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The Critical Phase Curve of Van Der Pol Equation

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Abstract

This article is concerned with the special trajectory $y_\infty(\chi;\mu)$ which is the leading term of the asymptotic solution of Van der Pol equation $x''+\mu x'(x^2-1)+x=0$ in the phase plane for some region. We show that in the phase plane, the difference of this asymptotic solution and the limit cycle of Van der Pol equation is not greater than $O(\mu^{-1/3})$ as $\mu\to+\infty$ for all $-1\le x\le 0$. Using this result, we can show that every trajectory of Van der Pol equation starting from y-axis with initial value bigger than that of the limit cycle gets close to the limit cycle by $O(\mu^{-1/3})$ from its first time on intersecting $x=1$ in the four quadrant as $\mu\to+\infty$.

Key words: Van der Pol equation, limit cycle

1 Introduction

Let us consider the well-know Van der Pol equation $x''+\mu x'(x^2-1)+x=0$, where $x'=dx/dt$. It is easily to see that the Van der Pol equation is equivalent to the following equation

$$\begin{cases} x'(t)=y(t) \\ y'(t)=-x(t)+\mu(-x(t)^2+1)y(t) \end{cases}$$

in the phase plane, called the Van der Pol equation in the phase plane.

The study of the unique periodic solution of the Van der Pol equation was started by Van der Pol. The behavior of the periodic solution of the Van der Pol equation exhibits a relaxation oscillation as the parameter $\mu>>1$. Therefore, there are many researchers investigating the asymptotic behavior and the numerical solution of the periodic solution of the Van der Pol equation with $\mu>>1$ (Ponzo and wax,1965; Andersen and Geer, 1982).

In this article, the asymptotic method was used to analysis the phase path of the Van der Pol equation in the phase plane. We define a special trajectory, called the critical phase curve $y_\infty(\chi;\mu)$ which is the solution of the following scalar equation

$$\frac{dy}{dx}=-\frac{x}{y}+\mu(1-x^2).$$

The critical phase curve $y_\infty(\chi;\mu)$ is related to the asymptotic behavior of $y_P(\chi;\mu)$ which is the corresponding phase path of the unique periodic solution of the Van der Pol equation. There are three major work in this study as following:

(1) $y_\infty(-1;\mu)-y_p(-1)=O(\mu^{-1/3})$. This result confirms that the leading term of the asymptotic solution $y_\infty(\chi;\mu)$ is not only a good asymptotic solution up to $x=-1+\mu^{-1/3}$, but also to x=−1.

(2) $y_\infty(x;\mu)-y_p(x;\mu)=O(\mu^{-1/3})$, for all $-1\le x\le 0$ as $\mu\to+\infty$.

(3) In the phase plane, every phase path of Van der Pol equation, say $y(\chi;\mu)$, with $y(0;\mu)>y_p(0;\mu)$, gets close to the limit cycle from its first time on intersecting x=1 in the fourth quadrant.

This article is organized in the following structure. In section 2, we prove the existence and the uniqueness of the critical phase curve $y_\infty(\chi;\mu)$ for all $\mu>0$. And then, we estimate $y_\infty(\chi;\mu)$, for $-1\le x\le 0$, and $\mu>>1$, in section 3. Moreover, we also prove that $y_\infty(x;\mu)-y_p(x;\mu)=O(\mu^{-1/3})$, for all $-1\le x\le 0$ as $\mu\to+\infty$. Finally, we have a concluding remark in section 4.

2 Some elementary properties of the critical curve $y_\infty(x;\mu)$

In this section, we look for positive solution of the scalar equation

$$\frac{dy}{dx}=-\frac{x}{y}+\mu(1-x^2) \tag{2.1}$$

with the condition $y_\infty(x;\mu)\ >0, \forall x\le 0$, and $\lim_{x\to-\infty} y(x;\mu)=0$. For simplicity, we set

$$z=-x, \tag{2.2}$$

so that the scalar equation (2.1) can be transformed into the following equivalent equation

$$\frac{dy}{dz}=-\frac{z}{y}+\mu(z^2-1). \tag{2.3}$$

We will use shooting argument to analysis this equation. Let $y(z,a;\mu)$ denote the solution of Eq. (2.3) with the initial condition $y(0,a;\mu)=a$. Let I_+ be the set of the initial value $y(0,a;\mu)=a>0$, for which $y'(z,a;\mu)$ vanishes at some finite number $R>0$ before $y(z,a;\mu)$ does. We also let I_- be the set of the positive initial values for which the corresponding solutions are monotone decreasing until they cross the z axis.

Firstly, we show that I_+ is not empty.

Lemma 2.1 Let $y(z;\mu)$ be a solution of equation (2.3), and there exists $z_0>1$ such that $y(z_0;u)=\dfrac{z_0}{\mu(z_0^2-1)}$. Then we have

(1) $y(z;\mu)$ strictly increases on $[z_0+\infty)$ and $\lim_{z\to+\infty} y(z;\mu)=+\infty$.

(2) $y(z;\mu)$ strictly decreases on $[0,z_0]$.

Proof.

(1) By the assumption and (2.3) we have

$$y'(z_0;\mu)=\frac{-z_0}{y(z_0;\mu)}+\mu(z_0^2-1)=0, \text{and}$$

$$y''(z_0;\mu)=-\frac{1}{y(z_0;\mu)}+2\mu z_0=\mu(z_0+\frac{1}{z_0})>0.$$

Hence there exists $1>>\varepsilon>0$ such that $y'(z;\mu)<0$, $\forall z\in(z_0-\varepsilon,z_0)$ and $y'(z;\mu)>0$, $\forall z\in(z_0,z_0+\varepsilon)$. Since $y(z_0;\mu)=z_0/[\mu(z_0^2-1)]$ and $z/[\mu(z^2-1)]$ decreases on $[1,\infty)$, it follows that

$$y(z;\mu)>\frac{z}{\mu(z^2-1)},\quad \forall z\in(z_0,z_0+\varepsilon).$$

On the other hand, $y'(z;\mu)>0$ as long as $y(z;\mu)>z/[\mu(z^2-1)]$ and $z>1$. Consequently, $y(z;\mu)$ strictly increases on $[z_0,b)$, where $y(z;\mu)$ is defined.

Next, we will prove that $y(z;\mu)$ is defined on $[z_0,+\infty)$. If not, then there exists $+\infty>b>z_0$ such that $\lim_{x\to b^-} y(z;\mu)=+\infty$, $\limsup_{x\to b^-} y'(z;\mu)=+\infty$.

However, from (2.3), we have

$$\limsup_{x\to b^-} y'(z;\mu)=\limsup_{x\to b^-}\frac{-z}{y(z;\mu)}+\mu(z^2-1)$$
$$=\mu(b^2-1)<\infty,$$

a contradiction. Hence, $y(z;\mu)$ is defined on $[z_0,+\infty)$.

Finally, we show that $\lim_{z\to+\infty} y(z;\mu)=+\infty$. If not, since $y(z;\mu)$ strictly increase on $[z_0,+\infty)$, then there exists $r>0$ such that

$$\lim_{x\to+\infty} y(z;\mu)=r, \liminf_{x\to+\infty} y'(z;\mu)=0.$$

On the other hand, by (2.3), we have

$$\liminf_{x\to+\infty} y'(z;\mu)=\liminf_{x\to+\infty}\frac{-z}{y(z;\mu)}+\mu(z^2-1)$$
$$=+\infty,$$

a contradiction. Therefore $\lim_{x\to+\infty} y(z;\mu)=+\infty$.

