \cdot p x_i^{p-1} d_i - (1-\alpha) \cdot p x_i^{p-1} \cdot \nabla F_i(x)^T d \\ = \ & p x_i^{p-1} \cdot (\nabla F_i(x)^T d - \alpha d_i - (1-\alpha) \cdot \nabla F_i(x)^T d) \\ = \ & p x_i^{p-1} \cdot (\alpha \nabla F_i(x)^T d - \alpha d_i) \\ = \ & p x_i^{p-1} \cdot \alpha \cdot (\nabla F_i(x)^T d - d_i) \geq 0. \end{aligned}$$

By (iii), (iv), and $\lambda_i + c x_i^p \leq \lambda_i + c(\overline{x}_{ub})^p \leq 0$, so we deduce that

$$\sum_{i \in I_{++}(x)} [\lambda_i + c x_i^p][\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}{}'(x; d) - H_i^T d] \leq 0.$$

(v) If $i \in I_{--}(x)$, $\alpha \in [0,1]$, and $d_i > \nabla F_i(x)^T d$, then

$$\begin{aligned} & -px_i^{p-1} \cdot d_i + \alpha \cdot px_i^{p-1} d_i + (1-\alpha) \cdot px_i^{p-1} \cdot \nabla F_i(x)^T d \\ = & -px_i^{p-1} \cdot (d_i - \alpha \cdot d_i - (1-\alpha) \cdot \nabla F_i(x)^T d) \\ = & -px_i^{p-1} \cdot (1-\alpha)(d_i - \nabla F_i(x)^T d) \leq 0. \end{aligned}$$

(vi) If $i \in I_{--}(x)$, $\alpha \in [0,1]$, and $d_i < \nabla F_i(x)^T d$, then

$$\begin{aligned} & -px_i^{p-1} \cdot \nabla F_i(x)^T d + \alpha \cdot px_i^{p-1} d_i + (1-\alpha) \cdot px_i^{p-1} \cdot \nabla F_i(x)^T d \\ = & -px_i^{p-1} \cdot (\nabla F_i(x)^T d - \alpha d_i - (1-\alpha) \cdot \nabla F_i(x)^T d) \\ = & -px_i^{p-1} \cdot (\alpha \nabla F_i(x)^T d - \alpha d_i) \\ = & -px_i^{p-1} \cdot \alpha \cdot (\nabla F_i(x)^T d - d_i) \leq 0. \end{aligned}$$

By (v), (vi), and $\lambda_i + cx_i^p \geq \lambda_i + c(\overline{x}_{lb})^p \geq 0$, so we deduce that

$$\sum_{i \in I_{--}(x)} [\lambda_i + cx_i^p][\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}{}'(x;d) - H_i^T d] \leq 0.$$

From all the above, we conclude that

$$\begin{aligned} & \sum_{i \in I_e(x)} [\lambda_i + cx_i^p][\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}{}'(x;d) - H_i^T d] \\ = & \sum_{i \in I_{++}(x)} [\lambda_i + cx_i^p][\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}{}'(x;d) - H_i^T d] + \sum_{i \in I_{--}(x)} [\lambda_i + cx_i^p][\Phi^p_{|\mathrm{S-NR}|-\mathrm{i}}{}'(x;d) - H_i^T d] \leq 0, \end{aligned}$$

hence we have

$$P'_c(z;q) - V^T d - \Phi^p_{|\mathrm{S-NR}|}(x)^T h \leq 0 \tag{24}$$

And we use the formula (24) and $q = [d,h]^T = [-V, -\Phi^p_{|\mathrm{S-NR}|}(x)]^T$, then

$$\begin{aligned} & P'_c(z;q) - V^T d - \Phi^p_{|\mathrm{S-NR}|}(x)^T h \leq 0 \\ \Rightarrow\ & P'_c(z;q) - V^T(-V) - \Phi^p_{|\mathrm{S-NR}|}(x)^T(-\Phi^p_{|\mathrm{S-NR}|}(x)) \leq 0 \\ \Rightarrow\ & P'_c(z;q) + ||V||^2 + ||\Phi^p_{|\mathrm{S-NR}|}(x)||^2 \leq 0 \\ \Rightarrow\ & P'_c(z;q) \leq -||V||^2 - ||\Phi^p_{|\mathrm{S-NR}|}(x)||^2 = -||q||^2 < 0. \end{aligned}$$

Thus, the desired result follows. □

Theorem 4.4. *Assume that $z = (x, \lambda)$ is not a critical point of $P_c(z)$ and that λ is nonpositive. Let*

$$q = \begin{bmatrix} -V \\ -\Psi^p_{|\mathrm{S-NR}|}(x) \end{bmatrix},$$

where $V \in \partial_x P_c(x,\lambda)$ *and* $\Psi^p_{|S-NR|}(x) = \Big[\psi^p_{|S-NR|}(x_1, F_1(x)), \cdots, \psi^p_{|S-NR|}(x_n, F_n(x))\Big]^T$.
If $\Psi^p_{|S-NR|}(x)$ *is continuously differentiable, Then,* $P_c'(z;q) < 0$.*That is,* q *is a descent direction of* P_c *at* z.

Proof. If $\Psi^p_{|S-NR|}(x)$ is continuously differentiable, Then $\Psi^p_{|S-NR|-i}{}'(x;d) = H_i^T d$. And by the formula (21), then

$$
\begin{aligned}
& P'_c(z;q) - V^T d - \Psi^p_{|S-NR|}(x)^T h \\
= \;& [\lambda + c\Psi^p_{|S-NR|}(x)]^T [\Psi^p_{|S-NR|}{}'(x;d) - H^T d] \\
= \;& \sum [\lambda_i + c\Psi^p_{|S-NR|-i}(x)]^T [\Psi^p_{|S-NR|-i}{}'(x;d) - H_i^T d] \\
= \;& \sum [\lambda_i + c\Psi^p_{|S-NR|-i}(x)]^T [H_i^T d - H_i^T d] = 0.
\end{aligned}
$$

And we have $q = [d,h]^T = [-V, -\Psi^p_{|S-NR|}(x)]^T$, then

$$
\begin{aligned}
& P'_c(z;q) - V^T d - \Psi^p_{|S-NR|}(x)^T h = 0 \\
\Rightarrow \;& P'_c(z;q) - V^T(-V) - \Psi^p_{|S-NR|}(x)^T(-\Psi^p_{|S-NR|}(x)) = 0 \\
\Rightarrow \;& P'_c(z;q) + ||V||^2 + ||\Psi^p_{|S-NR|}(x)||^2 = 0 \\
\Rightarrow \;& P'_c(z;q) = -||V||^2 - ||\Psi^p_{|S-NR|}(x)||^2 = -||q||^2 < 0.
\end{aligned}
$$

Thus, the desired result follows. □

Proposition 4.4. *The function* ϕ^p_{D-FB}

(a) *Let* $\Phi^p_{|D-FB|}(x) = \begin{bmatrix} \phi^p_{|D-FB|}(x_1, F_1(x)) \\ \vdots \\ \phi^p_{|D-FB|}(x_n, F_n(x)) \end{bmatrix}$ *and* $\begin{cases} \overline{I}(x) = I_1(x) \bigcup I_2(x), \\ I(x) = I_{0+}(x) \bigcup I_{+0}(x) \bigcup I_{00}(x), \end{cases}$

where $\begin{cases} I_1(x) = \{i | x_i > 0,\ F_i(x) > 0\}, \\ I_2(x) = \{i | x_i < 0 \text{ or } F_i(x) < 0\}, \end{cases}$ $\begin{cases} I_{0+}(x) = \{i | x_i = 0,\ F_i(x) > 0\}, \\ I_{+0}(x) = \{i | x_i > 0,\ F_i(x) = 0\}, \\ I_{00}(x) = \{i | x_i = 0,\ F_i(x) = 0\}. \end{cases}$

(b) *For any* $i \in \{1, 2, \cdots, n\}$, *each element* $H_i \in \partial \Phi^p_{|D-FB|}(x)^T$ *and* $V \in \partial_x P_c(x,\lambda)$ *can be written as follow:* $H_i = a_i(x)E_i + b_i(x)\nabla F_i(x)$ *and* $V = \nabla f(x) + H[\lambda + c\Phi^p_{|D-FB|}(x)]$ *where* E_i *is the ith column of the* $n \times n$ *unit matrix,* $H = (H_1, H_2, \cdots, H_n)$

$$
(a_i(x), b_i(x)) = \begin{cases} \nabla \phi^p_{|D-FB|}(x_i, F_i(x)) & \text{if} \quad i \in \overline{I}(x), \\ (\rho_i \cdot p(F_i(x))^{p-1}, 0) & \text{if} \quad i \in I_{0+}(x), \\ (0, \rho_i \cdot p x_i^{p-1}) & \text{if} \quad i \in I_{+0}(x), \\ (0,0) & \text{if} \quad i \in I_{00}(x), \end{cases}
$$

with $\rho_i \in [-1, 1]$.

