第四章　在 de Sitter 背景中的 Axial 微擾與純量粒子

de Sitter 空間滿足的 Einstein 方程式為

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}$$

de Sitter 度規

$$ds^2 = -\left(1 - r^2/r_0^{\ 2}\right) dt^2 + \left(1 - r^2/r_0^{\ 2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$\left(g_{\mu\nu}\right) = \begin{pmatrix} -\left(1 - r^2/r_0^{\ 2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - r^2/r_0^{\ 2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad \mu, \nu = 0, 1, 2, 3$$

$$\left(g^{\mu\nu}\right) = \begin{pmatrix} -\left(1 - r^2/r_0^{\ 2}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1 - r^2/r_0^{\ 2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}$$

其中$\Lambda = \frac{3}{r_0^{\ 2}}$ 。

4.1　在 de Sitter 背景中的 Axial 微擾

將總度規$\bar{g}_{\mu\nu}$分為背景度規$g_{\mu\nu}$與微擾度規$h_{\mu\nu}$

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

考慮 Regge-Wheeler axial 微擾度規

$$\left(h_{\mu\nu}\right) = \begin{pmatrix} 0 & 0 & -h_0(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi} & h_0(t,r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ 0 & 0 & -h_1(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi} & h_1(t,r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}$$

$$\overline{R}_{\mu}{}^{\nu} = \overline{g}^{\alpha\nu}\overline{R}_{\mu\alpha} = g^{\alpha\nu}R_{\mu\alpha} + g^{\alpha\nu}\delta R_{\mu\alpha} - h^{\alpha\nu}R_{\mu\alpha} + O(h^2) = R_{\mu}{}^{\nu} + \delta R_{\mu}{}^{\nu}$$
$$\Rightarrow \delta R_{\mu}{}^{\nu} = g^{\alpha\nu}\delta R_{\mu\alpha} - h^{\alpha\nu}R_{\mu\alpha} = g^{\alpha\nu}\delta R_{\mu\alpha} - g^{\alpha\kappa}g^{\lambda\nu}h_{\kappa\lambda}R_{\mu\alpha} \tag{4.1}$$

$$G_{\mu}{}^{\nu} = g^{\alpha\nu}G_{\mu\alpha} = R_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R = -\Lambda\delta_{\mu}^{\nu} \tag{4.2}$$

$$\overline{G}_{\mu}{}^{\nu} = \overline{g}^{\alpha\nu}\overline{G}_{\mu\alpha} = \overline{R}_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}\overline{R} = -\Lambda\delta_{\mu}^{\nu} \tag{4.3}$$

$(4.3)-(4.2)$ ，可得

$$\delta R_{\mu}{}^{\nu} = \frac{1}{2}\delta_{\mu}^{\nu}(\delta R)$$

$$\delta R_{\mu}{}^{\mu} = \delta R = \frac{1}{2}\delta_{\mu}^{\mu}(\delta R) = 2\delta R$$
$$\Rightarrow \delta R = 0$$

故在 de Sitter 背景中，微擾方程式為

$$\delta R_{\mu}{}^{\nu} = 0 \tag{4.4}$$

由(4.1)式及(4.4)式可知

$$\delta R_{1}{}^{2} = 0$$
$$\Rightarrow \frac{(-1+l)(2+l)}{r}h_1 + \frac{r_0{}^2}{(r-r_0)(r+r_0)}\left[-2\left(\frac{\partial h_0}{\partial t}\right) + r\left(\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0\right) - r\left(\frac{\partial^2 h_1}{\partial t^2}\right)\right] = 0 \tag{4.5}$$

$$\delta R_{2}{}^{3} = 0$$
$$\Rightarrow \frac{2r}{r_0{}^2}h_1 + \frac{(r-r_0)(r+r_0)}{r_0{}^2}\left(\frac{\partial h_1}{\partial r}\right) - \frac{r_0{}^2}{(r-r_0)(r+r_0)}\left(\frac{\partial h_0}{\partial t}\right) = 0 \tag{4.6}$$