(2) Note that $y'(z;\mu)<0$ on $[z_0-\varepsilon,z_0)$. Firstly, we prove that $y(z;\mu)$ is defined on $[1,z_0]$ and $y'(z;u)<0$ on $[1,z_0]$. If not, then there are two cases. One case is that there exists $z_0>c\ge 1$ such that $y'(z;\mu)<0$ in (c,z_0) and $y'(c;\mu)=0$. Then

$$y''(c;\mu)=\lim_{z\to c^+}\frac{y'(z;\mu)-y'(c;\mu)}{z-c}\le 0.$$

However, by (2.3), we have

$$y''(c;\mu) = -\frac{1}{y(c;\mu)} + 2\mu c \le \mu(c + \frac{1}{c}) > 0,$$

a contradiction.

By the same argument as in (1), the other case for which there exists $z_0 > c \ge 1$ such that $\lim_{z \to c^+} y(z;\mu) = +\infty$ is impossible. Consequently, $y(z;\mu)$ is defined and strictly decreases on $[1, z_0]$.

Next, we observe that $y'(z;\mu) < 0$ as long as $y(z;\mu) > 0$ and $1 \ge z \ge 0$. Also, by the same argument as in (1), there does not exist $1 > d \ge 0$ such that $\lim_{z \to d^+} y(z;\mu) = +\infty$. Hence $y(z;\mu)$ is defined and strictly decreases on $[0, z_0]$.

From Lemma 2.1 and continuous dependence on initial data, we have:

Corollary 2.2 I_+ is nonempty and open.

Lemma 2.3, There exists $C > 0$ depending on a positive number μ such that, for $a \in (0, C)$, the solution $y(z,a;\mu)$ of equation (2.3) with initial condition $y(0,a;\mu) = a$ vanishes at some finite number R, $1 \ge R > 0$ and $y'(z,a;\mu) < 0$ in $[0, R)$.

Proof.

Set $C = \frac{2}{3}\mu$. We will show that $\forall a \in (0, C)$, the solution $y(z,a;\mu)$ of (2.3) with initial condition $y(0,a;\mu) = a$ has the required property. From equation (2.3), it follows that $y'(z,a;\mu) < 0$ as long as $y(z,a;\mu) > 0$ and $1 \ge z \ge 0$. Thus if $y(z,a;\mu)$ does not vanish for $1 \ge R > 0$, then $y(z,a;\mu)$ is defined and $y'(z,a;\mu) < 0$ on $[0,1]$.

Integrating equation (2.3) from 0 to 1, we have

$$y(1,a;\mu) = a + \int_0^1 \frac{-z}{y(z,a;\mu)} - \frac{2}{3}\mu < a - \frac{2}{3}\mu < 0,$$

a contradiction. Therefore, $y(z,a;\mu)$ vanishes for $1 \ge R > 0$ and $y'(z,a;\mu) < 0$ in $[0, R)$.

Lemma 2.4 Let $y(z,a_0;\mu)$ be the solution of equation (2.3) with initial condition $y(0,a_0;\mu) = a_0 > 0$ such that $y(z,a_0;\mu)$ vanishes at a positive number $R_0 > 0$, and $y'(z,a_0;\mu) < 0$ in $[0, R_0)$.

Then there exists $\varepsilon > 0$ such that for $|a - a_0| < \varepsilon$, the solution $y(z,a;\mu)$ of equation (2.3) with initial condition $y(0,a;\mu) = a$ vanishes at some finite $R > 0$ and $y'(z,a;\mu) < 0$ in $[0, R)$.

Proof.

The proof consists of two cases.

Case1, consider $1 \ge R_0 > 0$.

By continuous dependence on initial data, there exists $\varepsilon > 0$ and $1 >> \delta > 0$ such that for all $|a - a_0| < \varepsilon$, we have

$y^2(R_0 - \delta, a;\mu) < [1 - (R_0 - \delta)^2]/2$, and

$y'(z,a;\mu) < 0$ in $[0, R_0 - \delta)$.

We will prove that $y(z,a;\mu)$ has to vanish for $1 \ge R > 0$. It follows from (2.3) that $y'(z,a;\mu) < 0$ as long as $y(z,a;\mu) > 0$ and $z \in [0,1]$. Hence, if $y(z,a;\mu)$ doesn't vanish at $1 \ge R > 0$, then we have $y'(z,a;\mu) < 0$ in $[0,1]$, and then

$$y(z,a;\mu) \le y(R_0 - \delta, a;\mu) \text{ in } [R_0 - \delta, 1]. \tag{2.5}$$

Integrating (2.3) from $R_0 - \delta$ to 1 and using (2.4), (2.5), we have

$$\begin{aligned} y(1,a;\mu) &= \\ & y(R_0 - \delta, a;\mu) + \int_{R_0 - \delta}^1 \frac{-s}{y(s,a;\mu)} ds + \int_{R_0 - \delta}^1 \mu(s^2 - 1) ds \\ &\le y(R_0 - \delta, a;\mu) + \int_{R_0 - \delta}^1 \frac{-s}{y(R_0 - \delta, a;\mu)} ds \\ &= \frac{y^2(R_0 - \delta, a;\mu) - [1 - (R_0 - \delta)^2]/2}{y(R_0 - \delta, a;\mu)} < 0, \end{aligned}$$

a contradiction. Hence $y(z,a;\mu)$ vanishes at $1 \ge R > 0$.

Case 2, consider $R_0 > 1$.

$k = \frac{R_0}{\mu(R_0^2 - 1)}$, and let $b > R_0$ such that

$$\frac{b}{\mu(b^2 - 1)} = k/2.$$

By continuous dependence on initial data and $y(R_0, a_0;\mu) = 0$, there exists $\varepsilon > 0$ and $1 >> \delta > 0$ such that for all $|a - a_0| < \varepsilon$, we have

$$y(R_0 - \delta, a;\mu) < k/4, \quad \text{and} \quad y(R_0 - \delta, a;\mu) + \mu(-R_0 + \frac{R_0^3}{3} + b - \frac{b^3}{3})(\frac{k}{2y(R_0 - \delta, a;\mu)} - 1) < 0, \tag{2.6}$$

and

$$y'(z,a;\mu) < 0 \text{ in } [0, R_0 - \delta].$$

We will show that $y(z,a;\mu)$ has to vanish at some finite number $b \ge R \ge 1$. From (2.3), it follows that $y'(z,a;\mu) < 0$ as long as $y(z,a;\mu) < z/\left[\mu(z^2-1)\right]$ and $z > 1$. Hence, if $y(z,a;\mu)$ does not vanish at $R > 1$, then from the definition of b, it follows that

$y'(z,a;\mu) < 0$ in $[0,b]$. Thus,

$$\frac{z/\left[\mu(z^2-1)\right]}{y(z,a;\mu)} \ge \frac{b/\left[\mu(b^2-1)\right]}{y(R_0-\delta,a;\mu)} \text{ in } [R_0-\delta,b].$$