(c) *The directional derivative of* $\Phi^p_{_{|D-FB|-i}}(x) = \phi^p_{_{|D-FB|}}(x_i, F_i(x))$ *at* x *in the direction* d *is given by*

$$\Phi^p_{_{|D-FB|-i}}{}'(x;d) = \begin{cases} H_i^T d & \text{if} \quad i \in \overline{I}(x), \\ |d_i| \cdot p(F_i(x))^{p-1} & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| \cdot p x_i^{p-1} & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x). \end{cases}$$

Proof. (b) and (c) The conclusion follows directly from Lemma 2.1 and Proposition 3.5 (b)-(d). □

Theorem 4.5. *Assume that* $z = (x, \lambda)$ *is not a critical point of* $P_c(z)$ *and that* λ *is nonpositive. Let*

$$q = \begin{bmatrix} -V \\ -\phi^p_{_{|D-FB|}}(x) \end{bmatrix},$$

where $V \in \partial_x P_c(x, \lambda)$ *and* $\rho_i \in [-1, 1]$*, Then,* $P_c'(z;q) < 0$*.That is,* q *is a descent direction of* P_c *at* z*.*

Proof. By the formula (21) and Proposition 4.4 (a).

Note that, $\begin{cases} \Phi^p_{_{|D-FB|-i}}(x) = 0 & \text{if} \quad i \in I(x), \\ \Phi^p_{_{|D-FB|-i}}{}'(x;d) = H_i^T d & \text{if} \quad i \in \overline{I}(x), \end{cases}$ then

$$\begin{aligned} & P'_c(z;q) - V^T d - \Phi^p_{_{|D-FB|}}(x)^T h \\ = \; & [\lambda + c\Phi^p_{_{|D-FB|}}(x)]^T [\Phi^p_{_{|D-FB|}}{}'(x;d) - H^T d] \\ = \; & \sum_{i \in I(x) \bigcup \overline{I}(x)} [\lambda_i + c\Phi^p_{_{|D-FB|-i}}(x)][\Phi^p_{_{|D-FB|-i}}{}'(x;d) - H_i^T d] \\ = \; & \sum_{i \in I(x)} [\lambda_i + c\Phi^p_{_{|D-FB|-i}}(x)][\Phi^p_{_{|D-FB|-i}}{}'(x;d) - H_i^T d] \\ + \; & \sum_{i \in \overline{I}(x)} [\lambda_i + c\Phi^p_{_{|D-FB|-i}}(x)][\Phi^p_{_{|D-FB|-i}}{}'(x;d) - H_i^T d], \end{aligned}$$

hence we have

$$P'_c(z;q) - V^T d - \Phi^p_{_{|D-FB|}}(x)^T h = \sum_{i \in I(x)} \lambda_i [\Phi^p_{_{|D-FB|-i}}{}'(x;d) - H_i^T d]. \tag{25}$$

From proposition 4.4 (b) and (c), we deduce that

$$\Phi^p_{_{|D-FB|-i}}{}'(x;d) - H_i^T d = \begin{cases} 0 & \text{if} \quad i \in \overline{I}(x), \\ |d_i| \cdot p(F_i(x))^{p-1} - \rho_i \cdot p(F_i(x))^{p-1} d_i & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| \cdot p x_i^{p-1} - \rho_i \cdot p x_i^{p-1} \nabla F_i(x)^T d & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x). \end{cases}$$

(i) If $i \in I_{0+}(x)$ and $\rho_i \in [-1,1]$, then

$$|d_i| \cdot p(F_i(x))^{p-1} - \rho_i \cdot p(F_i(x))^{p-1} d_i = (|d_i| - \rho_i d_i) \cdot p(F_i(x))^{p-1} \geq 0.$$

(ii) If $i \in I_{+0}(x)$ and $\rho_i \in [-1,1]$, then

$$|\nabla F_i(x)^T d| \cdot p x_i^{p-1} - \rho_i \cdot p x_i^{p-1} \nabla F_i(x)^T d = (|\nabla F_i(x)^T d| - \rho_i \nabla F_i(x)^T d) \cdot p x_i^{p-1} \geq 0.$$

Because λ is nonpositive and by (i) and (ii), so $\sum_{i \in I(x)} \lambda_i [\Phi^p_{{}_{|\text{D-FB}|-\text{i}}}{}'(x;d) - H_i^T d] \leq 0$.

And we use the formula (25) and $q = [d,h]^T = [-V, -\Phi^p_{{}_{|\text{D-FB}|}}(x)]^T$, then

$$\begin{aligned}
& P'_c(z;q) - V^T d - \Phi^p_{{}_{|\text{D-FB}|}}(x)^T h \leq 0 \\
\Rightarrow\ & P'_c(z;q) - V^T(-V) - \Phi^p_{{}_{|\text{D-FB}|}}(x)^T(-\Phi^p_{{}_{|\text{D-FB}|}}(x)) \leq 0 \\
\Rightarrow\ & P'_c(z;q) + ||V||^2 + ||\Phi^p_{{}_{|\text{D-FB}|}}(x)||^2 \leq 0 \\
\Rightarrow\ & P'_c(z;q) \leq -||V||^2 - ||\Phi^p_{{}_{|\text{D-FB}|}}(x)||^2 = -||q||^2 < 0.
\end{aligned}$$

Thus, the desired result follows. $\square$

Proposition 4.5. *The function* $\phi^p_{{}_{\text{A-NR}}}$

(a) *Let* $\Phi^p_{{}_{|\text{A-NR}|}}(x) = \begin{bmatrix} \phi^p_{{}_{|\text{A-NR}|}}(x_1, F_1(x)) \\ \vdots \\ \phi^p_{{}_{|\text{A-NR}|}}(x_n, F_n(x)) \end{bmatrix}$ *and* $\begin{cases} \overline{I}(x) = I_1(x) \bigcup I_2(x), \\ I(x) = I_{0+}(x) \bigcup I_{+0}(x) \bigcup I_{00}(x), \end{cases}$

where $\begin{cases} I_1(x) = \{i | x_i > 0,\ F_i(x) > 0\}, \\ I_2(x) = \{i | x_i < 0 \text{ or } F_i(x) < 0\}, \end{cases}$ $\begin{cases} I_{0+}(x) = \{i | x_i = 0,\ F_i(x) > 0\}, \\ I_{+0}(x) = \{i | x_i > 0,\ F_i(x) = 0\}, \\ I_{00}(x) = \{i | x_i = 0,\ F_i(x) = 0\}. \end{cases}$

(b) *For any* $i \in \{1, 2, \cdots, n\}$, *each element* $H_i \in \partial \Phi^p_{{}_{|\text{A-NR}|}}(x)^T$ *and* $V \in \partial_x P_c(x,\lambda)$ *can be written as follow:* $H_i = a_i(x) E_i + b_i(x) \nabla F_i(x)$ *and* $V = \nabla f(x) + H[\lambda + c\Phi^p_{{}_{|\text{A-NR}|}}(x)]$ *where* E_i *is the ith column of the* $n \times n$ *unit matrix,* $H = (H_1, H_2, \cdots, H_n)$

$$(a_i(x), b_i(x)) = \begin{cases} \nabla \phi^p_{{}_{|\text{A-NR}|}}(x_i, F_i(x)) & \text{if} \quad i \in \overline{I}(x), \\ (\rho_i \cdot p(\frac{F_i(x)}{2})^{p-1}, 0) & \text{if} \quad i \in I_{0+}(x), \\ (0, \rho_i \cdot p(\frac{x_i}{2})^{p-1}) & \text{if} \quad i \in I_{+0}(x), \\ (0,0) & \text{if} \quad i \in I_{00}(x), \end{cases}$$

with $\rho_i \in [-1, 1]$.

