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Critical Portraits and Fixed Point Portraits Large for Polynomial Maps

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Abstract

From the work of Goldberg-Milnor [GM], a candidate of fixed point portraits $\mathscr{P}$ under specific conditions can generate critical portraits Θ, though there are still many possible choices for Θ. Using Fisher's Main Theorem, there is a unique monic polynomial f which is critically periodic and realizes one of the critical portrait Θ and in turn realizes the fixed point portrait $\mathscr{P}$. At the end we pay our attention to the case of critical portraits of degree 3 and classify them.

Key Words. Complex dynamics, iteration for polynomial maps, rotation number, fixed point portraits.

1 Introduction

The subject of complex dynamics studies iterations of rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and discuss the behavior of forward orbits of each point $z \in \hat{\mathbb{C}}$. The Fatou set $\mathcal{F}$ is the maximal open set of $\hat{\mathbb{C}}$ on which $\{f^n\}$ is equicontinuous and the Julia set $\mathscr{J}$ is the complement of $\mathcal{F}$.

We only discuss rational maps whose degrees are greater or equal to 2. Constant maps are trivial cases without any prerequisites. Any Möbius map which fixes only one point has the property every forward orbit converges to this unique fixed point. For the other Möbius maps, any forward orbit of $z \in \hat{\mathbb{C}}$ other than those two fixed points either converges to one of the two fixed points, has a finite orbit, or is a dense subset of some circle.

The Fatou set and the Julia set are invariant under f. In particular, $\mathscr{J}$ is the closure of a backward orbit of any point in $\mathscr{J}$ and it is a nonempty perfect set. Attracting cycles lie in $\mathcal{F}$, while repelling ones and rationally indifferent ones lie in $\mathscr{J}$, but irrationally indifferent cycles may lie in $\mathcal{F}$ or $\mathscr{J}$. Moreover, the Julia set is the closure of the repelling periodic points.

Now, we consider a more limited situation where f is a polynomial map. Note that the point of ∞ is a superattracting fixed point of f, and the basin $\mathcal{A}(\infty)$ of ∞ is the component of $\mathcal{F}$ that contains ∞. The filled Julia set $\mathcal{K}$ is the complement of $\mathcal{A}(\infty)$ and its boundary is $\mathscr{J}$.

Finally, consider the family of quadratic functions $\{f_c(z) = z^2 + c \,|c \in \mathbb{C}\}$. We mention that the Mandelbrot set $\mathcal{M}$ is the set consisting of c whose orbit under f_c is bounded or, equivalently, never diverges to ∞. A more amazing fact is that $\mathcal{M}$ is also the set consisting of c with f_c has the connected filled Julia set, or equivalently, f_c has the connected Julia set.

In this article, we devote ourselves to the question that whether given a candidate of fixed point portrait $\mathscr{P}$ and d, there exists a polynomial degree $d \geq 2$ with connected Julia set. The organization of the article is as follow:

In Section 2, we introduce basic results about external rays in [J]. Most external rays land and every rational ray must land. A fixed ray must land at a repelling or parabolic fixed point. There is at least one rational ray landing at each repelling or parabolic periodic points.

In Section 3, we introduce the concepts of rotation numbers of a monotone degree one circle map and rotation sets. The main facts are Theorems 3.2 and 3.3.

In Section 4, we construct d monotone degree one circle maps ϕ_j's from a formal critical portrait Θ with properties C1-C5. Applying the Fisher's Main Theorem to a given critical portrait, we get a monic polynomial f which realizes it. Furthermore, we note that the fixed point portrait is the repelling periodic points of each ϕ_j.

In Section 5, in a special condition that a candidate fixed point portrait $\mathscr{P}$ with d rational types satisfying P1-P4, we may have a polynomial f to realize $\mathscr{P}$ by way of those results in Section 4.

In Section 6, given $d = 3$, we classify all possible fixed point portraits by way of critical portraits. We specifically carry out degree d $\frac{p}{q}$-rotation cycles with the aid of computer.

2 Extenal Rays

Standing Hypothesis 2.1. *Throughout this section, we assume that* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is a monic polynomial map*

$$f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0 \quad (d \geq 2)$$

with the Julia set $\mathscr{J}$ *connected, or equivalently,* $\mathcal{K}$ *is connected.*

Definition 2.1. *With Standing Hypothesis 2.1, there is the Böttcher isomorphism* $\phi : \hat{\mathbb{C}} \setminus \mathcal{K} \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *such that for any* z *in* $\hat{\mathbb{C}} \setminus \mathcal{K}, \phi(f(z)) = (\phi(z))^d$*. For any* t *in* $[0,1)$, $\mathcal{R}_t = \{\phi^{-1}(re^{2\pi it}) \,|r > 1\}$ *is called an* external ray *for* $\mathcal{K}$*. The orthogonal trajectories of the external rays for* $\mathcal{K}$ *are called* equipotential curves *around* $\mathcal{K}$*. Furthermore,* $\phi^{-1}(re^{2\pi it})$ *converges to a point, denoted by* $\gamma(t)$*, in* $\mathscr{J}$ *as* $r \searrow 1$ *and we say that* $\mathcal{R}_t$ lands at $\gamma(t)$.

