THE GENERALIZED PRINCIPAL IDEAL THEOREM IN IMAGINARY QUADRATIC FIELDS WITH CLASS NUMBER 1

推廣的主理想數定理在類數為1之二次虛體中之函數表法

Hsü Tao-ning

徐 道 寧

In a paper of T. Tannaka (Tannaka [1]), he proved the generalized ideal theorem due to him:

Th. 2. Let K be the ray class field mod. $\mathfrak{f}$ of the algebraic number field k and denote by $\mathfrak{f}'$ (K/k) the module of genus (Geschlechtermodul) of K/k. For every ideal $\mathfrak{m}$ prime to $\mathfrak{f}$ there exists an element $\theta(\mathfrak{m}) \in$ K with the following properties:

1. $\theta(\mathfrak{m})$ gives a representation of $\mathfrak{m}$ as a principal ideal mod. $\mathfrak{f}'$ of K:

$$\mathfrak{m} = (\theta(\mathfrak{m})) \text{ in K}, \ \theta(\mathfrak{m}) \equiv 1 \bmod. \mathfrak{f}'(\text{K/k}).$$

2. Let $\sigma(\mathfrak{m}) = \left(\frac{\text{K/k}}{\mathfrak{m}}\right)$ be the Artin automorphism and define the factor set $\varepsilon(\mathfrak{m}, \mathfrak{b})$ by

$$\varepsilon(\mathfrak{m}, \mathfrak{b}) = \frac{\theta(\mathfrak{m})\, \theta(\mathfrak{b})^{\sigma(\mathfrak{m})}}{\theta(\mathfrak{m}\mathfrak{b})}$$

Then $\varepsilon(\mathfrak{m}, \mathfrak{b})$ (which is obviously a unit $\equiv 1$ mod. $\mathfrak{f}'$ (K/k) of K) belongs to the ray mod. $\mathfrak{f}$ of the ground field k and is symmetric in $\mathfrak{m}$, $\mathfrak{b}$:

$$\varepsilon(\mathfrak{m}, \mathfrak{b}) \in \text{K}, \ \varepsilon(\mathfrak{b}, \mathfrak{m}) = \varepsilon(\mathfrak{m}, \mathfrak{b}), \ \varepsilon(\mathfrak{m}, \mathfrak{b}) \equiv 1 \bmod. \mathfrak{f}$$

In case of an imaginary quadratic ground field, H. Kempfert constructed the explicit expression for $\theta(\mathfrak{m})$ with the property 1 in ray class field (Kempfert [2])and T.N. Hsü constructed the explicit expression with the property 2 in absolute class field (Hsü [3]). Here in this paper a method is given to construct the function $\theta(\mathfrak{m})$ with both the properties 1 and 2 in case that the absolute class number is one, using the result of Kempfert.

Let $\Sigma = R(\sqrt{m})$ be the imaginary quadratic ground field and $K = K_{\mathfrak{f}}$ the ray class field mod. $\mathfrak{f}$. K is in the given case generated by the ray class invariant $\tau(\mathfrak{k})$, where $\mathfrak{k}$ is a ray class. The Artin Automorphism $\sigma(\mathfrak{m}) = \sigma(\mathfrak{k})$ is defined by

$$\sigma(\mathfrak{m}): \quad \tau(\mathfrak{k}) \longrightarrow \tau(\mathfrak{k}\ \mathfrak{k}_{\circ}),$$

if $\sigma(\mathfrak{m})$ denotes the Artin Automorphism corresponds the ray class $\mathfrak{k}$ of $\mathfrak{m}$ and $\mathfrak{k}_{\circ}$ any ray class. Let $\sigma(\mathfrak{k}_1), \sigma(\mathfrak{k}_2), \ldots, \sigma(\mathfrak{k}_n)$ be a basis of the Galois group of K/Σ, with the orders $f_1, f_2, \ldots, f_n$ respectively. The subgroup generated by $\sigma(\mathfrak{k}_1), \ldots, \sigma(\mathfrak{k}_{i-1}), \sigma(\mathfrak{k}_{i+1}), \ldots, \sigma(\mathfrak{k}_n)$ leaves a subfield K_i fixed. The Galois group $\mathfrak{P}_i$ of K_i with respect to Σ is isomorphic to the cyclic group generated by $\sigma(\mathfrak{k}_i)$. According to the class field theory of Weber-Takagi (Hasse [4]), K_i is a class field of Σ to an ideal group H_i, whose conductor (Führer) $\mathfrak{f}_i$ is a factor of $\mathfrak{f}$, the conductor of (K/Σ).