By the definition of b, this is equivalent to the following

$$\frac{z}{y(z,a;\mu)} \ge \frac{k\mu(z^2-1)}{2y(R_0-\delta,a;\mu)} \text{ in } [R_0-\delta,b]. \quad (2.7)$$

Integrating equation (2.3) from $R_0-\delta$ to b and using (2.6), (2.7), we have

$$\begin{aligned} y(b,a;\mu) &= y(R_0-\delta,a;\mu) \\ &+ \int_{R_0-\delta}^{b} \mu\frac{-s}{y(s,a;\mu)}ds + \int_{R_0-\delta}^{b}\mu(s^2-1)ds \\ &\le y(R_0-\delta,a;\mu) \\ &- \int_{R_0-\delta}^{b}\frac{k\mu(z^2-1)}{2y(R_0-\delta,a;\mu)} + \int_{R_0-\delta}^{b}\mu(s^2-1)ds \\ &\le y(R_0-\delta,a;\mu) \\ &- \mu\left(-R_0+\frac{R_0^3}{3}+b-\frac{b^3}{3}\right)\left(\frac{k}{2y(R_0-\delta,a;\mu)}-1\right) \\ &< 0, \end{aligned}$$

a contradiction. Hence $y(z,a;\mu)$ vanishes at $R > 1$.

Lemma 2.5 There exists $\alpha > 0$ such that the solution $y(z,\alpha;\mu)$ of equation (2.3) with the initial condition $y(0,\alpha;\mu) = \alpha$ satisfies the following conditions

$$\lim_{z\to\infty} y(z,\alpha;\mu) = 0 \text{ and}$$
$$y'(z,\alpha;\mu) < 0 \text{ in } [0,+\infty)$$

Proof.

Let $\alpha_1 = \inf I_+, \alpha_2 = \sup I_-$. By Corollary 2.2, Lemma 2.3 and 2.4, α_1,α_2 are finite numbers and I_+, I_- are open. Hence from the definition of I_+, I_- and Lemma 2.1, it follows that

$$\lim_{z\to\infty} y(z,\alpha;\mu) = 0, \quad i = 1,2, \text{ and}$$
$$y'(z,\alpha_i;\mu) < 0 \text{ in } [0,+\infty), \quad i = 1,2.$$

In the next lemma, we will show that $\alpha_1 = \alpha_2$.

Lemma 2.6 There is at most one positive number $\alpha > 0$ such that the solution $\lim_{z\to\infty} y(z,\alpha;\mu) = \alpha$ satisfies the following conditions

$$\lim_{z\to\infty} y(z,\alpha;\mu) = 0, \text{ and}$$
$$y'(z,\alpha;\mu) < 0 \quad in\ [0,+\infty).$$

Proof.

If not, then there exists $\alpha_1 > \alpha_2 > 0$ such that $y(z,\alpha_1;\mu)$, $y(z,\alpha_2;\mu)$ are solutions of (2.3) satisfying

$$\lim_{z\to\infty} y(z,\alpha_i;\mu) = 0, \ i = 1,2, \text{ and}$$
$$y'(z,\alpha_i;\mu) < 0 \text{ in } [0,+\infty), \ i = 1,2.$$

Then, we have

$$\frac{dy}{dz}(z,\alpha_1;\mu) = -\frac{z}{y(z,\alpha_1;\mu)} + \mu(z^2-1), \quad (2.8)$$

and

$$\frac{dy}{dz}(z,\alpha_2;\mu) = -\frac{z}{y(z,\alpha_2;\mu)} + \mu(z^2-1), \quad (2.9)$$

Set

$$f(z;\mu) = y(z,\alpha_1;\mu) - y(z,\alpha_2;\mu).$$

Subtraction (2.9) from (2.8), we have

$$\frac{df}{dz}(z;\mu) = \frac{z}{y(z,\alpha_1;\mu)y(z,\alpha_2;\mu)} f(z;\mu). \quad (2.10)$$

It follows from (2.10) that $df/dz > 0$ as long as $f(z;\mu) > 0$ and $z > 0$. Hence

$$y(z,\alpha_1;\mu) - y(z,\alpha_2;\mu) \ge y(0,\alpha_1;\mu) - y(0,\alpha_2;\mu) > 0.$$

On the other hand, by assumption, we have

$$\lim_{z\to+\infty} y(z,\alpha_1;\mu) - y(z,\alpha_2;\mu) = 0,$$

a contradiction.

From Lemma 2.5, 2.6 and equation (2.1), (2,2), (2.3), we have the following theorem. Note that the solution of equation (2.1) with initial condition a is denoted by $y(x,a;\mu)$.

Theorem 2.7 For each $\mu > 0$, there exists a unique positive number β such that $y(x,\beta;\mu)$ which we also denote by $y_\infty(x;\mu)$ satisfies the following conditions:

(1) $\lim_{x\to-\infty} y(x,\beta;\mu) = 0$, and $y'(x,\beta;\mu) > 0$ in $(-\infty,0]$.

(2) For each $a \in (0,\beta)$, $y(x,a;\mu)$ vanishes at some finite number $R < 0$ and $y'(x,a;\mu) > 0$ in $(R,0]$.

(3) For each $a > \beta$, there exists a unique number $r < 0$ such that $y(x,a;\mu)$ has the following properties:

(a) $y(r,a;\mu) = -r/\left[\mu(r^2-1)\right]$, $y'(r,a;\mu) = 0$.

(b) $y'(x,a;\mu) < 0$ in $(-\infty, r]$ and $\lim_{x\to-\infty} y(x,a;\mu) = +\infty$.

(c) $y'(x,a;\mu) > 0$ in $[r,0]$.

3 Estimation of $y_\infty(x;\mu)$

3.1 Some properties of a related differential equation

Since $y_\infty(x;\mu)$ would change sharply near $x=-1$, we introduce the following rescaled coordinates:

$$y = \mu^{1/3}\eta,\ x = -1 + \mu^{-2/3}\xi . \tag{3.1}$$

In this coordinate, Eq. (2.1) in phase plane can be transformed into

$$\eta\eta' = 2\xi\eta + 1 - \mu^{-2/3}(\xi + \xi^2\eta). \tag{3.2}$$

where $\eta' = d\eta/d\xi$. Note that the zero isocline of Eq. (3.2) is

$$f(\xi;\mu) = \frac{1+\mu^{-2/3}(-\xi)}{2(-\xi)+\mu^{-2/3}\xi^2}.$$

Since the coordinate transformation (3.1) is linear. Thus, there exists a unique solution of (3.2) which shares the same qualitative properties with $y_\infty(x;\mu)$. Actually, we have the following proposition. Note that the solution of Eq. (3.2) with initial condition a is denoted by $\eta(\xi,a;\mu)$.

Proposition 3.1 For each $\mu > 0$, there exists a unique positive number β such that $\eta(\xi,\beta;\mu)$ which is also denoted by $\eta_\infty(\xi;\mu)$ satisfies the following conditions

(1) $\lim_{\xi\to\infty} \eta(\xi,\beta;\mu) = 0$, and

$$\eta'(\xi,\beta;\mu) > 0 \text{ in } (-\infty,0].$$

(2) For each $a \in (0,\beta)$, $\eta(\xi,a;\mu)$ vanishes at some finite number $R<0$ and $\eta'(\xi,a;\mu) > 0$ in $(R,0]$.