The continuous function $G : \mathbb{C} \to \mathbb{R}$ *defined by*

$$G(z) = \begin{cases} \log|\phi(z)| > 0 & \textit{if} \quad z \in \mathbb{C} \setminus \mathcal{K}; \\ 0 & \textit{if} \quad z \in \mathcal{K} \end{cases}$$

is called the Green's function for $\mathcal{K}$*, and* $G(f(z)) = nG(z)$ *for all complex numbers* z*.*

Definition 2.2. *An external ray* $\mathcal{R}_t$ *is called* rational *if* $t \in \mathbb{Q}/\mathbb{Z}$*. It is called* periodic *if there is a positive number* p *such that* $d^p t \equiv t \pmod{\mathbb{Z}}$ *and is called* eventually periodic *if there is nonnegative number* q *such that* $d^q t \pmod{\mathbb{Z}}$ *is periodic.*

Remark In particular, $\mathcal{R}_t$ is rational if and only if $\mathcal{R}_t$ is eventually periodic. $\mathcal{R}_t$ is periodic if and only if t is rational with denominator relatively prime to d.

Lemma 2.1. *If the ray* $\mathcal{R}_t$ *lands at a point* $\gamma(t) \in \mathscr{J}$*, then the ray* $\mathcal{R}_{dt}$ *lands at the point* $f(\gamma(t))$*. Furthermore, each of the* d *rays of the form* $\mathcal{R}_{\frac{t+j}{d}}$ *lands at one of the points in* $f^{-1}(\gamma(t))$*, and every point in* $f^{-1}(\gamma(t))$ *is the landing point of at least one such ray.*

Proof. In case that $z = \gamma(t)$ is not a critical point, there exist a neighborhood N of z and a neighborhood N' of $f(z)$ such that $f : N \to N'$ is biholomorphic by the Inverse Function Theorem. Note that

$$f \circ \phi^{-1}\left(re^{i2\pi s}\right) = \phi^{-1}\left(\left(re^{i2\pi s}\right)^d\right) = \phi^{-1}\left(r^d e^{i2\pi ds}\right),$$

then $\gamma(ds) = f \circ \gamma(s)$ as $r \searrow 1$. Similarly, $f^{-1} \circ \gamma(ds) = \gamma(s)$. Hence $\mathcal{R}_{\frac{t+j}{d}}$ lands at $w \in f^{-1}(\gamma(t))$.

In the other case, there exist a neighborhood N of z, and a neighborhood N' of $f(z)$ such that $f : N \to N'$ is a branched covering with finite sheets.

A similar argument shows that $\mathcal{R}_{\frac{t+j}{d}}$ lands at $w \in f^{-1}(\gamma(t))$.

In the argument, we see that every point in $f^{-1}(\gamma(t))$ has such a ray landing at it. □

Lemma 2.2. *If a periodic ray lands at z_0, then only finite rays land at z_0, and these rays are all periodic of the same period.*

Proof. Here we consider the rays case by case, the first for the fixed rays, the second for the periodic rays.

Assume $\mathcal{R}_t$ is a fixed ray, say $t = \frac{j}{d-1}, 0 \leq j < d-1$. Clearly, z_0 is a fixed point of f. Let $X = \{x \in \mathbb{R}/\mathbb{Z} \quad | \mathcal{R}_x \quad \text{lands at} \quad z_0\}$. Assume that $x \in X$, f maps a neighborhood of z_0 diffeomorphically onto a neighborhood of z_0, preserving the cyclic order of the rays landing at z_0, then the d-map $\alpha \mapsto d\alpha \pmod{\mathbb{Z}}$ is injective and preserves the cyclic order.

Now, $x \not\equiv t \pmod{\mathbb{Z}}$ implies for any nonnegative integers $k, x^k \not\equiv t \pmod{\mathbb{Z}}$, then we can construct a sequence $\left\{x_k = d^k x \pmod{\mathbb{Z}}\right\}_{k=0}^{\infty}$. Neither $x_1 > x_0$ nor $x_1 < x_0$ hold, hence x is a fixed point of d-map; otherwise, $x_k \to \hat{x}$ which is a fixed point of d-map but there are only repelling fixed ones.

Hence X is the set consisting of fixed points of d-map. More precisely, these rays are also fixed rays.

Assume $\mathcal{R}_t$ is a periodic ray with period $p > 1$.

Note that $\mathcal{R}_t$ is a fixed ray of f^p, then the same argument shows the conclusion. □

Standing Hypothesis 2.2. *From Theorem 2.1 to Theorem 2.3, we assume $\psi : \mathbb{D} \to U$ is a conformal isomorphism and $U \subset \hat{\mathbb{C}}$ is a simply connected open set.*

We refer the proof of Theorem 2.1 to p. 177 of [J].

Theorem 2.1 ([J], p. 177)**.** *Suppose $\psi : \mathbb{D} \to U$ is a conformal isomorphism and $U \subset \hat{\mathbb{C}}$ is a simply connected open set. For each $e^{i\theta} \in \partial\mathbb{D}$, $\left\{re^{i\theta} | 0 \leq r < 1\right\}$ maps under ψ to a curve of finite spherical length in U. In particular, the radial limit $\lim_{r \nearrow 1} \psi\left(re^{i\theta}\right) \in \partial U$ exists for almost every θ.*

However, if we fix any particular point $u_0 \in \partial U$, then the set θ such that this radial limit is equal to u_0 has Lebesgue measure zero.