由(4.6)式可得

$$\frac{\partial h_0}{\partial t} = \frac{(r-r_0)(r+r_0)}{r_0{}^4}\left[2rh_1 + r^2\left(\frac{\partial h_1}{\partial r}\right) - r_0{}^2\left(\frac{\partial h_1}{\partial r}\right)\right] \tag{4.7}$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0 = \frac{\left(6r^2 - 2r_0{}^2\right)h_1 + \left(r^2 - r_0{}^2\right)\left[6r\left(\frac{\partial h_1}{\partial r}\right) + \left(r^2 - r_0{}^2\right)\left(\frac{\partial^2 h_1}{\partial r^2}\right)\right]}{r_0{}^4} \tag{4.8}$$

將(4.7)式及(4.8)式代入(4.5)式，可得

$$\left[\begin{aligned}&\frac{-2r^4 - r^2r_0{}^2l(l+1) + r_0{}^4[-2+l(l+1)]}{r^2r_0{}^4}h_1\\&+\frac{2\left(-2r^4 + r^2r_0{}^2 + r_0{}^4\right)}{rr_0{}^4}\left(\frac{\partial h_1}{\partial r}\right) - \frac{\left(r^2 - r_0{}^2\right)^2}{r_0{}^4}\left(\frac{\partial^2 h_1}{\partial r^2}\right) + \frac{\partial^2 h_1}{\partial t^2}\end{aligned}\right] = 0 \tag{4.9}$$

令 $h_1(t,r)=\frac{r}{\left(1-r^2/r_0^{\;2}\right)}Q_l(t,r)$ ，代入(4.9)式

$$-\frac{\left(r^2-r_0^{\;2}\right)l(l+1)}{r^2r_0^{\;2}}Q_l-\frac{2r\left(r^2-r_0^{\;2}\right)}{r_0^{\;4}}\left(\frac{\partial Q_l}{\partial r}\right)-\frac{\left(r^2-r_0^{\;2}\right)^2}{r_0^{\;4}}\left(\frac{\partial^2 Q_l}{\partial r^2}\right)+\frac{\partial^2 Q_l}{\partial t^2}=0 \qquad (4.10)$$

考慮 $r<r_0$ ，令

$$r=r_0\tanh(x/r_0)$$
$$\Rightarrow x=r_0\tanh^{-1}(r/r_0)$$

(4.10)式變成

$$\frac{l(l+1)\operatorname{csch}^2(x/r_0)}{r_0^{\;2}}Q_l(t,x)-\frac{\partial^2 Q_l}{\partial x^2}+\frac{\partial^2 Q_l}{\partial t^2}=0 \qquad (4.11)$$

令 $Q_l(t,x)=\exp(-i\omega t)\phi(x)$ ，代入(4.11)式

$$\frac{d^2\phi}{dx^2}+\left(\omega^2-\frac{l(l+1)\operatorname{csch}^2(x/r_0)}{r_0^{\;2}}\right)\phi(x)=0 \qquad (4.12)$$

其中位能

$$V(x)=\frac{l(l+1)\operatorname{csch}^2(x/r_0)}{r_0^{\;2}}$$

當 $l=1$ 時，(4.12)式變成

$$\frac{d^2\phi}{dx^2}+\left(\omega^2-\frac{2}{r_0^{\;2}}\operatorname{csch}^2(x/r_0)\right)\phi(x)=0$$

可解得

$$\phi(x)=C_1\left(i\omega-\frac{1}{r_0}\coth(x/r_0)\right)e^{i\omega x}+C_2\left(i\omega+\frac{1}{r_0}\coth(x/r_0)\right)e^{-i\omega x}$$

4.2 在 de Sitter 背景中的純量粒子

純量粒子在 de Sitter 背景中遵循的方程式為

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\partial^\mu\Phi\right)-m_0^{\;2}\Phi=0 \qquad (4.13)$$