K is the ray class field mod. $\mathfrak{f}$ over Σ, therefore the ray $\gamma = \{(\alpha) \mid \alpha \equiv 1 \text{ Mod. } \mathfrak{f}\}$ is the ideal group, to which K is the class field. The ideal group H_i can also be declared mod. $\mathfrak{f}$, and $\gamma \subset H_i$ must be true. Let A be the ideal group of all the ideals of Σ relative prime to $\mathfrak{f}$, the factor group A/H_i is then isomorphic to the group $\mathfrak{G}_i$, $i = 1, 2, \ldots, n$. Take a first ordered prime ideal $\mathfrak{P}_i$ from $\mathfrak{k}_i$ with $\mathfrak{P}_i$ relative prime to $2d\mathfrak{f}$ where d denotes the field discriminant of Σ, then the classes of $\mathfrak{P}_j^{r_j}$, $j \neq i$, $0 \leqq r_j \leqq f_j - 1$, lie in H_i, and $H_i = \bigcup_{i \neq j} (\mathfrak{P}_j^{r_j} \gamma)$. Factorize the ideal $\mathfrak{f}$ into prime ideal factors, $\mathfrak{f} = \prod \mathfrak{P}_i^{s_i}$, and pick out the $\mathfrak{P}_\nu^{s_\nu}$ of $\mathfrak{f}$, if a power of $\mathfrak{P}_\nu$ lies in H_i. Thus we get an ideal $\mathfrak{f}_i = \mathfrak{f} / \prod_\nu \mathfrak{P}_\nu$, and $\mathfrak{f}_i$ is the conductor of K_i.

In the field K_i we can take the value of the function $\theta(\mathfrak{P}_i)$ according to Kempfert. The following holds for $\theta(\mathfrak{P}_i)$

$$\theta(\mathfrak{P}_i) \equiv 1 \quad \text{mod. } \mathfrak{f}(K_i/\Sigma),$$

$$(\theta(\mathfrak{P}_i)) = \mathfrak{P}_i,$$

and

$$\theta(\mathfrak{P}_i)^{\prod_{i\neq j}\sigma_j(\mathfrak{P}_j^{r_i})} = \theta(\mathfrak{P}_i),\quad r_i = 0, 1, \ldots, f_j - 1.$$

In K holds $\theta(\mathfrak{P}_i) \equiv 1$ mod. $\mathfrak{f}(K/\Sigma)$, since $\mathfrak{f}(K/\Sigma)$ is a factor of $\mathfrak{f}(K_i/\Sigma)$.

To construct the function $\theta(\mathfrak{m})$ for any ideal $\mathfrak{m}$, we make use of the following lemmas.

Lemma 1. If $\theta(\mathfrak{m}) \in K$ has the property that $\theta(\mathfrak{m}) \equiv 1$ mod. $\mathfrak{f}(K/\Sigma)$ and $(\theta(\mathfrak{m})) = \mathfrak{m}$, then $\theta(\mathfrak{m})\sigma$ is an element of K with the same property for every $\sigma \in \mathfrak{g}(K/\Sigma)$, where $\mathfrak{g}(K/\Sigma)$ denotes the Galois group of K relative to Σ.

Proof:

$$\theta(\mathfrak{m}) \equiv 1 \qquad \text{mod. } \mathfrak{f}(K/\Sigma),$$

$$\theta(\mathfrak{m})\sigma \equiv 1 \qquad \text{mod. } \mathfrak{f}((K/\Sigma))\sigma,$$

$$(\theta(\mathfrak{m})\sigma) = (\theta(\mathfrak{m}))\sigma = \mathfrak{m}\sigma$$

The module of genus $\mathfrak{f}(K/\Sigma)$ is an invariant ideal in K with respect to Σ, and $\mathfrak{m}$ is a Σ-ideal, hence $(\mathfrak{f}(Kb\Sigma))\sigma = \mathfrak{f}(K/\Sigma)$ and $\mathfrak{m}\sigma = \mathfrak{m}$, and it is proved that

$$\theta(\mathfrak{m})\sigma \equiv 1 \qquad \text{mod. } \mathfrak{f}(K/\Sigma),$$

$$(\theta(\mathfrak{m})\sigma) = \mathfrak{m} \qquad \text{in } \Sigma.$$

Lemma 2. Let p_i be the norm of $\mathfrak{P}_i$, then $\dfrac{\theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots+\sigma(\mathfrak{P}_i^{f_i-1})}}{p_i^{f_i}} = E_i$ with E_i a ray unit of Σ.

Proof:

$$\theta(\mathfrak{P}_i) \in K_i,$$

$\theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots+\sigma(\mathfrak{P}_i^{f_i-1})}$ is the norm of $\theta(\mathfrak{P}_i)$ to Σ, hence an element of Σ. $(\theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots+\sigma(\mathfrak{P}_i^{f_i-1})}) = (p_i^{f_i})$. Since $\theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots+\sigma(\mathfrak{P}_i^{f_i-1})}$

$\equiv 1$ mod. $\mathfrak{f}$ (K_i/Σ) and is in Σ, $\theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots\sigma+(\mathfrak{P}_i^{f_i-1})} \equiv 1$ mod. $\mathfrak{f}$ and lemma 2 is proved.