Corollary 2.1. *If $\theta_1 \neq \theta_2$ land at the same point $u_0 \in \partial U$, then u_0 disconnects ∂U.*

Proof. Note that θ_1, θ_2 divide the unit circle into two open arcs A, B with nonzero length. It follows from Theorem 2.1 that there are $a \in A, b \in B$ such that $\mathcal{R}_a, \mathcal{R}_b$ landing at different points of ∂U. Hence u_0 does disconnect ∂U. □

We refer the proof of Theorem 2.2 to p. 183 of [J].

Theorem 2.2 ([J], p. 183)**.** *$\psi : \mathbb{D} \to U$ can be extended to a continuous map $\bar{\psi} : \bar{\mathbb{D}} \to \bar{U}$ if and only if ∂U is locally connected if and only if $\hat{\mathbb{C}} \setminus U$ is locally connected.*

Theorem 2.3. *If ∂U is a Jordan curve, then $\psi : \mathbb{D} \to U$ extends to a homeomorphism $\bar{\psi} : \bar{\mathbb{D}} \to \bar{U}$.*

Proof. It suffices to show that $\bar{\psi}$ is injective by Theorem 2.2.

Otherwise, there are two distinct points θ_1, θ_2 landing at the same point $u_0 \in \partial U$ which implies that u_0 disconnects ∂U that is a Jordan curve, which is a contradiction. □

Theorem 2.4. *Suppose X is a locally connected and compact space and $f : X \to Y$ is continuous and onto, where Y is a Haussdorff space, then Y is locally connected and compact.*

Proof. Since f is continuous from a compact space X onto Y, then Y is compact. It remains to show that Y is locally connected.

Suppose $y \in Y$, V is an open neighborhood of y. Clearly, $f^{-1}(V)$ is open in X. For all $x \in f^{-1}(y)$, there exists a connected open neighborhood V_x of x contained in $f^{-1}(V)$. It follows that $\bigcup_{x \in f^{-1}(y)} f(V_x)$ is connected.

Note that $f(X \setminus V_x)$ is closed in Y for each x in $f^{-1}(y)$ and

$$Y \setminus \bigcup_{x \in f^{-1}(y)} f(V_x) = \bigcap_{x \in f^{-1}(y)} (Y \setminus f(V_x)) \subset \bigcap_{x \in f^{-1}(y)} f(X \setminus V_x),$$

then $\bigcap_{x \in f^{-1}(y)} f(X \setminus V_x)$ is closed and

$$y \in Y \setminus \bigcap_{x \in f^{-1}(y)} f(X \setminus V_x) \subset \bigcup_{x \in f^{-1}(y)} f(V_x) \subseteq V.$$

This completes our proof. □

Theorem 2.5. *For almost every t in $\mathbb{R}/\mathbb{Z}$, the ray $\mathcal{R}_t$ has a well-defined landing point $\gamma(t) \in \mathscr{J}$. Furthermore, for any fixed $z_0 \in \mathscr{J}$, $X_0 = \{t \in \mathbb{R}/\mathbb{Z} \,|\, \gamma(t) = z_0\}$ has measure zero.*

Proof. Let $\varphi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \mathbb{D}$ be difined as $\varphi(w) = \frac{1}{w}$ $\forall w \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Now let $\psi : \mathbb{D} \to \hat{\mathbb{C}} \setminus \mathcal{K}$ as $\psi = \phi^{-1} \circ \varphi^{-1}$ and apply Theorem 2.1. For almost t in $\mathbb{R}/\mathbb{Z}$, $\lim_{r \nearrow 1} \phi^{-1} \circ \varphi^{-1} \left(re^{-i2\pi t}\right) \in \mathscr{J}$. Furthermore, X_0 is the set consisting of t in $\mathbb{R}/\mathbb{Z}$ whose radial limit equals z_0. This completes our proof. □

Theorem 2.6 ([J], p. 191)**.** *Given a polynomial f with connected $\mathscr{J}$, the following are equivalent:*

(1) *$\gamma(t)$ is continuous.*

(2) *$\mathscr{J}$ is locally connected.*

(3) *$\mathcal{K}$ is locally connected.*

(4) *$\phi^{-1} : \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{K}$ can be extended continuously over $\partial\mathbb{D}$, and the extension carries each $e^{2\pi i t} \in \partial\mathbb{D}$ to $\gamma(t) \in \mathscr{J}$.*

Proof. Consider ψ in the proof of Theorem 2.5 and Theorem 2.2, and this completes the proof the equivalence of the latter three statements.

It is straightforward that the fourth statement implies the first one. It remains to prove the converse is true.