(c) *The directional derivative of* $\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}(x) = \phi^p_{_{|\mathrm{A-NR}|}}(x_i, F_i(x))$ *at* x *in the direction* d *is given by*

$$\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) = \begin{cases} H_i^T d & \text{if} \quad i \in \overline{I}(x), \\ |d_i| \cdot p(\frac{F_i(x)}{2})^{p-1} & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| \cdot p(\frac{x_i}{2})^{p-1} & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x). \end{cases}$$

Proof. (b) and (c) The conclusion follows directly from Lemma 2.1 and Proposition 3.6 (b)-(d). □

Theorem 4.6. *Assume that* $z = (x, \lambda)$ *is not a critical point of* $P_c(z)$ *and that* λ *is nonpositive. Let*

$$q = \begin{bmatrix} -V \\ -\Phi^p_{_{|\mathrm{A-NR}|}}(x) \end{bmatrix},$$

where $V \in \partial_x P_c(x, \lambda)$ *and* $\rho_i \in [-1, 1]$*, Then,* $P_c'(z; q) < 0$*.That is,* q *is a descent direction of* P_c *at* z*.*

Proof. By the formula (21) and Proposition 4.5 (a).

Note that, $\begin{cases} \Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}(x) = 0 & \text{if} \quad i \in I(x), \\ \Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) = H_i^T d & \text{if} \quad i \in \overline{I}(x), \end{cases}$ then

$$\begin{aligned} & P'_c(z;q) - V^T d - \Phi^p_{_{|\mathrm{A-NR}|}}(x)^T h \\ = \; & [\lambda + c\Phi^p_{_{|\mathrm{A-NR}|}}(x)]^T [\Phi^p_{_{|\mathrm{A-NR}|}}{}'(x;d) - H^T d] \\ = \; & \sum_{i \in I(x) \bigcup \overline{I}(x)} [\lambda_i + c\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}(x)][\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) - H_i^T d] \\ = \; & \sum_{i \in I(x)} [\lambda_i + c\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}(x)][\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) - H_i^T d] \\ + \; & \sum_{i \in \overline{I}(x)} [\lambda_i + c\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}(x)][\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) - H_i^T d], \end{aligned}$$

hence we have

$$P'_c(z;q) - V^T d - \Phi^p_{_{|\mathrm{A-NR}|}}(x)^T h = \sum_{i \in I(x)} \lambda_i [\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) - H_i^T d]. \tag{26}$$

From proposition 4.5 (b) and (c), we deduce that

$$\Phi^p_{_{|\mathrm{A-NR}|-\mathrm{i}}}{}'(x;d) - H_i^T d = \begin{cases} 0 & \text{if} \quad i \in \overline{I}(x), \\ |d_i| \cdot p(\frac{F_i(x)}{2})^{p-1} - \rho_i \cdot p(\frac{F_i(x)}{2})^{p-1} d_i & \text{if} \quad i \in I_{0+}(x), \\ |\nabla F_i(x)^T d| \cdot p(\frac{x_i}{2})^{p-1} - \rho_i \cdot p(\frac{x_i}{2})^{p-1} \nabla F_i(x)^T d & \text{if} \quad i \in I_{+0}(x), \\ 0 & \text{if} \quad i \in I_{00}(x). \end{cases}$$

(i) If $i \in I_{0+}(x)$ and $\rho_i \in [-1, 1]$, then

$$|d_i| \cdot p(\frac{F_i(x)}{2})^{p-1} - \rho_i \cdot p(\frac{F_i(x)}{2})^{p-1} d_i = (|d_i| - \rho_i d_i) \cdot p(\frac{F_i(x)}{2})^{p-1} \geq 0.$$

(ii) If $i \in I_{+0}(x)$ and $\rho_i \in [-1, 1]$, then

$$|\nabla F_i(x)^T d| \cdot p(\frac{x_i}{2})^{p-1} - \rho_i \cdot p(\frac{x_i}{2})^{p-1} \nabla F_i(x)^T d = (|\nabla F_i(x)^T d| - \rho_i \nabla F_i(x)^T d) \cdot p(\frac{F_i(x)}{2})^{p-1} \geq 0.$$

Because λ is nonpositive and by (i) and (ii), so $\sum_{i \in I(x)} \lambda_i [\Phi^p_{|\mathrm{A-NR}|-\mathrm{i}}{}'(x; d) - H_i^T d] \leq 0$.

And we use the formula (26) and $q = [d, h]^T = [-V, -\Phi^p_{|\mathrm{A-NR}|}(x)]^T$, then

$$\begin{aligned} & P'_c(z; q) - V^T d - \Phi^p_{|\mathrm{A-NR}|}(x)^T h \leq 0 \\ \Rightarrow \ & P'_c(z; q) - V^T(-V) - \Phi^p_{|\mathrm{A-NR}|}(x)^T(-\Phi^p_{|\mathrm{A-NR}|}(x)) \leq 0 \\ \Rightarrow \ & P'_c(z; q) + ||V||^2 + ||\Phi^p_{|\mathrm{A-NR}|}(x)||^2 \leq 0 \\ \Rightarrow \ & P'_c(z; q) \leq -||V||^2 - ||\Phi^p_{|\mathrm{A-NR}|}(x)||^2 = -||q||^2 < 0. \end{aligned}$$

Thus, the desired result follows. □

In this section we state several basic assumptions. Based on these assumptions, we analyze the descentness of the generalized gradients of P_c in different NCP functions.

5 Conclusion

In this paper, we found that no matter which NCP function is used, these Theorems shows that at any noncritical point of P_c, some negative generalized gradient direction of P_c is its descent direction if suitable conditions hold. If we use the 3rd and 4th Sections in [19], then we can do P_c be well-defined and globally convergent. We used an LG-type algorithm model for solving the nonlinear complementarity problem. In particular our algorithm model does not depend on some specific NCP function and solve the difference of the nonlinear complementary problems. We may try to compare the convergence speed of different NCP functions, which may be a problem that can be considered in the future.

Meanwhile, we studied in details six NCP functions applicable to the LG method. Based on the observation to [19], we can construct some new NCP function possibly with the same properties. Set

$$
\begin{aligned}
\phi_1(a,b) &= \sqrt{[\phi(a,b)]^2 + \alpha(a_+ b_+)^2}, \\
\phi_2(a,b) &= \sqrt{[\phi(a,b)]^2 + \alpha((ab)_+)^2}, \\
\phi_3(a,b) &= \sqrt{[\phi(a,b)]^2 + \alpha((ab)_+)^4}, \\
\phi_4(a,b) &= \sqrt{\{[\phi(a,b)]_+\}^2 + \alpha((ab)_+)^2}, \\
\phi_5(a,b) &= \sqrt{\{[-\phi(a,b)]_+\}^2 + [(-a)_+]^2 + [(-b)_+]^2},
\end{aligned}
$$

where $\phi(a,b)$ is a NCP function, for any $\alpha > 0$.
We tried to replace ϕ with $\phi^p_{\rm FB}$, $\phi^p_{\rm D-FB}$, or $\phi^p_{\rm A-NR}$ to get some results, which we listed in Appendix. Maybe we can try to replace ϕ with other NCP functions to compare their pros and cons. We will leave these questions as further research topics.

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Appendix

A Some other forms to the ϕ^p_{FB} Function

The source of the idea is mainly based on [19].