(3) For each $a > \beta$, there exists a unique number $r<0$ such that $\eta(\xi,a;\mu)$ has the following properties

(a) $\eta(r,a;\mu) = \left[1+\mu^{-2/3}(-r)\right]/\left[2(-r)+\mu^{-2/3}\ r^2\right]$, $\eta'(r,a;\mu) = 0$.

(b) $\eta'(\xi,a;\mu) < 0$ in $(-\infty, r]$ and $\lim_{\xi\to-\infty} \eta(\xi,a;\mu) = +\infty$.

(c) $\eta'(\xi,a;\mu) > 0$ in $[r,0]$.

The proof is similar to Theorem 2.7.

Now, we are in position to get some information about $y_\infty(-1;\mu)$, and $\eta_\infty(0;\mu)$. It may be evident from equation (3.2) that the behaviour of solutions near $\xi = 0$ are dominated by the following equation

$$\eta\eta' = 2\xi\eta_1 + 1 \text{ where } \eta_1' = d\eta_1/d\xi . \tag{3.3}$$

Note that the zero isocline of Eq. (3.3) is $\dfrac{1}{2(-\xi)}$.

This equation can be solved in terms of Bessel functions or Airy functions. The interesting qualitative properties of Eq. (3.3) which are almost identical to the properties of the untransformed equation (2.1) are stated in the following proposition. Note that the solution of Eq. (3.3) with initial condition a is denoted by $\eta_1(\xi,a)$.

Proposition 3.2 There exists a unique positive number β_∞ such that $\eta_1(\xi,\beta_\infty)$, also denoted by $\eta_{1\infty}(\xi)$, satisfies the following conditions:

(1) $\lim_{\xi\to-\infty} \eta_1(\xi,\beta_\infty) = 0$, and

$$\eta_1'(\xi,\beta_\infty) > 0 \text{ in } (-\infty,0].$$

(2) For each $a \in (0,\beta_\infty)$, $\eta_1(\xi,a)$ vanishes at some finite number $R<0$ and $\eta_1'(\xi,a) > 0$ in $(R,0]$.

(3) For each $a > \beta_\infty$, there exists a unique number $r<0$ such that $\eta_1(\xi,a)$ has the following properties

(a) $\eta_1(r,a) = 1/(-2r)$, $\eta_1'(r,a) = 0$.

(b) $\eta_1'(\xi,a) < 0$ in $(-\infty, r]$ and $\lim_{\xi\to-\infty} \eta_1(\xi,a) = +\infty$.

(c) $\eta_1'(\xi,a) > 0$ in $[r,0]$.

Next, we state some quantitative propositions about Eq.(3.3).

Proposition 3.3

(1) β_∞ is equal to $Ai'(-\omega)$ at first zero, where $Ai(\omega)$ is the well-known special function, Airy funcion.

(2) $\int_0^{+\infty} 1/\eta_{1\infty}(\xi)d\xi = \alpha_\infty - \beta_\infty$, where α_∞ is the first zero of $Ai(-\omega)$.

(3)

$$\lim_{\beta\to\beta_\infty} \int_0^{+\infty} \frac{1}{\eta_1(\xi,\beta)} d\xi = \alpha_\infty - \beta_\infty .$$

Proof.

Since this equation can be solved by means of table, thus we just outline the procedure of how to solve it explicitly. Set $\eta_1 = \dfrac{d\xi}{d\omega}$.

Substituting this expression into (3.3), we have

$$\frac{d\xi}{d\xi}\frac{d\eta_1}{d\xi} = 2\xi\frac{d\xi}{d\omega} + 1.$$

Consequently,

$$\eta_1 = \frac{d\xi}{d\omega} = \xi^2 + \omega. \tag{3.4}$$

Eq. (3.4) is one of Riccati type and can be solved explicitly by means of the transformation

$$\xi = -\frac{\upsilon'(\omega)}{\upsilon(\omega)} \tag{3.5}$$

where, $\upsilon(\omega)$ satisfies the so-called Airy equation

$$\upsilon'' + \omega\upsilon = 0 \tag{3.6}$$

where, $\upsilon'' = d^2\upsilon/d\omega^2$.

This equation has two independent solutions, $Ai(-\omega)$ and $Bi(-\omega)$ in the standard notation, and its general solution is the linear combination of the above two special functions.

(1) we have

$$\eta_{1\infty}(\xi) = -\frac{Ai'(-\omega)}{Ai(-\omega)}.$$

Note that $Ai(-\omega)$ is exponentially small as $\omega \to -\infty$, rises to a maximum at $\omega = \beta_\infty = 1.019...$, and becomes zero at $\omega = \alpha_\infty = 2.338...$.

(2) $\int_0^{+\infty} 1/\eta_{1\infty}(\xi)d\xi$ is well-defined, since $\eta_{1\infty}(\xi) \sim \xi^2$ as $\xi \to +\infty$. Moreover, from equation (3.4), it follows that

$$\int_0^{+\infty}\frac{1}{\eta_{1\infty}(\xi)}d\xi = \int_0^{+\infty}\frac{d\xi}{\xi^2+\omega} = \int_{\beta_\infty}^{\alpha_\infty} d\omega = \alpha_\infty - \beta_\infty.$$

(3) By Eq. (3.4) and (3.5), we set

$$\xi = \frac{Ai'(-\omega) + bBi'(-\omega)}{Ai(-\omega) + bBi(-\omega)}.$$

Note that $\eta_1(\xi,\beta)$ is the solution of (3.3) with the initial condition $\eta(0,\beta) = \beta$, and that for all $|\beta - \beta_\infty| << 1$, $\eta_1(\xi,\beta)$ can be defined on $[0,+\infty)$. Hence, from Eq. (3.3), (3.4) and (3.5), we have $Ai'(-\beta) + bBi'(-\beta) = 0, \quad Ai(-\alpha) + bBi(-\alpha) = 0$. Where, α is a real number such that $\xi \to +\infty$ as $\omega \to \alpha$. Consequently, we have

$$\frac{Ai(-\alpha)}{Bi(-\alpha)} = \frac{Ai'(-\beta)}{Bi'(-\beta)}. \tag{3.7}$$

Equation (3.7) can be viewed as an equation with argument β, where α is viewed as a given parameter. Consequently, from the regularity of $Ai(-\omega)/Bi(-\omega)$ and $Ai'(-\omega)/Bi'(-\omega)$ at α_∞, β_∞ respectively, and $\alpha = \alpha_\infty, \beta = \beta_\infty$ satisfying (3.7), we have $\alpha \to \alpha_\infty$ as $\beta \to \beta_\infty$, and this implies

$$\lim_{\beta\to\beta_\infty}\int_0^{+\infty}\frac{1}{\eta_1(\xi,\beta)}d\xi = \alpha_\infty - \beta_\infty.$$

3.2 Estimation of $y_\infty(-1;\mu)$

The following lemmas are mainly based on the comparison theorem for differential equation and some special functions. Thus for convenience, we state the comparison theorem for differential equation in the following.