$\gamma\left(\mathbb{R}/\mathbb{Z}\right)$ is compact and locally connected by Theorem 2.4. Without loss of generality, say $\gamma(0) \in \mathscr{J}$, then $O^{-}\left(\gamma(0)\right) \subseteq \gamma\left(\mathbb{R}/\mathbb{Z}\right)$. Hence $\mathscr{J} = \overline{O^{-}\left(\gamma(0)\right)} \subseteq \overline{\gamma\left(\mathbb{R}/\mathbb{Z}\right)} \subseteq \mathscr{J}$ implies $\gamma\left(\mathbb{R}/\mathbb{Z}\right) = \mathscr{J}$. □

Corollary 2.2. *$\mathscr{J}$ is a Jordan curve if and only if $\gamma : \mathbb{R}/\mathbb{Z} \to \mathscr{J}$ homeomorphically.*

Proof. The necessary condition is a direct consequence of Theorem 2.3 and the sufficient one is the verification of what a Jordan curve is. □

We refer the proof of Theorem 2.3 to p. 168 of [J], which is called the Snail lemma.

Lemma 2.3 ([J],p. 168)**.** *Suppose* $g(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$. *A path* $p : [0, \infty)$ *in* $V \setminus \{0\}$ *where* V *is neighborhood of* 0 *converges to* 0 *and*

$$g(p(t)) = p(t+1), \quad \forall \quad t \geq 0, \tag{2.1}$$

then either $|\lambda| < 1$ *or* $\lambda = 1$.

Corollary 2.3. *If (2.1) is replaced with*

$$g(p(t)) = p(t-1), \quad \forall \quad t \geq 1, \tag{2.2}$$

then either $|\lambda| > 1$ *or* $\lambda = 1$.

Proof. Note that the orbit

$$\cdots \mapsto p(2) \mapsto p(1) \mapsto p(0)$$

is repelled by the origin, then $\lambda \neq 0$.

We have the holomorphic inverse $g^{-1} : W \to V$ of f where $W \subset f(V)$ is a neighborhood of 0 with the derivative λ^{-1} at 0. From (2.2), we see that

$$g^{-1}(p(t)) = p(t+1), \quad \forall \quad t \quad \text{sufficiently large.}$$

Apply Lemma 2.3 to g^{-1} on W, we see either $\lambda = 1$ or $|\lambda| > 1$. □

Lemma 2.4. *If a fixed ray* $\mathcal{R}_t$ *lands at* z_0*, then* z_0 *is either a repelling or parabolic fixed point.*

Proof. It is straightforward that z_0 is a fixed point.

Since $\phi : \hat{\mathbb{C}} \setminus \mathcal{K} \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is a conformally isomorphism and the Green's function $G : \hat{\mathbb{C}} \setminus \mathcal{K} \to \mathbb{R}^+$, then

$$\forall s \in \mathbb{R}, e^s > 0, \exp(e^s) > 1, \exists! z \in \hat{\mathbb{C}} \setminus \mathcal{K} \ni \phi(z) = \exp(e^s + 2\pi i t), \tag{2.3}$$

i.e., $z \in \mathcal{R}_t$.

Define $p : \mathbb{R} \to \hat{\mathbb{C}} \setminus \mathcal{K}$ as follows:

$$p(s) \mapsto z \ni \phi(z) = \exp(e^s + 2\pi i t) \Rightarrow G(z) = e^s.$$

Hence p is well-defined and continuous by (2.3). Also, note

$$G(f(p(s))) = \log |\phi(p(s))|^n = nG(p(s)),$$

then we see that

$$p(s + \log n) = f(p(s))$$

after taking the logarithm on both sides. Let $p' : [k, \infty) \to V, p'(t) = p(-t)$ where V contains $p((-\infty, -k])$, for some $k > 0$.

Hence,by Corollary 2.3 , either $|f'(z_0)| > 1$ or $f'(z_0) = 1$. □

Finally, we state Theorem 2.7 and Theorem 2.8 without proof. Their proofs are in Chapter 18 of [J]. They state that rational rays must land and repelling and parabolic points are landing points.

Theorem 2.7 ([J],p. 195)**.** *Every periodic external ray lands at a either repelling or parabolic periodic point. Moreover, a rational but non-periodic ray lands at a strictly eventually periodic point.*

Theorem 2.8 ([J],p. 195)**.** *Every repelling or parabolic periodic point is the landing point of at least one periodic ray.*

3 Rotation Numbers

Definition 3.1. $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *is called a* monotone degree one circle map *if there is a lift* $\Psi : \mathbb{R} \to \mathbb{R}$*, monotone, continuous and* $\Psi(u+1) = \Psi(u) + 1, \quad u \in \mathbb{R}$.

Proposition 3.1. *Given a monotone degree one circle map* $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *, there exists a unique lift* $\Psi : \mathbb{R} \to \mathbb{R}$ *up to addition of an integer constant.*

Theorem 3.1. *Given a lift* Ψ *of a monotone degree one circle map* ψ *and a real number* $u \in \mathbb{R}$*, the limit*

$$\lim_{n\to\infty} \frac{\Psi^n(u) - u}{n} \tag{3.1}$$

exists. This limit is independent of u*.*

Proof. For any two real numbers u, v, there exists an integer k such that such that $u \leq v + k < u + 1$. Apply the monotonicity of Ψ and use the induction, then $\Psi^n(u) \leq \Psi^n(v+k) < \Psi^n(u+1)$. Substracting it with $u+1$, we see that $\Psi^n(u) - u - 1 < \Psi^n(v) - v < \Psi^n(u) - u + 1$. Hence,

$$\forall u, v \in \mathbb{R}, \forall n \in \mathbb{N}, \Psi^n(u) - u - 1 < \Psi^n(v) - v < \Psi^n(u) - u + 1. \tag{3.2}$$

Uniqueness follows from (3.2). It remains to show the existence of the limit. Assume that u is a real number. For any two natural numbers p, q, set $v = \Psi^q(u)$.