Let ϕ^p_{FB} be defined as in (9). Then, for any $\alpha > 0$, the following variants of ϕ^p_{FB} are also NCP function,

$$\phi^p_{\text{FB}-1}(a,b) = \sqrt{(\phi^p_{\text{FB}}(a,b))^2 + \alpha(a_+b_+)^2}, \tag{27}$$

$$\phi^p_{\text{FB}-2}(a,b) = \sqrt{(\phi^p_{\text{FB}}(a,b))^2 + \alpha((ab)_+)^2}, \tag{28}$$

$$\phi^p_{\text{FB}-3}(a,b) = \sqrt{(\phi^p_{\text{FB}}(a,b))^2 + \alpha((ab)_+)^4}, \tag{29}$$

$$\phi^p_{\text{FB}-4}(a,b) = \sqrt{\{[\phi^p_{\text{FB}}(a,b)]_+\}^2 + \alpha((ab)_+)^2}, \tag{30}$$

$$\phi^p_{\text{FB}-5}(a,b) = \sqrt[p]{\{[-\phi^p_{\text{FB}}(a,b)]_+\}^p + [(-a)_+]^p + [(-b)_+]^p}. \tag{31}$$

We are now ready to deduce the generalized gradients and the directional derivatives of function $\phi^p_{\text{FB-i}}$ for every $i \in \{1,2,3,4,5\}$. We first rewrite function $\phi^p_{\text{FB-i}}$ for every $i \in \{1,2,3,4,5\}$ as follows:

$$\phi^p_{\text{FB}-1}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{FB}}(a,b)]^2 + \alpha(ab)^2} & \text{if} \quad a > 0,\ b > 0, \\ \phi^p_{|\text{FB}|}(a,b) = \phi^p_{\text{FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{FB}-2}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{FB}}(a,b)]^2 + \alpha(ab)^2} & \text{if} \quad ab > 0, \\ \phi^p_{|\text{FB}|}(a,b) = \phi^p_{\text{FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{FB}-3}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{FB}}(a,b)]^2 + \alpha(ab)^4} & \text{if} \quad ab > 0, \\ \phi^p_{|\text{FB}|}(a,b) = \phi^p_{\text{FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{FB}-4}(a,b) = \begin{cases} \sqrt{\alpha}ab & \text{if} \quad a > 0,\ b > 0, \\ \sqrt{\{[\phi^p_{\text{FB}}(a,b)]_+\}^2 + \alpha(ab)^2} & \text{if} \quad a < 0,\ b < 0, \\ \phi^p_{|\text{FB}|}(a,b) = \phi^p_{\text{FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{FB}-5}(a,b) = \begin{cases} (a+b) - ||(a,b)||_p & \text{if} \quad a \geq 0,\ b \geq 0, \\ -b & \text{if} \quad a \geq 0,\ b < 0, \\ -a & \text{if} \quad a < 0,\ b \geq 0, \\ ||(a,b)||_p & \text{if} \quad a < 0,\ b < 0. \end{cases}$$

We next list the generalized gradients and directional derivatives that currently compute these NCP functions.

Proposition A.1. *Let $\phi^p_{\text{FB}-1}$ be defined as in (27) with $p > 1$ being an arbitrary fixed real number. Then, the following hold.*

(a) *If* $\phi^p_{\rm FB-1}(a,b)\neq 0$, *then* $\phi^p_{\rm FB-1}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\rm FB-1}(a,b)=\begin{cases}\frac{1}{\phi^p_{\rm FB-1}(a,b)}\left(\phi^p_{\rm FB}(a,b)\cdot\nabla\phi^p_{\rm FB}(a,b)+\alpha\begin{bmatrix}ab^2\\ a^2b\end{bmatrix}\right) & \text{if}\quad a>0,\ b>0,\\ \nabla\phi^p_{\rm FB}(a,b) & \text{otherwise.}\end{cases}$$

If $\phi^p_{\rm FB-1}(a,b)=0$, *then the generalized gradient of* $\phi^p_{\rm FB-1}$ *at* (a,b) *is*

$$\partial\phi^p_{\rm FB-1}(a,b)=\begin{cases}\{(\rho,0)|\rho\in[-1,\sqrt{1+\alpha b^2}]\} & \text{if}\quad a=0,\ b>0,\\ \{(0,\rho)|\rho\in[-1,\sqrt{1+\alpha a^2}]\} & \text{if}\quad a>0,\ b=0,\\ \Xi_1 & \text{if}\quad a=0,\ b=0,\end{cases}$$

where $\Xi_1\subseteq conv\{\Xi_{11}\bigcup\Xi_{12}\}$, *here*

$$\begin{cases}\Xi_{11}=\{(1-\xi,1-\zeta)|\xi\geq 0\ and\ \zeta\geq 0,\ |\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1\},\\ \Xi_{12}=\{(\xi-1,\zeta-1)|\xi\leq 0\ or\ \zeta\leq 0,\ |\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1\}.\end{cases}$$

where $\Xi_1=\Xi$.

(b) *The directional derivative of* $\phi^p_{\rm FB-1}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{\rm FB-1}{}'((a,b);v)=\begin{cases}\langle\nabla\phi^p_{\rm FB-1}(a,b),v\rangle & \text{if}\quad \phi^p_{\rm FB-1}(a,b)\neq 0,\\ -v_1 & \text{if}\quad a=0,\ b>0,\ v_1<0,\\ \sqrt{1+\alpha b^2}v_1 & \text{if}\quad a=0,\ b>0,\ v_1>0,\\ -v_2 & \text{if}\quad a>0,\ b=0,\ v_2<0,\\ \sqrt{1+\alpha a^2}v_2 & \text{if}\quad a>0,\ b=0,\ v_2>0,\\ \phi^p_{|\rm FB|}(v_1,v_2) & \text{if}\quad a=0,\ b=0.\end{cases}$$

Proposition A.2. *Let* $\phi^p_{\rm FB-2}$ *be defined as in (28) with* $p>1$ *being an arbitrary fixed real number. Then, the following hold.*

(a) *If* $\phi^p_{\rm FB-2}(a,b)\neq 0$, *then* $\phi^p_{\rm FB-2}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\rm FB-2}(a,b)=\begin{cases}\frac{1}{\phi^p_{\rm FB-2}(a,b)}\left(\phi^p_{\rm FB}(a,b)\cdot\nabla\phi^p_{\rm FB}(a,b)+\alpha\begin{bmatrix}ab^2\\ a^2b\end{bmatrix}\right) & \text{if}\quad ab>0,\\ \nabla\phi^p_{\rm FB}(a,b) & \text{otherwise.}\end{cases}$$

If $\phi^p_{\rm FB-2}(a,b)=0$, *then the generalized gradient of* $\phi^p_{\rm FB-2}$ *at* (a,b) *is*

$$\partial\phi^p_{\rm FB-2}(a,b)=\begin{cases}\{(\rho,0)|\rho\in[-1,\sqrt{1+\alpha b^2}]\} & \text{if}\quad a=0,\ b>0,\\ \{(0,\rho)|\rho\in[-1,\sqrt{1+\alpha a^2}]\} & \text{if}\quad a>0,\ b=0,\\ \Xi_2 & \text{if}\quad a=0,\ b=0,\end{cases}$$

where $\Xi_2 \subseteq conv\{\Xi_{21} \bigcup \Xi_{22}\}$, *here*

$$\begin{cases} \Xi_{21} = \{(1-\xi, 1-\zeta) | \xi \geq 0 \text{ and } \zeta \geq 0,\ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}, \\ \Xi_{22} = \{(\xi - 1, \zeta - 1) | \xi \leq 0 \text{ or } \zeta \leq 0,\ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}, \end{cases}$$

where $\Xi_2 = \Xi$.

(b) *The directional derivative of* $\phi^p_{_{\rm FB-2}}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{_{\rm FB-2}}{}'((a,b);v) = \begin{cases} \langle \nabla \phi^p_{_{\rm FB-2}}(a,b), v\rangle & \text{if} \quad \phi^p_{_{\rm FB-2}}(a,b) \neq 0, \\ -v_1 & \text{if} \quad a = 0,\ b > 0,\ v_1 < 0, \\ \sqrt{1+\alpha b^2} v_1 & \text{if} \quad a = 0,\ b > 0,\ v_1 > 0, \\ -v_2 & \text{if} \quad a > 0,\ b = 0,\ v_2 < 0, \\ \sqrt{1+\alpha a^2} v_2 & \text{if} \quad a > 0,\ b = 0,\ v_2 > 0, \\ \phi^p_{_{|\rm FB|}}(v_1, v_2) & \text{if} \quad a = 0,\ b = 0. \end{cases}$$

Proposition A.3. *Let* $\phi^p_{_{\rm FB-3}}$ *be defined as in (29) with* $p > 1$ *being an arbitrary fixed real number. Then, the following hold.*

(a) *If* $\phi^p_{_{\rm FB-3}}(a,b) \neq 0$, *then* $\phi^p_{_{\rm FB-3}}$ *is continuously differentiable at* (a,b) *and*

$$\nabla \phi^p_{_{\rm FB-3}}(a,b) = \begin{cases} \frac{1}{\phi^p_{_{\rm FB-3}}(a,b)} \left(\phi^p_{_{\rm FB}}(a,b) \cdot \nabla \phi^p_{_{\rm FB}}(a,b) + 2\alpha \begin{bmatrix} a^3 b^4 \\ a^4 b^3 \end{bmatrix} \right) & \text{if} \quad ab > 0, \\ \nabla \phi^p_{_{\rm FB}}(a,b) & \text{otherwise.} \end{cases}$$