Lemma 3.1 (Comparison Theorem)

Suppose $f_1(t,x), f_2(t,x)$ are real-valued functions of the scalars t,x in some open connected set Ω of R^2, satisfying $f_1(t,x) \ge f_2(t,x)$, and $\ell_1(t), \ell_2(t)$ are the solutions of the following differential equations

$$\frac{dx}{dt} = f_1(t,x), \frac{dx}{dt} = f_2(t,x)$$

respectively, with the same initial condition

$$\ell_1(t_0) = \ell_2(t_0) = x_0$$

where ℓ_1, ℓ_2 are all defined on the interval $a < t < b$ $(-\infty \le a < b \le +\infty)$ and $t_0 \in (a,b)$, then we have $\ell_1(t) \ge \ell_2(t)$ for all $t \ge t_0$ in (a,b). Similarly, we have $\ell_1(t) \le \ell_2(t)$ for all $t \le t_0$ in (a,b).

With the aid of Proposition 3.2, we can estimate the lower bound of $\eta_\infty(0;\mu)$. Recall that $\eta_\infty(\xi;\mu)$ is the solution of (3.2) which satisfies Proposition 3.1.

Lemma 3.2 For each $\mu > 0$, we have

$$\eta_\infty(0;\mu) > \beta_\infty.$$

Proof.

If $\xi < 0, \eta < 1/(1-\xi)$, then

$$2\xi\eta + 1 - \mu^{-2/3}\xi(\xi\eta + 1) > 2\xi\eta + 1.$$

Consequently, if $\xi<0$ and $-\xi$ small, then it follows from Lemma 3.1, Eq. (3.2), (3.3) and Proposition 3.2 that

$$\eta(\xi,\beta_\infty;\mu)<\eta_1(\xi,\beta_\infty)=\eta_{1\infty}(\xi)<\frac{1}{-2\xi} \tag{3.8}$$

Moreover, (3.8) is always true as long as $1/(-\xi)>\eta(\xi,\beta_\infty;\mu)>0$ and $\xi\le 0$. Hence for all $\xi\le 0$ such that $\eta(\xi,\beta_\infty;\mu)$, $\eta_{1\infty}(\xi)$ are defined, we always have

$$\eta(\xi,\beta_\infty;\mu)<\frac{1}{-2\xi} \tag{3.9}$$

Recall that $\left[1+\mu^{-2/3}(-\xi)\right]/\left[2(-\xi)+\mu^{-2/3}\xi^2\right]$ and $1/(-2\xi)$ are the zero isoclines of Eq. (3.2), (3.3) respectively, and for all $\xi\le 0$, we have

$$\frac{1}{-2\xi}<\frac{1+\mu^{2/3}(-\xi)}{2(-\xi)+\mu^{-2/3}\xi^2}.$$

Thus, it follows from (3.9) and Proposition 3.1 that $\eta(\xi,\beta_\infty;\mu)$ has to intersect with ξ axis at some finite number R<0. Hence

$$\eta_\infty(0;\mu)>\eta(0,\beta_\infty;\mu)=\beta_\infty$$

is achieved.

For estimating the upper bound of $\eta_\infty(0;\mu)$, we draw our attention to the following equation for a moment

$$\eta_2\eta_2'=2\xi\eta_2+1+\mu^{-1/3} \tag{3.10}$$

where $\eta_2'=d\eta_2/d\xi$. Note that the zero isocline of Eq. (3.10) is

$$\frac{1+\mu^{-1/3}}{2(-\xi)}.$$

It is easily convinced that Eq. (3.10) has similar properties as equation (3.3). Thus let $\eta_{2\infty}(\xi;\mu)$ has similar meaning as $\eta_{1\infty}(\xi)$. The following estimate is our required upper bound for $\eta_\infty(0;\mu)$.

Lemma 3.3 For each $\mu>0$, we have

$$\gamma_\mu>\eta_\infty(0;\mu) \tag{3.11}$$

where γ_μ is a positive number such that

$$\eta_2\left(-\mu^{1/3},\gamma_\mu;\mu\right)=\left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)\right]/2.$$

Proof.

If $\eta>0$ and $-\mu^{1/3}<\xi<0$, then we have

$2\xi\eta+1-\mu^{-2/3}\xi(\xi\eta+1) < 2\xi\eta+1-\mu^{-2/3}\xi < 2\xi\eta+1+\mu^{-1/3}$. It follows from Lemma 3.1, Eq. (3.2) and (3.10) that $\eta(\xi,\gamma_\mu;\mu)>\eta_2(\xi,\gamma_\mu;\mu)$ where $-\mu^{1/3}\le\xi<0$ and γ_μ is a positive number such that $\eta_2\left(-\mu^{1/3},\gamma_\mu;\mu\right)=\left[\mu\ \left(1+\mu^{-1/3}\right)\right]/2$. On the otherhand,

$$f\left(-\mu^{1/3};\mu\right)=\frac{1+\mu^{-1/3}}{1+2\mu^{1/3}}<\left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)\right]/2$$
$$=\eta_2\left(-\mu^{1/3},\gamma_\mu;\mu\right)$$
$$<\eta\left(-\mu^{1/3},\gamma_\mu;\mu\right)$$

where $f(\xi;\mu)$ is the zero isocline of Eq. (3.2). Hence, by Proposition 3.1, $\eta(\xi,\gamma_\mu;\mu)$ tends monotonically to $+\infty$, as $\xi\to-\infty$. Thus we have

$$\gamma_\mu=\eta(0,\gamma_\mu;\mu)>\eta_\infty(0;\mu).$$

Combining the above two lemmas, we can estimate $\eta_\infty(0;\mu)$ for any $\mu>0$. However, it is not a easy task to calculate the magnitudes of γ_μ and β_∞. But to our surprise, we have the following qualitative result.

Lemma 3.4

$$\lim_{\mu\to+\infty}\eta_\infty(0;\mu)=\beta_\infty, \text{ and}$$
$$\lim_{\mu\to+\infty}\gamma_\mu=\lim_{\mu\to+\infty}\delta_\mu=\beta_\infty$$

where, $\delta_\mu=\left(1+\mu^{-1/3}\right)^{-2/3}\gamma_\mu$.

Proof.

Set $\delta_\mu=\left(1+\mu^{-1/3}\right)^{-2/3}\gamma_\mu$. (3.12)

It is easy to check that $\eta_2(\xi,\gamma_\mu;\mu)$ and $\left(1+\mu^{-1/3}\right)^{-2/3}\eta_1\left(\left(1+\mu^{-1/3}\right)^{-1/3}\xi,\delta_\mu\right)$ are all solutions of Eq. (3.10) with the same initial condition γ_μ. Hence, we have

$$\eta_2(\xi,\gamma_\mu;\mu)=\left(1+\mu^{-1/3}\right)^{2/3}\eta_1\left(\left(1+\mu^{-1/3}\right)^{-1/3}\xi,\delta_\mu\right).$$

From Lemma 3.3 and the above equality, we have

$$\left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)\right]/2=\eta_2\left(-\mu^{1/3},\gamma_u;\mu\right)$$
$$=\left(1+\mu^{-1/3}\right)^{2/3}\eta_1\left(-\mu^{1/3}\left(1+\mu^{-1/3}\right)^{-1/3},\delta_\mu\right).$$

Thus

$$\eta_1\left(-\mu^{1/3}\left(1+\mu^{-1/3}\right)^{-1/3},\delta_\mu\right)$$
$$=\left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)^{-1/3}\right]/2.$$