From (3.2), we see that

$$\Psi^{p+q}(u) < \Psi^q(u) + \Psi^p(u) - u + 1, \tag{3.3}$$

and $\Psi^{2p}(u) < 2\Psi^p(u) - u + 1$ as $p = q$. Inductively, we have

$$\Psi^{kp}(u) - u < k\left(\Psi^p(u) - u\right) + (k-1) \tag{3.4}$$

for each postive integer k. At last, for each positive integer n, there are integers $k, i, k \geq 0, 0 \leq i < p$ such that $n = kp + i$.

First, we see $\Psi^n(u) - u < \Psi^{kp}(u) + \Psi^i(u) - u + 1 - u$ in (3.3) and deduce that $\Psi^n(u) - u < k\left(\Psi^p(u) - u + 1\right) + \Psi^i(u) - u$ via (3.4). Note that

$$\frac{\Psi^n(u) - u}{n} < \frac{\left(\Psi^p(u) - u + 1\right)}{p} + \frac{\Psi^i(u) - u}{n}$$

and fix p, let $n \to \infty$, and let $p \to \infty$, then we see that

$$\limsup_{n\to\infty} \frac{F^n(u) - u}{n} \leq \liminf_{p\to\infty} \frac{F^p(u) - u}{p}. \tag{3.5}$$

□

The following definition is well-defined via Theorem 3.1.

Definition 3.2. *Given a lift Ψ of a monotone degree one circle map ψ, we define*

$$\mathrm{Trans}\,(\Psi) = \lim_{n\to\infty} \frac{\Psi^n(u) - u}{n}, \quad \textit{for some} \quad u \in \mathbb{R}. \tag{3.6}$$

The limit is called the translation number *of* Ψ.

The following two propositions are easy calculation rules for the translation number.

Proposition 3.2. *For any lifts $\Psi, \Psi + k$ of a monotone degree one circle map ψ where $k \in \mathbb{Z}$,*

$$\mathrm{Trans}\,(\Psi + k) = \mathrm{Trans}\,(\Psi) + k. \tag{3.7}$$

Proposition 3.3. *For any lift Ψ of a monotone degree one circle map ψ, assume $m \in \mathbb{N}, k \in \mathbb{Z}$*

$$\mathrm{Trans}\,(\Psi^m + k) = m\,\mathrm{Trans}\,(\Psi) + k. \tag{3.8}$$

Proof. Choose $u \in \mathbb{R}$.

Note that, for any positive number n, $(\Psi^m + k)^n(u) - u = (\Psi^m)^n(u) + nk - u$, then we may take the limit after dividing the identity by n. □

The following definition is well-defined via Proposition 3.3.

Definition 3.3. *Given a monotone degree one circle map $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, define the* rotation number *of ψ to be*

$$\rho(\psi) = \mathrm{Trans}\,(\Psi) \mod \mathbb{Z} \quad \textit{, where} \quad \Psi \quad \textit{is a lift of} \quad \psi. \tag{3.9}$$

Theorem 3.2. *Assume that ψ is a monotone degree one circle map. The rotation number of ψ is rational, say $\rho(\psi) = \frac{p}{q}$ in reduced form, if and only if ψ has a periodic point with period q. Furthermore, every orbit under ψ is either periodic or tends asymptotically to a one-sided attracting or attracting periodic orbit.*

Proof. It is clear that the sufficient condition holds. For the necessary condition, given $\rho(\psi) = \frac{p}{q}$, let $\Psi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Trans}(\Psi) = \frac{p}{q}$ and $G(t) = \Psi^q(t) - p$ implies $\mathrm{Trans}(G) = 0$. Consider the value $G(0)$, there are three possiblities $G(0) = 0$, $G(0) > 0$, and $G(0) < 0$.

The first condition $G(0) = 0$ implies that 0 is a fixed point of G. For the second possibility, we have $0 < G(0) < G^2(0) < \cdots \leq 1$ which achieves our proof. The same reason for the third possibility. In fact, we can have $G(t) = t$, $t < G(t) < G^2(t) < \cdots \leq t + 1$, or $t > G(t) > G^2(t) > \cdots \geq t - 1$ in this fashion. This finishes our last assertion. □

Definition 3.4. *A d-fold covering map $f_d : S^1 \to S^1$ where $f_d(\theta) = d\theta \mod \mathbb{Z}$ is given. Suppose $\Theta = \{0 \le \theta_0 < \theta_1 < \cdots < \theta_{n-1} < 1\}$ where $0 \le m \le n$ satisfies*

$$\forall i = 0, 1, 2, \ldots, n, \quad f_d(\theta_i) = \theta_{(i+m) \mod n},$$

then Θ is called a degree d $\frac{m}{n}$-rotation set.

