2. Preliminary Results

First we introduce the definition of elliptical distribution, which we will use to develop our methods in the sequent.

Definition 1. *An $m \times 1$ random vector X is said to have an elliptical distribution with parameter $\mu\,(m \times 1)$ and $V\,(m \times m)$ if its density function has the form:*

$$c_m\,(\det V)^{1/2}\, h\left[(x-\mu)'\,V^{-1}\,(x-\mu)\right]$$

for some function h, where c_m is a normalizing constant, and V is positive definite.

To achieve our goal, first we need to study the asymptotic distribution of r. The following Theorems 2 and 3 are the well-known results for this, which can be found, for example, in Muirhead (1982), section 5.1.

Theorem 2. *Let r be the correlation coefficient formed from a sample of size n from a bivariate elliptical distribution with correlation coefficient ρ and kurtosis parameter κ. Then the asymptotic distribution, as $n \to \infty$, of $\sqrt{n-1}\frac{r-\rho}{1-\rho^2}$ is $N(0, 1+\kappa)$.*

Theorem 2 states that r is asymptotically normal as $n \to \infty$. A more accurate result can be obtained by considering $z = \tanh^{-1} r = \frac{1}{2}\log\frac{1+r}{1-r}$. Fisher (1921) showed that the distribution of z converges to the normal distribution much faster than r itself.

Theorem 3. *Let r be the correlation coefficient formed from a sample of size n, $n > 1$, from a bivariate elliptical distribution with correlation coefficient ρ and kurtosis parameter κ. Define $z = \tanh^{-1} r = \frac{1}{2}\log\frac{1+r}{1-r}$ and $\xi = \tanh^{-1}\rho = \frac{1}{2}\log\frac{1+\rho}{1-\rho}$. Then, as $n \to \infty$, the asymptotic distribution of $\sqrt{n-1}\,(z-\xi)$ is $N\,(0, 1+\kappa)$.*

Proof. We use the fact that if $\{X_n\}$ is a sequence of random variables such that $\sqrt{n-1}\,(X_n-\mu) \to N\,(0, \sigma^2)$ in distribution as $n \to \infty$, and if $f\,(x)$ is a function which is differentiable at $x = \mu$, then $\sqrt{n-1}\,\{f\,(X_n) - f\,(\mu)\} \to N\left[0, f'\,(\mu)^2\,\sigma^2\right]$ in distribution as $n \to \infty$; see, e.g., Bichel and Doksum (1977), p.461. Let $f(x) = \frac{1}{2}\log\frac{1+x}{1-x}$. Note that $f'\,(x)^2\,(1-x^2)^2 = 1$. According to Theorem 2,

$$\sqrt{n-1}\,(r-\rho) \to N\left(0, (1+\kappa)\left(1-\rho^2\right)^2\right) \quad \text{as } n \to \infty,$$

and hence

$$\sqrt{n-1}\,\{f(r)-f(\rho)\} \to N\left(0, f'(\rho)^2 (1+\kappa)\left(1-\rho^2\right)^2\right) \quad \text{as } n \to \infty,$$

namely

$$\sqrt{n-1}\,(z-\xi) \to N(0,(1+\kappa)) \qquad \text{as } n \to \infty$$

$\square$