If $\phi^p_{_{\rm FB-3}}(a,b) = 0$, *then the generalized gradient of* $\phi^p_{_{\rm FB-3}}$ *at* (a,b) *is*

$$\partial \phi^p_{_{\rm FB-3}}(a,b) = \begin{cases} \{(\rho, 0) | \rho \in [-1,1]\} & \text{if} \quad a = 0,\ b > 0, \\ \{(0, \rho) | \rho \in [-1,1]\} & \text{if} \quad a > 0,\ b = 0, \\ \Xi_3 & \text{if} \quad a = 0,\ b = 0, \end{cases}$$

where $\Xi_3 \subseteq conv\{\Xi_{31} \bigcup \Xi_{32}\}$, *here*

$$\begin{cases} \Xi_{31} = \{(1-\xi, 1-\zeta) | \xi \geq 0 \text{ and } \zeta \geq 0,\ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}, \\ \Xi_{32} = \{(\xi - 1, \zeta - 1) | \xi \leq 0 \text{ or } \zeta \leq 0,\ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\}, \end{cases}$$

where $\Xi_3 = \Xi$.

(b) *The directional derivative of* $\phi^p_{_{\rm FB-3}}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{_{\rm FB-3}}{}'((a,b);v) = \begin{cases} \langle \nabla \phi^p_{_{\rm FB-3}}(a,b), v\rangle & \text{if} \quad \phi^p_{_{\rm FB-3}}(a,b) \neq 0, \\ |v_1| & \text{if} \quad a = 0,\ b > 0, \\ |v_2| & \text{if} \quad a > 0,\ b = 0, \\ \phi^p_{_{|\rm FB|}}(v_1, v_2) & \text{if} \quad a = 0,\ b = 0. \end{cases}$$

Proposition A.4. *Let $\phi^p_{_{\rm FB-4}}$ be defined as in (30) with $p > 1$ being an arbitrary fixed real number. Then, the following hold.*

(a) *If $\phi^p_{_{\rm FB-4}}(a,b) \neq 0$, then $\phi^p_{_{\rm FB-4}}$ is continuously differentiable at (a,b) and*

$$\nabla\phi^p_{_{\rm FB-4}}(a,b) = \begin{cases} (\sqrt{\alpha}b, \sqrt{\alpha}a) & \text{if } \ a>0, \ b>0, \\ \frac{1}{\phi^p_{_{\rm FB-4}}(a,b)}\left(\phi^p_{_{\rm FB}}(a,b)\cdot\nabla\phi^p_{_{\rm FB}}(a,b) + \alpha\begin{bmatrix} ab^2 \\ a^2b \end{bmatrix}\right) & \text{if } \ a<0, \ b<0, \\ \nabla\phi^p_{_{\rm FB}}(a,b) & \text{otherwise.} \end{cases}$$

If $\phi^p_{_{\rm FB-4}}(a,b) = 0$, then the generalized gradient of $\phi^p_{_{\rm FB-4}}$ at (a,b) is

$$\partial\phi^p_{_{\rm FB-4}}(a,b) = \begin{cases} \{(\rho,0)|\rho\in[-1,\sqrt{\alpha}b]\} & \text{if } \ a=0, \ b>0, \\ \{(0,\rho)|\rho\in[-1,\sqrt{\alpha}a]\} & \text{if } \ a>0, \ b=0, \\ \Xi_4 & \text{if } \ a=0, \ b=0, \end{cases}$$

where $\Xi_4 = conv\{\{(0,0)\}\bigcup\Xi_{42}\}$, here

$$\Xi_{42} = \{(\xi-1,\zeta-1)|\xi\le 0 \ or \ \zeta\le 0, \ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \le 1\}.$$

(b) *The directional derivative of $\phi^p_{_{\rm FB-4}}$ at (a,b) in the direction $v=(v_1,v_2)^T$ is*

$$\phi^p_{_{\rm FB-4}}{}'((a,b);v) = \begin{cases} \langle\nabla\phi^p_{_{\rm FB-4}}(a,b), v\rangle & \text{if } \ \phi^p_{_{\rm FB-4}}(a,b)\neq 0, \\ -v_1 & \text{if } \ a=0, \ b>0, \ v_1<0, \\ \sqrt{\alpha}bv_1 & \text{if } \ a=0, \ b>0, \ v_1>0, \\ -v_2 & \text{if } \ a>0, \ b=0, \ v_2<0, \\ \sqrt{\alpha}av_2 & \text{if } \ a>0, \ b=0, \ v_2>0, \\ \left[\phi^p_{_{\rm FB}}(v_1,v_2)\right]_+ & \text{if } \ a=0, \ b=0. \end{cases}$$

Proposition A.5. *Let $\phi^p_{_{\rm FB-5}}$ be defined as in (31) with $p > 1$ being an arbitrary fixed real number. Then, the following hold.*

(a) *If $\phi^p_{_{\rm FB-5}}(a,b) \neq 0$, then $\phi^p_{_{\rm FB-5}}$ is continuously differentiable at (a,b) and*

$$\nabla\phi^p_{_{\rm FB-5}}(a,b) = \begin{cases} -\nabla\phi^p_{_{\rm FB}}(a,b) & \text{if } \ a>0, \ b>0, \\ (0,-1)^T & \text{if } \ a\ge 0, \ b<0, \\ (-1,0)^T & \text{if } \ a<0, \ b\ge 0, \\ \left(\frac{-a^{p-1}}{\|(a,b)\|_p^{p-1}}, \frac{-b^{p-1}}{\|(a,b)\|_p^{p-1}}\right) & \text{if } \ a<0, \ b<0. \end{cases}$$

If $\phi^p_{_{\rm FB-5}}(a,b)=0$*, then the generalized gradient of* $\phi^p_{_{\rm FB-5}}$ *at* (a,b) *is*

$$\partial\phi^p_{_{\rm FB-5}}(a,b)=\begin{cases}\{(\rho,0)|\rho\in[-1,1]\} & \text{if} \quad a=0,\ b>0,\\ \{(0,\rho)|\rho\in[-1,1]\} & \text{if} \quad a>0,\ b=0,\\ \Xi_5 & \text{if} \quad a=0,\ b=0,\end{cases}$$

where $\Xi_5\subseteq conv\{\Xi_{51}\bigcup\Xi_{52}\}$*, here*

$$\begin{cases}\Xi_{51}=\{(1-\xi,1-\zeta)|\xi\geq 0 \ and \ \zeta\geq 0,\ |\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1\},\\ \Xi_{52}=\{(\xi,\zeta)|\xi\leq 0 \ or \ \zeta\leq 0,\ |\xi|^{\frac{p}{p-1}}+|\zeta|^{\frac{p}{p-1}}\leq 1\}.\end{cases}$$

(b) *The directional derivative of* $\phi^p_{_{\rm FB-5}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{_{\rm FB-5}}{}'((a,b);v)=\begin{cases}\langle\nabla\phi^p_{_{\rm FB-5}}(a,b),v\rangle & \text{if} \quad \phi^p_{_{\rm FB-5}}(a,b)\neq 0,\\ |v_1| & \text{if} \quad a=0,\ b>0,\\ |v_2| & \text{if} \quad a>0,\ b=0,\\ \phi^p_{_{\rm FB-5}}(v_1,v_2) & \text{if} \quad a=0,\ b=0.\end{cases}$$

B Some other forms to the $\phi^p_{\text{D-FB}}$ Function

The source of the idea is mainly based on [19].