Also, if $m = kp, \quad n = kq$ where $k \in \mathbb{N}, p, q \in \mathbb{Z}$, then Θ will be composed of k q-cycles. Each q-cycle is called a degree d $\frac{p}{q}$-rotation cycle. *And the rotation number of Θ is $\frac{p}{q}$.*

Lemma 3.1. *$\forall q \ge 2$, the q-cycles under f_d are in one-to-one correspondence with q-orbits consisting of numbers with base d numeral system in $(0, 1)$ under d-shift.*

Proof. For each $i = 0, 1, \ldots, d-1$, let $A_i = \left(\frac{i}{d}, \frac{i+1}{d}\right)$. Define $\gamma : \bigcup_{i=0}^{d-1} A_i \to \{0, 1, \ldots, d-1\}$ as follows: $\theta \in A_i \mapsto i$. Suppose $\{\theta_0 < \theta_1 < \cdots < \theta_{q-1}\}$ is a q-cycle under f_d, then, for any $j = 0, 1, \ldots, q-1$, let $a_j = 0.\overline{\gamma(\theta_j)\gamma\left(f_d(\theta_j)\right)\ldots\gamma\left(f_d^{q-1}(\theta_j)\right)}$. In fact,

$$a_j = \left(\frac{\gamma(\theta_j)}{d} + \frac{\gamma\left(f_d(\theta_j)\right)}{d^2} + \cdots + \frac{\gamma\left(f_d^{q-1}(\theta_j)\right)}{d^q}\right) \cdot \frac{d^q}{d^q - 1}.$$

□

Definition 3.5. *Let $\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_{n-1}\} \subseteq S^1$. $\forall j = 1, 2, \ldots, d-1, s_j = \#\left(\left\{\theta_i \,\middle|\, \theta_i \in \left[0, \frac{j}{d-1}\right)\right\}\right)$. Clearly, $s_1 \le s_2 \le \cdots \le s_{d-1} = n$. Then $(s_1, s_2, \ldots, s_{d-1})$ is called the degree d* deployment sequence *of Θ.*

Definition 3.6. *Let $\theta \in S^1$. θ is* advancing *if $f_d(\theta) > \theta$; otherwise, θ is* retreating.

Lemma 3.2. *A degree d rotation set is completely determined by its rotation number $\frac{p}{q}$ together with its deployment sequence $s_1 \le s_2 \le \cdots \le s_{d-1} = kq$.*

Proof. For each $j = 1, 2, \ldots, d-1$, let $U_j = \left(\frac{j-1}{d-1}, \frac{j}{d-1}\right)$.

$$U_{j,adv} = \left(\frac{j-1}{d-1}, \frac{j}{d}\right) \subset \left(\frac{j-1}{d}, \frac{j}{d}\right) \tag{3.10}$$

$$U_{j,adv} = \left(\frac{j}{d}, \frac{j}{d-1}\right) \subset \left(\frac{j}{d}, \frac{j+1}{d}\right) \tag{3.11}$$

Suppose $\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_{kq-1}\} \subset S^1$. Clearly, we see that f_d advances $\theta_0, \theta_1, \ldots, \theta_{kq-kp-1}$ and retreats the others. For each element in Θ, follow the proof of Lemma 3.1 and (3.10) and (3.11), and we may write down the cyclic fractions with base d. $\square$

Definition 3.7. *Let $\Theta = \{\theta_0 \le \theta_1 \le \cdots \le \theta_{kq-1}\} \subset S^1$ be points which are not fixed by f_d. Then the complement of Θ is composed of kq arcs joining θ_j and $\theta_{j+1 \mod kq}$ which is denoted by A_j whose length $\ell(A_j) = \theta_{j+1} - \theta_j$ for $j = 0, 1, \cdots, kq-2$ and $\ell(A_{kq-1}) = 1 + \theta_0 - \theta_{kq-1}$, and whose weight $\omega(A_j) = \#\left(\left\{\frac{i}{d-1} \,\middle|\, \frac{i}{d-1} \in A_j\right\}\right)$. In the case $\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_{kq-1}\}$, we may have the length of every arc is nonzero.*

Lemma 3.3. *Let $\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_{kq-1}\} \subset S^1$ be a degree d rotation set with rotation number $\frac{p}{q}$ and complementary arcs $A_0, A_1, \ldots A_{kq-1}$.*

Then

$$d\ell(A_i) = \ell\left(A_{i+kp \mod kq}\right) + \omega(A_i) \quad \forall i = 0, 1, \ldots, kq-1. \tag{3.12}$$

Also, $f_d : A_i \xrightarrow{\cong} A_{i+kp \mod kq}$ if and only if $\omega(A_i) = 0$.