其中 $g=\det\left(g_{\mu\nu}\right)=-r^4\sin^2\theta$ ，令波函數 $\Phi=\exp(-i\omega t)f(r)Y_{lm}(\theta,\varphi)$ ，代入(4.13)式，整理可得

$$\left(-m_0{}^2-\frac{l(l+1)}{r^2}-\frac{r_0{}^2\omega^2}{r^2-r_0{}^2}\right)f(r)+\left(\frac{2}{r}-\frac{4r}{r_0{}^2}\right)\left(\frac{df}{dr}\right)+\left(1-\frac{r^2}{r_0{}^2}\right)\left(\frac{d^2f}{dr^2}\right)=0 \qquad (4.14)$$

考慮 $r<r_0$，令

$$r=r_0\tanh(x/r_0)$$
$$\Rightarrow x=r_0\tanh^{-1}(r/r_0)$$

$$f=\frac{\phi}{r}$$

(4.14)式變成

$$\frac{d^2\phi}{dx^2}+\left[\omega^2-\mathrm{sech}^2(x/r_0)\left(\frac{l(l+1)}{r_0{}^2\tanh^2(x/r_0)}-\frac{2}{r_0{}^2}+m_0{}^2\right)\right]\phi(x)=0 \qquad (4.15)$$

其中位能

$$V(x)=\mathrm{sech}^2(x/r_0)\left(\frac{l(l+1)}{r_0{}^2\tanh^2(x/r_0)}-\frac{2}{r_0{}^2}+m_0{}^2\right)$$

當 $m_0=0$ ， $l=0$ 時，(4.15)式變成

$$\frac{d^2\phi}{dx^2}+\left(\omega^2+\frac{2\,\mathrm{sech}^2(x/r_0)}{r_0{}^2}\right)\phi(x)=0$$

可解得

$$\phi(x)=C_1\left(i\omega-\frac{1}{r_0}\tanh(x/r_0)\right)e^{i\omega x}+C_2\left(i\omega+\frac{1}{r_0}\tanh(x/r_0)\right)e^{-i\omega x}$$

當 $m_0=0$ ， $l=1$ 時，(4.15)式變成

$$\frac{d^2\phi}{dx^2}+\left(\omega^2-\frac{8\,\mathrm{csch}^2(2x/r_0)}{r_0{}^2}\right)\phi(x)=0$$

可解得

$$\phi(x)=C_1\left(i\omega-\frac{2}{r_0}\coth(2x/r_0)\right)e^{i\omega x}+C_2\left(i\omega+\frac{2}{r_0}\coth(2x/r_0)\right)e^{-i\omega x}$$

4.3 de Sitter 度規的另一形式

$$
\begin{aligned}
ds^2 &= -\left(1-r^2/r_0{}^2\right)dt^2+\left(1-r^2/r_0{}^2\right)^{-1}dr^2+r^2d\Omega^2 \\
&= -dT^2+\exp\left(2T/r_0\right)dR^2+R^2\exp\left(2T/r_0\right)d\Omega^2
\end{aligned}
\qquad (4.16)
$$

(4.16)式比較之後可得

$$r = R\exp\left(T/r_0\right) \qquad (4.17)$$

由(4.17)式可得

$$R = r\exp\left(-T/r_0\right)$$

$$dR = \exp\left(-T/r_0\right)dr - \frac{r}{r_0}\exp\left(-T/r_0\right)dT$$

$$
\begin{aligned}
-dT^2+\exp\left(2T/r_0\right)dR^2 &= -dT^2+dr^2-2\frac{r}{r_0}dTdr+\frac{r^2}{r_0{}^2}dT^2 \\
&= -\left(1-r^2/r_0{}^2\right)dT^2+dr^2-2\frac{r}{r_0}dTdr
\end{aligned}
$$