Let $\phi^p_{\text{D-FB}}$ be defined as in (13). Then, for any $\alpha > 0$, the following variants of $\phi^p_{\text{D-FB}}$ are also NCP function,

$$\phi^p_{\text{D-FB-1}}(a,b) = \sqrt{(\phi^p_{\text{D-FB}}(a,b))^2 + \alpha(a_+ b_+)^2}, \tag{32}$$

$$\phi^p_{\text{D-FB-2}}(a,b) = \sqrt{(\phi^p_{\text{D-FB}}(a,b))^2 + \alpha((ab)_+)^2}, \tag{33}$$

$$\phi^p_{\text{D-FB-3}}(a,b) = \sqrt{(\phi^p_{\text{D-FB}}(a,b))^2 + \alpha((ab)_+)^4}, \tag{34}$$

$$\phi^p_{\text{D-FB-4}}(a,b) = \sqrt{\{[\phi^p_{\text{D-FB}}(a,b)]_+\}^2 + \alpha((ab)_+)^2}, \tag{35}$$

We are now ready to deduce the generalized gradients and the directional derivatives of function $\phi^p_{\text{D-FB-i}}$ for every $i \in \{1,2,3,4\}$. We first rewrite function $\phi^p_{\text{D-FB-i}}$ for every $i \in \{1,2,3,4\}$ as follows:

$$\phi^p_{\text{D-FB-1}}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha(ab)^2} & \text{if } a > 0,\ b > 0, \\ \phi^p_{|\text{D-FB}|}(a,b) = \phi^p_{\text{D-FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{D-FB-2}}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha(ab)^2} & \text{if } ab > 0, \\ \phi^p_{|\text{D-FB}|}(a,b) = \phi^p_{\text{D-FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{D-FB-3}}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha(ab)^4} & \text{if } ab > 0, \\ \phi^p_{|\text{D-FB}|}(a,b) = \phi^p_{\text{D-FB}}(a,b) & \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{D-FB-4}}(a,b) = \begin{cases} \sqrt{\alpha}ab & \text{if } a > 0,\ b > 0, \\ \sqrt{\{[\phi^p_{\text{D-FB}}(a,b)]_+\}^2 + \alpha(ab)^2} & \text{if } a < 0,\ b < 0, \\ \phi^p_{|\text{D-FB}|}(a,b) = \phi^p_{\text{D-FB}}(a,b) & \text{otherwise}, \end{cases}$$

We next list the generalized gradients and directional derivatives that currently compute these NCP functions.

Proposition B.1. *Let $\phi^p_{\text{D-FB-1}}$ be defined as in (32) with $p > 1$ being a positive odd integer. Then, the followings hold.*

(a) *If $\phi^p_{\text{D-FB-1}}(a,b) \neq 0$, then $\phi^p_{\text{D-FB-1}}$ is continuously differentiable at (a,b) and*

$$\nabla\phi^p_{\text{D-FB-1}}(a,b) = \begin{cases} \frac{1}{\phi^p_{\text{D-FB-1}}(a,b)}\left(\phi^p_{\text{D-FB}}(a,b)\cdot\nabla\phi^p_{\text{D-FB}}(a,b) + \alpha\begin{bmatrix} ab^2 \\ a^2b \end{bmatrix}\right) & \text{if } a > 0,\ b > 0, \\ \nabla\phi^p_{\text{D-FB}}(a,b) & \text{otherwise}. \end{cases}$$

If $\phi^p_{\text{D-FB-1}}(a,b)=0$ *with* $(a,b)\neq(0,0)$*, then the generalized gradient of* $\phi^p_{\text{D-FB-1}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{D-FB-1}}(a,b)=\begin{cases}\left\{(\rho,0)|\rho\in[-pb^{p-1},\sqrt{p^2b^{2p-2}+\alpha b^2}\ ]\right\} & \text{if} \quad a=0,\ b>0,\\ \left\{(0,\rho)|\rho\in[-pa^{p-1},\sqrt{p^2a^{2p-2}+\alpha a^2}\ ]\right\} & \text{if} \quad a>0,\ b=0.\end{cases}$$

(b) *The function* $\phi^p_{\text{D-FB-1}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{D-FB-1}}(0,0)=\begin{bmatrix}0\\0\end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{D-FB-1}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{\text{D-FB-1}}{}'((a,b);v)=\begin{cases}\langle\nabla\phi^p_{\text{D-FB-1}}(a,b),v\rangle & \text{if} \quad \phi^p_{\text{D-FB-1}}(a,b)\neq0,\\ -pb^{p-1}v_1 & \text{if} \quad a=0,\ b>0,\ v_1<0,\\ \sqrt{p^2b^{2p-2}+\alpha b^2}v_1 & \text{if} \quad a=0,\ b>0,\ v_1>0,\\ -pa^{p-1}v_2 & \text{if} \quad a>0,\ b=0,\ v_2<0,\\ \sqrt{p^2a^{2p-2}+\alpha a^2}v_2 & \text{if} \quad a>0,\ b=0,\ v_2>0,\\ 0 & \text{if} \quad a=0,\ b=0,\end{cases}$$

Proposition B.2. *Let* $\phi^p_{\text{D-FB-2}}$ *be defined as in (33) with* $p>1$ *being a positive odd integer. Then, the followings hold.*

(a) *If* $\phi^p_{\text{D-FB-2}}(a,b)\neq0$*, then* $\phi^p_{\text{D-FB-2}}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\text{D-FB-2}}(a,b)=\begin{cases}\frac{1}{\phi^p_{\text{D-FB-2}}(a,b)}\left(\phi^p_{\text{D-FB}}(a,b)\cdot\nabla\phi^p_{\text{D-FB}}(a,b)+\alpha\begin{bmatrix}ab^2\\a^2b\end{bmatrix}\right) & \text{if} \quad ab>0,\\ \nabla\phi^p_{\text{D-FB}}(a,b) & \text{otherwise.}\end{cases}$$

If $\phi^p_{\text{D-FB-2}}(a,b)=0$ *with* $(a,b)\neq(0,0)$*, then the generalized gradient of* $\phi^p_{\text{D-FB-2}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{D-FB-2}}(a,b)=\begin{cases}\left\{(\rho,0)|\rho\in[-pb^{p-1},\sqrt{p^2b^{2p-2}+\alpha b^2}]\right\} & \text{if} \quad a=0,\quad b>0,\\ \left\{(0,\rho)|\rho\in[-pa^{p-1},\sqrt{p^2a^{2p-2}+\alpha a^2}]\right\} & \text{if} \quad a>0,\quad b=0.\end{cases}$$

(b) *The function* $\phi^p_{\text{D-FB-2}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{D-FB-2}}(0,0)=\begin{bmatrix}0\\0\end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{D-FB-2}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$
{\phi^p_{\text{D-FB-2}}}'((a,b);v) = \begin{cases} \langle \nabla \phi^p_{\text{D-FB-2}}(a,b), v\rangle & \text{if} \quad \phi^p_{\text{D-FB-2}}(a,b) \neq 0, \\ -pb^{p-1}v_1 & \text{if} \quad a=0,\ b>0,\ v_1<0, \\ \sqrt{p^2b^{2p-2}+\alpha b^2}\,v_1 & \text{if} \quad a=0,\ b>0,\ v_1>0, \\ -pa^{p-1}v_2 & \text{if} \quad a>0,\ b=0,\ v_2<0, \\ \sqrt{p^2a^{2p-2}+\alpha a^2}\,v_2 & \text{if} \quad a>0,\ b=0,\ v_2>0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}
$$

Proposition B.3. *Let* $\phi^p_{\text{D-FB-3}}$ *be defined as in (34) with* $p>1$ *being a positive odd integer. Then, the followings hold.*

(a) *If* $\phi^p_{\text{D-FB-3}}(a,b)\neq 0$*, then* $\phi^p_{\text{D-FB-3}}$ *is continuously differentiable at* (a,b) *and*

$$
\nabla \phi^p_{\text{D-FB-3}}(a,b) = \begin{cases} \frac{1}{\phi^p_{\text{D-FB-3}}(a,b)}\left(\phi^p_{\text{D-FB}}(a,b)\cdot \nabla \phi^p_{\text{D-FB}}(a,b) + 2\alpha \begin{bmatrix} a^3b^4 \\ a^4b^3 \end{bmatrix}\right) & \text{if} \quad ab>0, \\ \nabla \phi^p_{\text{D-FB}}(a,b) & \text{otherwise.} \end{cases}
$$

If $\phi^p_{\text{D-FB-3}}(a,b)=0$*, then the generalized gradient of* $\phi^p_{\text{D-FB-3}}$ *at* (a,b) *is*

$$
\partial \phi^p_{\text{D-FB-3}}(a,b) = \begin{cases} \{(\rho,0)\,|\,\rho\in[-pb^{p-1},pb^{p-1}]\} & \text{if} \quad a=0,\ b>0, \\ \{(0,\rho)\,|\,\rho\in[-pa^{p-1},pa^{p-1}]\} & \text{if} \quad a>0,\ b=0. \end{cases}
$$