Note that $\left(-\mu^{1/3}\left(1+\mu^{-1/3}\right)^{-1/3},\left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)^{1/3}\right]/2\right)$ lies on the curve $2\xi\eta+1=0$, and $-\mu^{1/3}\left(1+\mu^{-1/3}\right)^{-1/3}\to-\infty$. Thus it follows from Proposition 3.2 that

$$\delta_\mu\to\beta_\infty \text{ as } \mu\to+\infty.$$

Finally, since $\gamma_\mu=\left(1+\mu^{-1/3}\right)^{2/3}\delta_\mu$, and the inequalities for $\eta_\infty(0;\mu)$ in Lemma 3.2 and Lemma 3.3, we have

$$\eta_\infty(0;\mu) \to \beta_\infty \quad \text{as} \quad \mu \to +\infty .$$

Corollary 3.5 For $\mu >> 1$, we have

(a)

$$\gamma_\mu - \beta_\infty = \mathrm{O}\left(\mu^{-1/3}\right).$$

(b)

$$\eta_\infty(0;\mu) - \beta_\infty = \mathrm{O}\left(\mu^{-1/3}\right), \text{ that is,}$$

$$y_\infty(-1;\mu) - \beta_\infty \mu^{-1/3} = \mathrm{O}\left(\mu^{-2/3}\right).$$

Proof.

(a) For $\eta_1(\xi,\delta_\mu)$, we can proceed as in the proof of Proposition 3.3 and write

$$\xi = \xi(w) = \frac{a\mathrm{A}i'(-w) + bBi'(w)}{a\mathrm{A}i'(-w) + bBi'(w)}.$$

Moreover, we have

$$-\frac{b}{a} = \frac{Ai'(-\delta_\mu)}{Bi'(-\delta_\mu)} \qquad (3.13)$$

where, δ_μ is defined by (3.12). Set

$$\xi_1 = \mu^{1/3}, \quad \xi_2 = \left(1 + \mu^{-1/3}\right)^{1/3},$$

$$t_\mu = \left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)^{1/3}\right]/2 - \mu^{2/3}\left(1+\mu^{-1/3}\right)^{-2/3}.$$

Since δ_μ satisfies the equality

$$\eta_1\left(-\mu^{1/3}\left(1+\mu^{-1/3}\right)^{-1/3}, \delta_\mu\right)$$

$= \left[\mu^{-1/3}\left(1+\mu^{-1/3}\right)^{1/3}\right]/2$, then from Eq. (3.4) and (3.5), the following equality is held

$$-\frac{\xi_1}{\xi_2} = \frac{aAi'(-t_\mu) + bBi'(-t_\mu)}{aAi'(-t_\mu) + bBi'(-t_\mu)} \qquad (3.14)$$

where,

$$a = -\frac{\xi_1 Bi(-t_\mu) + \xi_2 Bi'(-t_u)}{\Delta},$$

$$b = \frac{\xi_1 Ai(-t_\mu) + \xi_2 Ai'(-t_\mu)}{\Delta},$$

$$\Delta = Ai'(-t_\mu)Bi(-t_\mu) - Bi'(-t_\mu)Ai(-t_\mu).$$

Therefore, it follows from (3.13) and (3.14) that we have

$$\frac{Ai'(-\delta_\mu)}{Bi'(-\delta_\mu)} = -\frac{b}{a} = \frac{\xi_1 Ai(-t_\mu) + \xi_2 Ai'(-t_\mu)}{\xi_1 Bi(-t_\mu) + \xi_2 Bi'(-t_\mu)}. \qquad (3.15)$$

Since t_μ approaches $-\infty$ as $\mu \to +\infty$, thus we consider the asymptotic series of $Ai(x)$, $Bi(x)$ for $x >> 1$ as stated in the following

$$Ai(x) \sim \frac{e^{-\zeta}}{2\pi^{1/2}x^{1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{u_s}{\zeta^s},$$

$$Ai'(x) \sim -\frac{e^{-\zeta}}{2\pi^{1/2}x^{-1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{v_s}{\zeta^s},$$

$$Bi(x) \sim \frac{e^{\zeta}}{\pi^{1/2}x^{1/4}} \sum_{s=0}^{\infty} \frac{u_s}{\zeta^s},$$

$$Bi'(x) \sim -\frac{e^{\zeta}}{\pi^{1/2}x^{-1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{v^s}{\zeta^s}$$

where

$$u_s = \frac{(2s+1)(2s+3)(2s+5)\ldots(6s-1)}{216^s s!}, \quad v_s = -\frac{6s+1}{6s+1},$$

$$u_0 = v_0 = 1, \quad s \geq 1, \quad \zeta = \frac{2}{3}x^{2/3}.$$

With the aid of this device, we can rewrite (3.15) as

$$\frac{Ai'(-\delta_\mu)}{Bi'(-\delta_\mu)} = \frac{\xi_1 Ai(-t_\mu) + \xi_2 Ai'(-t_\mu)}{\xi_1 Bi(-t_\mu) + \xi_2 Bi'(-t_\mu)} = \mathrm{O}\left(e^{-4\mu/3}\right). \qquad (3.16)$$

Note that the Taylor series of $\dfrac{Ai'(-t)}{Bi'(-t)}$ at β_∞ is

$$-\frac{-\beta_\infty Ai(-\beta_\infty)}{Bi'(-\beta_\infty)}(t-\beta_\infty) + \left(-\left(\frac{\beta_\infty^2 Ai(-\beta_\infty)Bi(-\beta_\infty)}{Bi'(-\beta_\infty)^2}\right) + \frac{Ai(-\beta_\infty)}{2Bi'(-\beta_\infty)}\right)(t-\beta_\infty)^{2^3} + \mathrm{O}(t-\beta_\infty). \qquad (3.17)$$

Thus if follows from Lemma 3.4, (3.16) and (3.17) that

$$\delta_\mu = \beta_\infty + \mathrm{O}\left(e^{-4\mu/3}\right), \text{ and}$$

$$\gamma_\mu = \beta_\infty + \mathrm{O}\left(\mu^{-1/3}\right).$$

(b) Combining the result of (a) with the inequality

$$\gamma_\infty > \eta_\infty(0;\mu) > \beta_\infty,$$

we have

$$\eta_\infty(0;\mu) - \beta_\infty = \mathrm{O}\left(\mu^{-1/3}\right).$$

Moreover, if follows from (3.1) that

$$y_\infty(-1;\mu) - \mu^{-1/3}\beta_\infty = \mathrm{O}\left(\mu^{-2/3}\right).$$

(c) Combining the result of Cartwrigh(1952), we have

$$y_p(-1;\mu) - \mu^{-1/3}\beta_\infty = \mathrm{O}\left(\mu^{-1/3}\right).$$

Hence, we arrive at the surprised result

$$y_p(-1;\mu)-y_\infty(-1;\mu)=O(\mu^{-1/3}).$$

This result confirms that this leading term of the asymptotic solution, $y_\infty(x;\mu)$, is not only a good asymptotic solution up to $x=-1+\mu^{-1/3}$, but also to $x=-1$. Moreover, from the theorem of continuous dependence on initial value of solution of differential equation, we can expect that the behaviour of the limit cycle of Van der Pol equation is closely related to that of $y_\infty(x;\mu)$ when $\mu>>1$.