由(4.16)式可知

$$-\left(1-r^2/r_0{}^2\right)dt^2+\left(1-r^2/r_0{}^2\right)^{-1}dr^2 = -\left(1-r^2/r_0{}^2\right)dT^2+dr^2-2\frac{r}{r_0}dTdr$$

$$dt^2 = dT^2+\frac{2r/r_0}{\left(1-r^2/r_0{}^2\right)}dTdr+\frac{r^2/r_0{}^2}{\left(1-r^2/r_0{}^2\right)^2}dr^2 = \left[dT+\frac{r/r_0}{\left(1-r^2/r_0{}^2\right)}dr\right]^2$$

$$\Rightarrow dt = dT+\frac{r/r_0}{\left(1-r^2/r_0{}^2\right)}dr$$

積分之後可得 (R,T) 與 (r,t) 間的座標變換

$$
\begin{cases}
T = t+\dfrac{r_0}{2}\ln\left(1-r^2/r_0{}^2\right) & (4.18) \\
R = r\exp\left(-T/r_0\right) = \dfrac{r}{\sqrt{1-r^2/r_0{}^2}}\exp\left(-t/r_0\right) & (4.19)
\end{cases}
$$

$$
\begin{cases}
t = T-\dfrac{r_0}{2}\ln\left(1-r^2/r_0{}^2\right) = T-\dfrac{r_0}{2}\ln\left[1-\dfrac{R^2}{r_0{}^2}\exp\left(2T/r_0\right)\right] \\
r = R\exp\left(T/r_0\right)
\end{cases}
\qquad (4.20)
$$

令

$$Q_l(t,r) = Q_l(T,R)$$

$$\frac{\partial Q_l}{\partial r} = \frac{r_0{}^2\exp\left(-T/r_0\right)}{r_0{}^2-R^2\exp\left(2T/r_0\right)}\left(\frac{\partial Q_l}{\partial R}\right)-\frac{Rr_0\exp\left(T/r_0\right)}{r_0{}^2-R^2\exp\left(2T/r_0\right)}\left(\frac{\partial Q_l}{\partial T}\right)$$

$$
\frac{\partial^2 Q_l}{\partial r^2}=\left[\begin{array}{l}
\frac{3Rr_0^{\ 2}}{\left[r_0^{\ 2}-R^2\exp(2T/r_0)\right]^2}\left(\frac{\partial Q_l}{\partial R}\right)+\frac{r_0^{\ 4}\exp(-2T/r_0)}{\left[r_0^{\ 2}-R^2\exp(2T/r_0)\right]^2}\left(\frac{\partial^2 Q_l}{\partial R^2}\right)\\
-\frac{2Rr_0^{\ 3}}{\left[r_0^{\ 2}-R^2\exp(2T/r_0)\right]^2}\left(\frac{\partial}{\partial T}\frac{\partial}{\partial R}Q_l\right)-\frac{r_0\left[r_0^{\ 2}+R^2\exp(2T/r_0)\right]}{\left[r_0^{\ 2}-R^2\exp(2T/r_0)\right]^2}\left(\frac{\partial Q_l}{\partial T}\right)\\
+\frac{R^2 r_0^{\ 2}\exp(2T/r_0)}{\left[r_0^{\ 2}-R^2\exp(2T/r_0)\right]^2}\left(\frac{\partial^2 Q_l}{\partial T^2}\right)
\end{array}\right]
$$

$$
\frac{\partial^2 Q_l}{\partial t^2}=\frac{R}{r_0^{\ 2}}\left(\frac{\partial Q_l}{\partial R}\right)+\frac{R^2}{r_0^{\ 2}}\left(\frac{\partial^2 Q_l}{\partial R^2}\right)-\frac{2R}{r_0}\left(\frac{\partial}{\partial T}\frac{\partial}{\partial R}Q_l\right)+\frac{\partial^2 Q_l}{\partial T^2}
$$

代入(4.10)式，可得

$$
r_0\frac{l(l+1)}{R^2}Q_l(T,R)-r_0\left(\frac{\partial^2 Q_l}{\partial R^2}\right)+\exp(2T/r_0)\left(\frac{\partial Q_l}{\partial T}\right)+r_0\exp(2T/r_0)\left(\frac{\partial^2 Q_l}{\partial T^2}\right)=0 \quad (4.21)
$$