(b) *The function* $\phi^p_{\text{D-FB-3}}$ *is differentiable at* $(0,0)$ *with* $\nabla \phi^p_{\text{D-FB-3}}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{D-FB-3}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$
{\phi^p_{\text{D-FB-3}}}'((a,b);v) = \begin{cases} \langle \nabla \phi^p_{\text{D-FB-3}}(a,b), v\rangle & \text{if} \quad \phi^p_{\text{D-FB-3}}(a,b) \neq 0, \\ |v_1|pb^{p-1} & \text{if} \quad a=0,\ b>0, \\ |v_2|pa^{p-1} & \text{if} \quad a>0,\ b=0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}
$$

Proposition B.4. *Let* $\phi^p_{\text{D-FB-4}}$ *be defined as in (35) with* $p>1$ *being a positive odd integer. Then, the followings hold.*

(a) *If* $\phi^p_{\text{D-FB-4}}(a,b) \neq 0$*, then* $\phi^p_{\text{D-FB-4}}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\text{D-FB-4}}(a,b) = \begin{cases} (\sqrt{\alpha}b, \sqrt{\alpha}a) & \text{if} \quad a>0,\ b>0, \\ \frac{1}{\phi^p_{\text{D-FB-4}}(a,b)}\left(\phi^p_{\text{D-FB}}(a,b)\cdot\nabla\phi^p_{\text{D-FB}}(a,b) + \alpha\begin{bmatrix} ab^2 \\ a^2b \end{bmatrix}\right) & \text{if} \quad a<0,\ b<0, \\ \nabla\phi^p_{\text{D-FB}}(a,b) & \quad \text{otherwise.} \end{cases}$$

If $\phi^p_{\text{D-FB-4}}(a,b) = 0$*, then the generalized gradient of* $\phi^p_{\text{D-FB-4}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{D-FB-4}}(a,b) = \begin{cases} \{(\rho,0) | \rho \in [-pb^{p-1}, \sqrt{\alpha}b]\} & \text{if} \quad a=0,\ b>0, \\ \{(0,\rho) | \rho \in [-pa^{p-1}, \sqrt{\alpha}a]\} & \text{if} \quad a>0,\ b=0. \end{cases}$$

(b) *The function* $\phi^p_{\text{D-FB-4}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{D-FB-4}}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{D-FB-4}}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{\text{D-FB-4}}{}'((a,b);v) = \begin{cases} \langle \nabla\phi^p_{\text{D-FB-4}}(a,b), v\rangle & \text{if} \quad \phi^p_{\text{D-FB-4}}(a,b) \neq 0, \\ -pb^{p-1}v_1 & \text{if} \quad a=0,\ b>0,\ v_1<0, \\ \sqrt{\alpha}bv_1 & \text{if} \quad a=0,\ b>0,\ v_1>0, \\ -pa^{p-1}v_2 & \text{if} \quad a>0,\ b=0,\ v_2<0, \\ \sqrt{\alpha}av_2 & \text{if} \quad a>0,\ b=0,\ v_2>0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}$$

C Some other forms to the $\phi^p_{\text{A-NR}}$ Function

The source of the idea is mainly based on [19].

Let $\phi^p_{\text{A-NR}}$ be defined as in (14). Then, for any $\alpha > 0$, the following variants of $\phi^p_{\text{A-NR}}$ are also NCP function,

$$\phi^p_{\text{A-NR-1}}(a,b) = \sqrt{(\phi^p_{\text{A-NR}}(a,b))^2 + \alpha(a_+b_+)^2}, \tag{36}$$

$$\phi^p_{\text{A-NR-2}}(a,b) = \sqrt{(\phi^p_{\text{A-NR}}(a,b))^2 + \alpha((ab)_+)^2}, \tag{37}$$

$$\phi^p_{\text{A-NR-3}}(a,b) = \sqrt{(\phi^p_{\text{A-NR}}(a,b))^2 + \alpha((ab)_+)^4}, \tag{38}$$

$$\phi^p_{\text{A-NR-4}}(a,b) = \sqrt{\{[-\phi^p_{\text{A-NR}}(a,b)]_+\}^2 + \alpha((ab)_+)^2}. \tag{39}$$

We are now ready to deduce the generalized gradients and the directional derivatives of function $\phi^p_{\text{A-NR-i}}$ for every $i \in \{1,2,3,4\}$. We first rewrite function $\phi^p_{\text{A-NR-i}}$ for every $i \in \{1,2,3,4\}$ as follows:

$$\phi^p_{\text{A-NR-1}}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha(ab)^2} & \text{if } \ a>0,\ b>0, \\ \phi^p_{|\text{A-NR}|}(a,b) = -\phi^p_{\text{A-NR}}(a,b) & \ \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{A-NR-2}}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{A-NR}}(a,b)]^2 + \alpha(ab)^2} & \text{if } \ ab>0, \\ \phi^p_{|\text{A-NR}|}(a,b) = -\phi^p_{\text{A-NR}}(a,b) & \ \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{A-NR-3}}(a,b) = \begin{cases} \sqrt{[\phi^p_{\text{A-NR}}(a,b)]^2 + \alpha(ab)^4} & \text{if } \ ab>0, \\ \phi^p_{|\text{A-NR}|}(a,b) = -\phi^p_{\text{A-NR}}(a,b) & \ \text{otherwise}, \end{cases}$$

$$\phi^p_{\text{A-NR-4}}(a,b) = \begin{cases} \sqrt{\alpha}ab & \text{if } \ a>0,\ b>0, \\ \sqrt{\{[\phi^p_{\text{A-NR}}(a,b)]_+\}^2 + \alpha(ab)^2} & \text{if } \ a<0,\ b<0, \\ \phi^p_{|\text{A-NR}|}(a,b) = -\phi^p_{\text{A-NR}}(a,b) & \ \text{otherwise}, \end{cases}$$

We next list the generalized gradients and directional derivatives that currently compute these NCP functions.

Proposition C.1. *Let $\phi^p_{\text{A-NR-1}}$ be defined as in (36) with $p > 1$ being a positive odd integer. Then, the followings hold.*

(a) *If $\phi^p_{\text{A-NR-1}}(a,b) \neq 0$, then $\phi^p_{\text{A-NR-1}}$ is continuously differentiable at (a,b) and*

$$\nabla\phi^p_{\text{A-NR-1}}(a,b) = \begin{cases} \frac{1}{\phi^p_{\text{A-NR-1}}(a,b)}\left(\phi^p_{\text{A-NR}}(a,b)\cdot\nabla\phi^p_{\text{A-NR}}(a,b) + \alpha\begin{bmatrix} ab^2 \\ a^2b \end{bmatrix}\right) & \text{if } \ a>0,\ b>0, \\ -\nabla\phi^p_{\text{A-NR}}(a,b) & \ \text{otherwise}. \end{cases}$$

If $\phi^p_{\text{A-NR-1}}(a,b)=0$ *with* $(a,b)\neq(0,0)$*, then the generalized gradient of* $\phi^p_{\text{A-NR-1}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{A-NR-1}}(a,b)=\begin{cases}\left\{(\rho,0)\,|\,\rho\in[-p(\frac{b}{2})^{p-1},\sqrt{p^2(\frac{b}{2})^{2p-2}+\alpha b^2}]\right\} & \text{if} \quad a=0,\ b>0,\\ \left\{(0,\rho)\,|\,\rho\in[-p(\frac{a}{2})^{p-1},\sqrt{p^2(\frac{a}{2})^{2p-2}+\alpha a^2}]\right\} & \text{if} \quad a>0,\ b=0.\end{cases}$$

(b) *The function* $\phi^p_{\text{A-NR}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{A-NR-1}}(0,0)=\begin{bmatrix}0\\0\end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{A-NR-1}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{\text{A-NR-1}}{}'((a,b);v)=\begin{cases}\langle\nabla\phi^p_{\text{A-NR-1}}(a,b),v\rangle & \text{if} \quad \phi^p_{\text{A-NR-1}}(a,b)\neq 0,\\ -p(\frac{b}{2})^{p-1}v_1 & \text{if} \quad a=0,\ b>0,\ v_1<0,\\ \sqrt{p^2(\frac{b}{2})^{2p-2}+\alpha b^2}\,v_1 & \text{if} \quad a=0,\ b>0,\ v_1>0,\\ -p(\frac{a}{2})^{p-1}v_2 & \text{if} \quad a>0,\ b=0,\ v_2<0,\\ \sqrt{p^2(\frac{a}{2})^{2p-2}+\alpha a^2}\,v_2 & \text{if} \quad a>0,\ b=0,\ v_2>0,\\ 0 & \text{if} \quad a=0,\ b=0.\end{cases}$$