Every ideal $\mathfrak{a}$ can be represented in one and only one way as

$$\mathfrak{a} = (\alpha) \Pi \mathfrak{P}_i^{r_i}, \qquad 0 \leqq r_i < f_i,$$

with α in the ray mod. $\mathfrak{f}$. Define

$$\theta(\mathfrak{a}) = \alpha \prod_i \theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots+\sigma(\mathfrak{P}_i^{r_i-1})}$$

then $\theta(\mathfrak{a}) \equiv 1$ mod. $\mathfrak{f}$ (K/Σ) and $(\theta(\mathfrak{a})) = \mathfrak{a}$. Now we can prove the following

Theorem. The elements $\theta(\mathfrak{a})$ of K just defined have both the properties 1 and 2 of theorem 2 of Tannaka.

Proof. The property 1 is already proved. To prove that property 2 is also ture, let

$$\text{and} \quad \begin{aligned} \mathfrak{a} &= (\alpha) \prod_i \mathfrak{P}_i^{r_i} \\ \mathfrak{b} &= (\beta) \Pi \mathfrak{P}_i^{s_i} \end{aligned} \qquad 0 \leqq r_i,\ s_i < f_i$$

be two arbitrary ideals in Σ, and

$$\theta(\mathfrak{a}) = \alpha \prod_i \theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots+\sigma(\mathfrak{P}_i^{r_i-1})},$$

$$\theta(\mathfrak{b}) = \beta \prod_i \theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots\cdots+\alpha(\mathfrak{P}_i^{si-1})},$$

$$\mathfrak{a}\mathfrak{b} = (\alpha\beta) \prod_i \mathfrak{P}_i^{r_i+s_i}.$$

Put $t_i = r_i + s_i - f_i$, $n_i = 1$ if $r_i + s_i \geqq f_i$,

$t_i = r_i + s_i$, $n_i = 0$ if $r_i + s_i < f_i$,

then $(\alpha\beta \prod_i p_i^{n_i}) \Pi \mathfrak{P}_i^{r_i}$ represents $\mathfrak{a}\mathfrak{b}$, and

$$\theta(\mathfrak{a}\mathfrak{b}) = \alpha\beta \prod_i p_i^{n_i} \prod_i \theta(\mathfrak{P}_i)^{1+\sigma(\mathfrak{P}_i)+\cdots\cdots+\sigma(\mathfrak{P}_i^{t_i-1})}.$$

But

$$\sigma(\mathfrak{a}) = \prod_i \sigma(\mathfrak{p}_i^{r_i}),$$

$$\sigma(\mathfrak{b}) = \prod_i \sigma(\mathfrak{p}_i^{s_i}),$$

here

$$\varepsilon(\mathfrak{a},\mathfrak{b}) = \frac{\theta(\mathfrak{a})\ \theta(\mathfrak{b})^{\sigma(\mathfrak{a})}}{\theta(\mathfrak{a},\mathfrak{b})}$$

$$= \frac{\alpha \prod \theta(\mathfrak{p}_i)^{1+\sigma(\mathfrak{p}_i)+\cdots+\sigma(\mathfrak{p}_i^{r_i-1})} \beta \prod \theta(\mathfrak{p}_i)^{(1+\sigma(\mathfrak{p}_i)+\cdots+\sigma(\mathfrak{p}_i^{s_i-1}))\sigma(\mathfrak{p}_i^{r_i})}}{\alpha\beta \prod p_i^{n_i} \prod \theta(\mathfrak{p}_i)^{1+\sigma(\mathfrak{p}_i)+\cdots\cdots+\sigma(\mathfrak{p}_i^{t_i-1})}}$$

$$= \frac{\alpha\beta \prod_i \theta(\mathfrak{p}_i)^{1+\sigma(\mathfrak{p}_i)+\cdots\cdots+\sigma(\mathfrak{p}_i^{r_i})+\cdots\cdots+\sigma(\mathfrak{p}_i^{r_i+s_i-1})}}{\alpha\beta \prod_{p_i}{}^{n_i} \prod \theta(\mathfrak{p}_i)^{1+\sigma(\mathfrak{p}_i)+\cdots\cdots+\sigma(\mathfrak{p}_i^{t_i-1})}}$$

$$= \frac{\alpha \prod \theta(\mathfrak{p}_i)^{(1+\sigma(\mathfrak{p}_i)+\cdots+\sigma(\mathfrak{p}_i^{r_i}))\sigma(\mathfrak{p}_i^{s_i})} \beta \prod \theta(\mathfrak{p}_i)^{1+\sigma(\mathfrak{p}_i)+\cdots+\sigma(\mathfrak{p}_i^{s_i-1})}}{\alpha\beta \prod p_i^{n_i} \prod \theta(\mathfrak{p}_i)^{1+\sigma(\mathfrak{p}_i)+\cdots\cdots+\sigma(\mathfrak{p}_i^{t_i-1})}}$$

$$= \frac{\theta(\mathfrak{a})^{\sigma(\mathfrak{b})}\theta(\mathfrak{b})}{\theta(\mathfrak{a}\mathfrak{b})}$$

and

$$\varepsilon(\mathfrak{a},\mathfrak{b}) = \frac{\prod \theta(\mathfrak{p}_i)^{n_i}}{\prod p_i^{n_i}}$$ is a ray unit in Σ.

References

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