3.3 Estimation of $y_\infty(x;\mu)$ for $-1\le x\le 0$

Lemma 3.6 For $\mu>>1$, we have

(1)

$$\lim_{\mu\to+\infty}\int_0^{\mu^{1/3}}\frac{1}{\eta_\infty(\xi;\mu)}d\xi=\alpha_\infty-\beta_\infty .$$

(2)

$$\eta_\infty(\mu^{1/3};\mu)=\alpha_\infty-\mu^{1/3}/3+\mu^{2/3}+O(1).$$

Proof.

(1) If $\xi>0,\eta>0$, then we have

$$2\xi\eta+1>2\xi\eta+1-\mu^{-2/3}(\xi^2\eta+\xi).$$

Thus by Lemma 3.1, Eq. (3.2) and (3.3), we have

$$\eta_\infty(\xi;\mu)<\eta_1(\xi,\eta_\infty(0;\mu))\ \text{ in }[0,+\infty). \quad (3.18)$$

Let us consider the following equation

$$\eta_3\eta_3'=(2-\mu^{-1/3})\xi\eta_3+(1-\mu^{-1/3}). \quad (3.19)$$

where $\eta_3'=d\eta_3/d\xi$ and $\mu>1$. It is easily convinced that Eq. (3.19) has similar properties as Eq. (3.3). Let $\eta_{3\infty}(\xi;\mu)$ has similar meaning as $\eta_{1\infty}(\xi)$ and $\eta_3(\xi,a,\mu)$ be the solution of Eq. (3.19) with the initial condition $\eta_3(\xi,a,\mu)=a$.

When $\mu^{1/3}\ge\xi>0$ and $\eta>0$, we have

$$2\xi\eta+1-\mu^{-2/3}(\xi^2\eta+\xi)>(2-\mu^{-1/3})\xi\eta+(1-\mu^{-1/3}).$$

Therefore by Lemma 3.1, Eq. (3.2) and (3.19), we have

$$\eta_3(\xi,\eta_\infty(0;\mu);\mu)<\eta_\infty(\xi;\mu)\ \text{ in }(0,\mu^{1/3}]. \quad (3.20)$$

It follows from Eq. (3.18) and (3.20) that

$$\int_0^{\mu^{1/3}}\frac{1}{\eta_1(\xi,\eta_\infty(0;\mu))}d\xi<\int_0^{\mu^{1/3}}\frac{1}{\eta_\infty(\xi;\mu)}d\xi$$
$$<\int_0^{\mu^{1/3}}\frac{1}{\eta_3(\xi,\eta_\infty(0;\mu);\mu)}d\xi. \quad (3.21)$$

Dividing equation (3.3) by η_1 and noting that $\eta_1(\xi,\eta_\infty(0;\mu))$ is a solution of (3.3), we have

$$\frac{d\eta_1(\xi,\eta_\infty(0;\mu))}{d\xi}=2\xi+\frac{1}{\eta_1(\xi,\eta_\infty(0;\mu))}>2\xi,$$

$$\forall\xi\ge 0, \quad (3.22)$$

and so $\eta_1(\xi,\eta_\infty(0;\mu))>\xi^2$, $\forall\xi\ge 0$. It follows from Proposition 3.3, Lemma 3.4 and (3.22) that

$$\left|\int_0^{\mu^{1/3}}\frac{1}{\eta_1(\xi,\eta_\infty(0;\mu))}d\xi-(\alpha_\infty-\beta_\infty)\right|$$
$$\le\left|\int_0^{\infty}\frac{1}{\eta_1(\xi,\eta_\infty(0;\mu))}d\xi-(\alpha_\infty-\beta_\infty)\right|\xi$$
$$+\int_{\mu^{1/3}}^{\infty}\frac{1}{\eta_1(\xi,\eta_\infty(0;\mu))}d$$
$$\le O(1)+\int_{\mu^{1/3}}^{\infty}\frac{1}{\xi^2}d\xi=O(1). \quad (3.23)$$

We also have

$$\eta_3(\xi,\eta_\infty(0;\mu);\mu)$$
$$=(1-\mu^{-1/3}/2)^{-1/3}(1-\mu^{-1/3})^{2/3}\times\eta_1\times$$
$$\left(\omega(1-\mu^{-1/3})^{-1/3}\xi,\frac{\eta_\infty(0;\mu)}{\omega(1-\mu^{-1/3})^{2/3}}\right) \quad (3.24)$$

where, $\omega=\left(1-\dfrac{\mu^{-1/3}}{2}\right)^{2/3}$.

Since both sides of the above equality are solutions of (3.19) with the same nitial condition $\eta_\infty(0;\mu)$. Then, proceeding as in (3.23) and noting (3.24), we have

$$\left|\int_0^{\mu^{1/3}}\frac{d\xi}{\eta_3(\xi,\eta_\infty(0;\mu);\mu)}-(\alpha_\infty-\beta_\infty)\right|=O(1) \quad (3.25)$$

Hence, it follows from (3.21), (3,23) and (3.25) that

$$\int_0^{\mu^{1/3}}\frac{1}{\eta_\infty(\xi;\mu)}d\xi=\alpha_{\infty-\beta_\infty+O(1)}\quad as\quad \mu\to+\infty .$$

(2) Note that $\eta_\infty(\xi;\mu)$ satisfies (3.2), that is

$$\eta_\infty(\xi;\mu)\eta_\infty'(\xi;\mu)$$
$$=2\xi\eta_\infty(\xi;\mu)+1-\mu^{-2/3}(\xi+\xi^2\eta_\infty(\xi;\mu)). \quad (3.26)$$

Dividing equation (3.26) by $\eta_\infty(\xi;\mu)$ and integrating it from 0 to $\mu^{1/3}$, we have

$$\eta_\infty(\mu^{1/3};\mu)=\eta_\infty(0,\mu)+\mu^{2/3}-\mu^{1/3}/3d\xi .$$
$$+\int_0^{\mu^{1/3}}\frac{1}{\eta_\infty(\xi;\mu)}d\xi-\int_0^{\mu^{1/3}}\frac{\mu^{-2/3}\xi}{\eta_\infty(\xi;\mu)} \quad (3.27)$$

Next, by part (b) of Corollary 3.5, we have

$$\eta_\infty(0;\mu)=\beta_\infty+O(\mu^{-1/3}). \quad (3.28)$$

Moreover, from part (1) of this Lemma, the following inequalities are held:

$$\int_0^{\mu^{1/3}}\frac{1}{\eta_\infty(\xi;\mu)}d\xi=\alpha_\infty-\beta_\infty+O(1), \quad (3.29)$$

and

$$\left|\int_0^{\mu^{1/3}} \frac{\mu^{-2/3}\xi}{\eta_\infty(\xi;\mu)} d\xi\right| \le \mu^{-1/3}\int_0^{\mu^{1/3}} \frac{1}{\eta_\infty(\xi;\mu)} d\xi = O\left(\mu^{-1/3}\right). \quad (3.30)$$

Substituting (3.28), (3.29) and (3.30) into (3.27), we have

$$\eta_\infty\left(\mu^{1/3};\mu\right) = \alpha_\infty - \mu^{1/3}/3 + \mu^{2/3} + O(1).$$

Corollary 3.7 For $\mu >> 1$, we have

$$y_\infty\left(-1+\mu^{-1/3};\mu\right) = -1/3 + \alpha_\infty\mu^{-1/3} + \mu^{1/3} + O\left(\mu^{-1/3}\right).$$

Proof.