Proposition C.2. *Let* $\phi^p_{\text{A-NR-2}}$ *be defined as in (37) with* $p>1$ *being a positive odd integer. Then, the followings hold.*

(a) *If* $\phi^p_{\text{A-NR-2}}(a,b)\neq 0$*, then* $\phi^p_{\text{A-NR-2}}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\text{A-NR-2}}(a,b)=\begin{cases}\frac{1}{\phi^p_{\text{A-NR-2}}(a,b)}\left(\phi^p_{\text{A-NR}}(a,b)\cdot\nabla\phi^p_{\text{A-NR}}(a,b)+\alpha\begin{bmatrix}ab^2\\a^2b\end{bmatrix}\right) & \text{if} \quad ab>0,\\ -\nabla\phi^p_{\text{A-NR}}(a,b) & \text{otherwise.}\end{cases}$$

If $\phi^p_{\text{A-NR-2}}(a,b)=0$ *with* $(a,b)\neq(0,0)$*, then the generalized gradient of* $\phi^p_{\text{A-NR-2}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{A-NR-2}}(a,b)=\begin{cases}\left\{(\rho,0)\,|\,\rho\in[-p(\frac{b}{2})^{p-1},\sqrt{p^2(\frac{b}{2})^{2p-2}+\alpha b^2}]\right\} & \text{if} \quad a=0,\ b>0,\\ \left\{(0,\rho)\,|\,\rho\in[-p(\frac{a}{2})^{p-1},\sqrt{p^2(\frac{a}{2})^{2p-2}+\alpha a^2}]\right\} & \text{if} \quad a>0,\ b=0.\end{cases}$$

(b) *The function* $\phi^p_{\text{A-NR-2}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{A-NR-2}}(0,0)=\begin{bmatrix}0\\0\end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{A-NR-2}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{\text{A-NR-2}}{}'((a,b);v)=\begin{cases}\langle\nabla\phi^p_{\text{A-NR-2}}(a,b),v\rangle & \text{if} \quad \phi^p_{\text{A-NR-2}}(a,b)\neq 0,\\ -p(\frac{b}{2})^{p-1}v_1 & \text{if} \quad a=0,\ b>0,\ v_1<0,\\ \sqrt{p^2(\frac{b}{2})^{2p-2}+\alpha b^2}v_1 & \text{if} \quad a=0,\ b>0,\ v_1>0,\\ -p(\frac{a}{2})^{p-1}v_2 & \text{if} \quad a>0,\ b=0,\ v_2<0,\\ \sqrt{p^2(\frac{a}{2})^{2p-2}+\alpha a^2}v_2 & \text{if} \quad a>0,\ b=0,\ v_2>0,\\ 0 & \text{if} \quad a=0,\ b=0.\end{cases}$$

Proposition C.3. *Let* $\phi^p_{\text{A-NR-3}}$ *be defined as in (38) with* $p>1$ *being a positive odd integer. Then, the followings hold.*

(a) *If* $\phi^p_{\text{A-NR-3}}(a,b)\neq 0$*, then* $\phi^p_{\text{A-NR-3}}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\text{A-NR-3}}(a,b)=\begin{cases}\frac{1}{\phi^p_{\text{A-NR-3}}(a,b)}\left(\phi^p_{\text{A-NR}}(a,b)\cdot\nabla\phi^p_{\text{A-NR}}(a,b)+2\alpha\begin{bmatrix}a^3b^4\\ a^4b^3\end{bmatrix}\right) & \text{if} \quad ab>0,\\ -\nabla\phi^p_{\text{A-NR}}(a,b) & \text{otherwise.}\end{cases}$$

If $\phi^p_{\text{A-NR-3}}(a,b)=0$*, then the generalized gradient of* $\phi^p_{\text{A-NR-3}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{A-NR-3}}(a,b)=\begin{cases}\{(\rho,0)|\rho\in[-p(\frac{b}{2})^{p-1},p(\frac{b}{2})^{p-1}]\} & \text{if} \quad a=0,\ b>0,\\ \{(0,\rho)|\rho\in[-p(\frac{a}{2})^{p-1},p(\frac{a}{2})^{p-1}]\} & \text{if} \quad a>0,\ b=0.\end{cases}$$

(b) *The function* $\phi^p_{\text{A-NR-3}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{A-NR-3}}(0,0)=\begin{bmatrix}0\\0\end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{A-NR-3}}$ *at* (a,b) *in the direction* $v=(v_1,v_2)^T$ *is*

$$\phi^p_{\text{A-NR-3}}{}'((a,b);v)=\begin{cases}\langle\nabla\phi^p_{\text{A-NR-3}}(a,b),v\rangle & \text{if} \quad \phi^p_{\text{A-NR-3}}(a,b)\neq 0,\\ |v_1|p(\frac{b}{2})^{p-1} & \text{if} \quad a=0,\ b>0,\\ |v_2|p(\frac{a}{2})^{p-1} & \text{if} \quad a>0,\ b=0,\\ 0 & \text{if} \quad a=0,\ b=0.\end{cases}$$

Proposition C.4. *Let* $\phi^p_{\text{A-NR-4}}$ *be defined as in (39) with* $p>1$ *being a positive odd integer. Then, the followings hold.*

(a) *If* $\phi^p_{\text{A-NR-4}}(a,b) \neq 0$, *then* $\phi^p_{\text{A-NR-4}}$ *is continuously differentiable at* (a,b) *and*

$$\nabla\phi^p_{\text{A-NR-4}}(a,b) = \begin{cases} (\sqrt{\alpha}b, \sqrt{\alpha}a) & \text{if} \quad a>0,\ b>0, \\ \frac{1}{\phi^p_{\text{A-NR-4}}(a,b)}\left(\phi^p_{\text{A-NR}}(a,b)\cdot\nabla\phi^p_{\text{A-NR}}(a,b) + \alpha\begin{bmatrix} ab^2 \\ a^2b \end{bmatrix}\right) & \text{if} \quad a<0,\ b<0, \\ -\nabla\phi^p_{\text{A-NR}}(a,b) & \quad \text{otherwise.} \end{cases}$$

If $\phi^p_{\text{A-NR-4}}(a,b) = 0$, *then the generalized gradient of* $\phi^p_{\text{A-NR-4}}$ *at* (a,b) *is*

$$\partial\phi^p_{\text{A-NR-4}}(a,b) = \begin{cases} \{(\rho,0) | \rho \in [-p(\frac{b}{2})^{p-1}, \sqrt{\alpha}b]\} & \text{if} \quad a=0,\ b>0, \\ \{(0,\rho) | \rho \in [-p(\frac{a}{2})^{p-1}, \sqrt{\alpha}a]\} & \text{if} \quad a>0,\ b=0. \end{cases}$$

(b) *The function* $\phi^p_{\text{A-NR-4}}$ *is differentiable at* $(0,0)$ *with* $\nabla\phi^p_{\text{A-NR-4}}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) *The directional derivative of* $\phi^p_{\text{A-NR-4}}$ *at* (a,b) *in the direction* $v = (v_1, v_2)^T$ *is*

$$\phi^p_{\text{A-NR-4}}{}'((a,b);v) = \begin{cases} \langle\nabla\phi^p_{\text{A-NR-4}}(a,b), v\rangle & \text{if} \quad \phi^p_{\text{A-NR-4}}(a,b) \neq 0, \\ -p(\frac{b}{2})^{p-1}v_1 & \text{if} \quad a=0,\ b>0,\ v_1<0, \\ \sqrt{\alpha}bv_1 & \text{if} \quad a=0,\ b>0,\ v_1>0, \\ -p(\frac{a}{2})^{p-1}v_2 & \text{if} \quad a>0,\ b=0,\ v_2<0, \\ \sqrt{\alpha}av_2 & \text{if} \quad a>0,\ b=0,\ v_2>0, \\ 0 & \text{if} \quad a=0,\ b=0. \end{cases}$$

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