Proof. Note that

$$\begin{aligned} d\ell(A_i) &= d(\theta_{i+1 \mod kq} - \theta_i) \\ &= f_d(\theta_{i+1 \mod kq}) - f_d(\theta_i) + z \\ &= \theta_{(i+1+kp) \mod kq} - \theta_{(i+kp) \mod kq} + z \\ &= \ell(A_{(i+kp) \mod kq}) + z \end{aligned}$$

for some $z \in \mathbb{Z}$ and $z = \omega(A_i)$. $\square$

Lemma 3.4. *For each $i = 0, 1, 2, \ldots, d-1$,*

$$\begin{aligned} &(d-1)\left(\ell(A_i) + \ell(A_{i+k}) + \cdots + \ell(A_{i+k(q-1)})\right) \\ &\quad = \omega(A_i) + \omega(A_{i+k}) + \cdots + \omega(A_{i+k(q-1)}). \end{aligned} \tag{3.13}$$

Given the rotation set $\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_{kq-1}\}$, this equation is greater than zero. Therefore there is a nonnegative number z such that $\omega(A_{i+kz}) > 0$. If $\omega(A_i) = 0$, then $A_i \cong f_d(A_i)$; otherwise, $\omega(A_i) > 0$.

Proof. Given this i, apply (3.12) to $i, i+k, i+2k, \ldots, i+k(q-1)$. This completes our proof. $\square$

Lemma 3.5. *A sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d-1} = kq$ is realized by a degree d rotation set if and only if $\{[0], [1], \ldots [k-1]\} = \{[s_1], [s_2], \ldots [s_{d-1}]\}$.*

Proof. Because (3.13) holds, it is positive if and only if one of the length of some arc is positive which means the strict inequality holds. □

Corollary 3.1. $k \leq d-1$. *Furthermore, a degree d rotation set with rotation number $\frac{p}{q}$ has at most $(d-1)q$ points.*

Proof. Otherwise, the deployment sequence cannot be realized . □

Theorem 3.3. *A degree d rotatation set is uniquely determined by its rotation number $\frac{p}{q}$ and its deployment sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d-1} = kq$. Conversely, a indivisible fraction $\frac{p}{q}$ and candidate deployment sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d-1} = kq$ determine a rotation set if and only if $\{[0], [1], \ldots [k-1]\} = \{[s_1], [s_2], \ldots [s_{d-1}]\}$*

Proof. This is the consequence of Lemma 3.2 and Lemma 3.5. □

Theorem 3.4. *f_d has $\binom{d+q-2}{q}$ rotation cycles with rotation number $\frac{p}{q}$.*

Proof. A deployment sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d-1} = q$ represents a way of the permutation on

$$q \quad \text{triangles} \quad \triangle, \triangle, \ldots, \triangle \quad \text{and} \quad d-2 \quad \text{sticks} \quad |, |, \ldots, |.$$

This completes the proof. □

We use Maple to find out all possible degree 3 5-rotation cycles, see Appendix. In fact, we demostrate the algorithm in Algorithm 6.1. One may follow the procedure to find out what he needs.

4 Critical Portraits

In this section, we assume Standing Hypothesis 2.1 and the following hypothesis.

Standing Hypothesis 4.1. *Assume f is a monic polynomial of degree $d \geq 2$ and every critical point of f must be a landing point of at least one external ray.*

Definition 4.1. *Two subsets $A, B \subset \mathbb{R}/\mathbb{Z}$ are* unlinked *if there are two disjoint components U, V of $\mathbb{R}/\mathbb{Z}$ such that $A \subset U, B \subset V$.*

Definition 4.2. *A* critical portrait *of f is a finite set $\Theta = \{\Theta_1, \Theta_2, \dots, \Theta_s\}$ consisting of those finite types of all critical points of f satisfying three conditions:*

(1) *Every external ray $\mathcal{R}_t$, where $t \in \Theta_j$ for some j, must land at the corresponding critical point ω_j.*

(2) *For any two $x, y \in \Theta_j$, $x = y \pmod{\frac{1}{d}}$.*

(3) *For any $j = 1, 2, \dots, s$, $\#\Theta_j = \mu_j + 1$ where μ_j is the multiplicity of ω_j.*

From the Böttcher conformal isomorphism, we see that the Θ_j are disjoint and pairwise unlinked. And this contributes to our next step.

Lemma 4.1. *Given a critical portrait $\Theta = \{\Theta_1, \Theta_2, \dots, \Theta_s\}$ of f, the following conditions hold:*

C1 *The Θ_j are disjoint and pairwise unlinked.*

C2 *For any two $x, y \in \Theta_j$, $x = y \pmod{\frac{1}{d}}$.*

C3 *For any $j = 1, 2, \dots, s$, $\#\Theta_j \geq 2$ and $\sum_{j=1}^{s} (\#\Theta_j - 1) = d - 1$.*

Moreover, we call any finite collection of finite sets of angles which satisfies these three conditions a formal critical portrait.

Before we proceed to the Fisher's Main Theorem, we illustrate how to write down the itinerary of t.

Given a formal critical portrait $\Theta = \{\Theta_1, \Theta_2, \dots, \Theta_s\}$, we draw all the external rays and the landing points. This divide the complex plane into d non-overlapping regions, say $\Omega_1, \Omega_2, \dots, \Omega_d$, and the Julia set $\mathscr{J} = \mathscr{J}_1 \bigcup \mathscr{J}_2 \bigcup \dots \bigcup \mathscr{J}_d$ where $\mathscr{J}_j = \mathscr{J} \bigcap \Omega_j$ for $j = 1, 2, \dots, d$.