It follows from part (2) of Lemma 3.6 and the transformation rule (3.1) relating the $x-y$ and $\xi-\eta$ planes.

Lemma 3.8 For $\mu >> 1$, we have

(a) $y_\infty(0;\mu) = \alpha_\infty\mu^{-1/3} + 2\mu/3 + O\left(\mu^{-1/3}\right)$.

(b) $y_\infty(0;\mu) - y_\infty(0;\mu) = O\left(\mu^{-1/3}\right)$.

(c) $y_\infty(x;\mu) - y_\infty(x;\mu) = O\left(\mu^{-1/3}\right), -1 \le x \le 0$.

Proof.

(a) Since $y_\infty(x;\mu)$ is a solution of (2.1) and decreases on $(-\infty,0]$, then we have

$$\mu\left(1-x^2\right) \le \frac{dy_\infty(x;\mu)}{dx} \le \frac{-x}{y_\infty\left(-1+\mu^{-1/3};\mu\right)}.$$

$$\mu\left(1-x^2\right),\ \forall x \in \left[-1+\mu^{-1/3},0\right]$$

Integrating this inequality from $-1+\mu^{-1/3}$ to 0, we have

$$2\mu/3 + 1/3 - \mu^{1/3} \le y_\infty(0;\mu) - y_\infty\left(-1+\mu^{-1/3};\mu\right) \le 2\mu/3 + 1/3 - \mu^{1/3} + \frac{\left(-1+\mu^{-1/3}\right)^2}{2y\left(-1+\mu^{-1/3};\mu\right)}.$$

Combining the above inequality with Corollary 3.7, we have

$$y_\infty(0;\mu) = \alpha_\infty\mu^{-1/3} + 2\mu/3 + O\left(\mu^{-1/3}\right).\ \text{and}$$

$$y_p(0;\mu) = \alpha_\infty\mu^{-1/3} + 2\mu/3 + O\left(\mu^{-2/3}\right).$$

(b) Hence, it follows from part (a) of this Lemma with the above equation, we arrive at our result.

(c) Since $y_\infty(x;\mu)$ and $y_p(x;\mu)$ are all solutions of Eq. (2.1), then it follows from Eq. (2.1) that the difference between $y_\infty(x;\mu)$ and $y_p(x;\mu)$ is dominated by the following equation

$$\left(y_\infty(x;\mu) - y_p(x;\mu)\right)' = x\frac{y_\infty(x;\mu) - y_p(x;\mu)}{-y_\infty(x;\mu)y_p(x;\mu)}.$$

Hence $y_\infty(x;\mu) - y_p(x;\mu)$ decreases on $(-\infty,0]$. Then it follows from part (b) of this Lemma that

$$y_\infty(x;\mu) - y_p(x;\mu) = O\left(\mu^{-1/3}\right),\ \forall x \in [-1,0].$$

3.4 The proof of the main theorem

Firstly, we state a lemma from Ponzo and Wax (1965) in the following Lemma.

Lemma 3.9 For $\mu >> 1$, every solution $y\ (x,a;\mu)$ of Eq. (2.1) with initial value $a = y_p(0;\mu) + O\left(\mu^{-1/3}\right)$ vanishes at $2 + \frac{1}{3}\alpha_\infty\mu^{-4/3} + O\left(\mu^{-4/3}\right)$.

By Lemma 3.8, 3.9 and similar arguments as in part (c) of Lemma 3.8, it is easily convinced that we have the following corollary.

Corollary 3.10 For $\mu >> 1$,

(a) $y_\infty(x;\mu)$ vanishes at $2 + \alpha_\infty\mu^{-4/3}/3 + O\left(\mu^{-4/3}\right)$.

(b) $y_\infty(x;\mu) - y_p(x;\mu) = O\left(\mu^{-1/3}\right)$ for all $x \ge 0$ such that $y_\infty(x;\mu)$ and $y_p(x;\mu)$ are defined.

Now, we have enough information to get our main result, that is,

Theorem 3.11 In the phase plane, every trajectory $y(x,\beta;\mu)$ of Eq. (1.1) with initial value β bigger than that of limit cycle, $y_p(0;\mu)$, will get close to the limit cycle with error not greater than $O\left(\mu^{-1/3}\right)$ from its first time on intersecting $x=1$ in the fourth quadrant.

Proof.

Since equation (1.1) and (1.2) are symmetric with respect to origin,we have results similar to all the above lemmas for $y \le 0$ .Hence from Theorem 2.7 and Corollary 3.5, $y(x,\beta;\mu)$ is only bigger than $y_p(x)$ by $O\left(\mu^{-1/3}\right)$. Furthermore, from Theorem 2.7, part (c) of Lemma 3.8 and part (b) of Corollary 3.10, this theorem is proved.

4 Conclusion

In this paper, we discuss some properties of a special trajectory, $y_\infty(x;\mu)$, of Van der Pol equation in the phase plane, and use these to see that a solution of Van der Pol equation with initial value bigger than $y_p(0)$ will get close to the limit cycle for $\mu >> 1$ from the point at which this trajectory cut $x=1$ in the fourth quadrant. However, we don't calculate how much time this solution takes to get close to the limit cycle. Concerning this question which is also the stability of Van der Pol equation, the first problem we meet is how much time the solution with initial value bigger than

$y_p(0)$ takes to intersect the positive x-axis. For this, it concerns the behaviour of the differential equation (2.1) in the first quadrant, and needs finer analysis for any solution curve with initial value bigger than $y_p(0)$. The second one is that the solution curve of equation (2.1) starting from the positive x-axis has to confine between $y_\infty(x;\mu)$ which is meant to be the corresponding part of $y_\infty(x;\mu)$ for $y \le 0$, and the implicit curve $2\mu x y^3 + y^2 + \mu x(x^2 - 1)y + x^2 = 0$, at points of which the second derivative of the solution of equation (2.1) is equal to zero. And it is easily convinced that when going through the negative x values, every solution curve of equation (2.1) starting from the positive y-axis and touching this implicit curve will intersect the negative x-axis, and these two curve are very close for $x \le -1$. However, it is not a easy task to calculate the approximating time, that is, the integral $\int dx/y$ where (x,y) satisfies the equation of the implicit curve.

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Van Der Pol 方程式在相平面上臨界曲線之研究

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本文探討 Van der Pol 方程式 $x''+\mu x'(x^2-1)+x=0$ 在相平面上一條特殊臨界曲線，記為 $y_\infty(\chi;\mu)$。它是 Van der Pol 方程式在相平面上特定區域中對於極限環的漸進解。本研究證明在相平面的上半平面中，Van der Pol 方程式的極限環與臨界曲線 $y_\infty(\chi;\mu)$ 之差至多為 $O(\mu^{-1/3})$，當 $-1\le x\le 0$, $\mu\to+\infty$。更進一步，可以利用這個結果，證明當 $\mu\to+\infty$ 時，相平面上任一條 Van der Pol 方程式的解軌線從 y 軸出發且在極限環外部時，當第一次與 x=1 相交於第四象限之後，其與極限環的差至多為 $O(\mu^{-1/3})$。

關鍵字：Van der Pol 方程式、極限環