令 $Q_l(T,R)=R^c F(T,R)$ ，其中 c 為某一常數

$$
\frac{\partial^2 Q_l}{\partial R^2}=c(c-1)R^{c-2}F(T,R)+2cR^{c-1}\frac{\partial F}{\partial R}+R^c\frac{\partial^2 F}{\partial R^2}
$$

$$
\frac{\partial Q_l}{\partial T}=R^c\frac{\partial F}{\partial T}
$$

$$
\frac{\partial^2 Q_l}{\partial T^2}=R^c\frac{\partial^2 F}{\partial T^2}
$$

代入(4.21)式，整理可得

$$
\left[\begin{array}{l}
\left[\frac{c(c-1)}{R^2}F(T,R)+\frac{2c}{R}\left(\frac{\partial F}{\partial R}\right)+\frac{\partial^2 F}{\partial R^2}-\frac{l(l+1)}{R^2}F(T,R)\right]\\
-\frac{\exp(2T/r_0)}{r_0}\left(\frac{\partial F}{\partial T}\right)-\exp(2T/r_0)\left(\frac{\partial^2 F}{\partial T^2}\right)
\end{array}\right]=0 \quad (4.22)
$$

我們希望將上式的微分項化成 Laplacian 的形式，故取 $c=1$，$Q_l(T,R)=RF(T,R)$，(4.22)式變成

$$
\left[\begin{array}{l}
\left[\frac{2}{R}\left(\frac{\partial F}{\partial R}\right)+\frac{\partial^2 F}{\partial R^2}-\frac{l(l+1)}{R^2}F(T,R)\right]\\
-\frac{\exp(2T/r_0)}{r_0}\left(\frac{\partial F}{\partial T}\right)-\exp(2T/r_0)\left(\frac{\partial^2 F}{\partial T^2}\right)
\end{array}\right]=0 \quad (4.23)
$$

$\Phi(T,R,\theta,\varphi)=F(T,R)Y_{lm}(\theta,\varphi)$ ，(4.23)式可再變成

$$
\nabla^2\Phi-\frac{\exp(2T/r_0)}{r_0}\left(\frac{\partial \Phi}{\partial T}\right)-\exp(2T/r_0)\left(\frac{\partial^2 \Phi}{\partial T^2}\right)=0 \quad (4.24)
$$

現在用直角座標來看

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}$$

利用分離變數法，令 $\Phi = \Phi(T, X, Y, Z) = f_1(T) f_2(X) f_3(Y) f_4(Z)$ ，代入(4.24)式

$$\frac{1}{f_2}\frac{d^2 f_2}{dX^2} + \frac{1}{f_3}\frac{d^2 f_3}{dY^2} + \frac{1}{f_4}\frac{d^2 f_4}{dZ^2} - \frac{\exp(2T/r_0)}{r_0}\frac{1}{f_1}\left(\frac{df_1}{dT}\right) - \exp(2T/r_0)\frac{1}{f_1}\left(\frac{d^2 f_1}{dT^2}\right) = 0 \quad (4.25)$$

令 $\frac{1}{f_2}\frac{d^2 f_2}{dX^2} = -k_X{}^2$ ，其中 k_X 為一常數，可得

$$f_2(X) = \exp(ik_X X)$$

同理

$$f_3(Y) = \exp(ik_Y Y)$$
$$f_4(Z) = \exp(ik_Z Z)$$

其中 k_Y 與 k_Z 均為常數，可得

$$f_2(X) f_3(Y) f_4(Z) = \exp[i(k_X X + k_Y Y + k_Z Z)] \equiv \exp(i\vec{k} \cdot \vec{R})$$

令 $k^2 = k_X{}^2 + k_Y{}^2 + k_Z{}^2$ ，(4.25)式變成

$$k^2 f_1(T) + \frac{\exp(2T/r_0)}{r_0}\left(\frac{df_1}{dT}\right) + \exp(2T/r_0)\left(\frac{d^2 f_1}{dT^2}\right) = 0 \quad (4.26)$$

從(4.26)式解得

$$f_1(T) = C_1 \sin[kr_0 \exp(-T/r_0)] + C_2 \cos[kr_0 \exp(-T/r_0)]$$

故得

$$\Phi(T, X, Y, Z) = C \exp[ikr_0 \exp(-T/r_0) + i\vec{k} \cdot \vec{R}]$$