We define an equivalence relation $\sim$ on $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{j=1}^{s} \Theta_j$ by $t \sim t'$ if and only if $\{t, t'\}, \Theta_1, \ldots, \Theta_s$ are pairwise unlinked. It is straightforward that we get the equivalence classes $L_1, L_2, \ldots, L_d$. By the itinerary of t, we mean a sequence, $p_1, p_2, \ldots,$ of positive integers between 1 and d where $f_d^n(t) = d^n t \mod \mathbb{Z} \in \overline{L_{p_n}}$ for each n.

Lemma 4.2. *Each L_i or even $\overline{L_i}$ has length $\frac{1}{d}$.*

Proof. For each L_i, the closure $\overline{L_i}$ has at most finite points which belong to $\bigcup_{j=1}^{s} \Theta_j$ more than L_i. From C2, $\ell(L_i) = \frac{k_i}{d}$ and $\sum_{i=1}^{d} \ell(L_i) = 1$ implies $k_1 = k_2 = \cdots = k_d = 1$. Moreover, only one L_i is disconnected. □

Lemma 4.3. *For each L_i, there is one and only one continuous monotone degree one circle map $\phi_i : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ where $\phi_i(t) \equiv dt \pmod{\mathbb{Z}}$ for any $t \in \overline{L_i}$ and ϕ_i is constant on every component of $\mathbb{R}/\mathbb{Z} \setminus \overline{L_i}$.*

Proof. Since $\ell\left(\overline{L_i}\right) = \frac{1}{d}$ and the boundary points of $\overline{L_i}$ satisfies C2, then we can define the ϕ_i by drawing a line with slope d and horizontal lines between the adjacent critical angles outside L_i. This construction is unique. □

There are two more necessarty conditions for a critical portrait:

- C4 For any $\theta \in \bigcup_{j=1}^{s} \Theta_j$, θ is strictly preperiodic.
- C5 $\theta \in \Theta_i, \theta' \in \Theta_j$ with $i \neq j$, then θ and θ' do not have any itinerary in common.

In Section §4 of [P], we see an example for a critical portrait of a polynomial with C1-C3 but not C4. That is why we do not put the five properties together. Furthermore, Fisher's work asserts that a formal critical portrait satisfying C4 is realized by a polynomial whose Julia set is connected if and only if the portrait satisfies C5.

We will state the following two conclusions Lemma 4.4 [GM] and Theorem 4.1 [BFH] without proof.

Lemma 4.4. *A formal critical portrait satisfying* C4 *is given, then for any $s, t \in S^1$, the external rays $\mathcal{R}_s, \mathcal{R}_t$ lands at a common point of $\mathscr{J}$ if and only if for any $n \geq 0$, there is p_n with $1 \leq p_n \leq d$ such that $s_n, t_n \in \overline{L_{p_n}}$. And each $\mathscr{J}_j$ contains a unique fixed point of f.*

Theorem 4.1. *Suppose a formal critical portrait Θ sastifying* C4 *and* C5 *is given. Then there is one and only one polynomial f which has connected $\mathscr{J}$ and realizes Θ. Furthermore, f is critically pre-periodic, i.e., every orbit of critical point is strictly pre-periodic, and f has d repelling fixed points.*

Lemma 4.5. *Assume that a formal critical portrait sastisfies* C4*. For each associated map ϕ_j, we have*

(a) $\rho(\phi_j)$ *is rational, say* $\frac{p_j}{q_j}$.

(b) *Every periodic point is either repelling or ultra-attracting.*

(c) *These two kinds of periodic points alternate around the circle , and the number of orbits of each kind is between* 1 *and* $d-1$.

(d) *Every point of* $\mathbb{R}/\mathbb{Z}$ *is either periodic or pre-periodic.*

Proof. (a) This follows from that the endpoints of $\overline{L_j}$ are strictly preperiodic under ϕ_j.

(b) Clearly, every periodic point is either repelling, ultra-attracting , or repelling on one side and ultra-attracting on the other side. But the endpoints of each L_j are strictly preperiodic. This completes our second conclusion.

(c) Note that $\Phi^{q_j} - p_j$ crosses from above the diagonal to below at every ultra-attracting periodic point and it crosses from below the diagonal to above at every repelling periodic point. Note that each interval of constancy of ϕ_j has at most an orbit of one ultra-attracting periodic points. This shows the third assertion.

(d) For the last assertion, clearly every point of $\mathbb{R}/\mathbb{Z}$ is either periodic or not periodic. For those non-periodic points, their orbits tend asymptotically to an one-sided attracting periodic orbit by Theorem 3.2.

□

5 Fixed Point Portraits

In this section, we assume Standing Hypothesis 2.1.

Definition 5.1. *With Standing Hypothesis 2.1, given a fixed point z of f, let $T = T(f,z) = \{t \in \mathbb{Q}/\mathbb{Z}\,|\mathcal{R}_t \ \text{ lands at } \ z\}$ denote the* rational type *of z. If $T = \emptyset$, then we say that z is* rationally invisible*; otherwise, z is called* rationally visible. *By the* fixed point portrait *of f, we mean the set consisting of the rational types of all rationally visible fixed points.*

Lemma 5.1. *A fixed point z of f is rationally visible if and only if it is either repelling or parabolic.*

Proof. The sufficient condition is a direct consequence of Theorem 2.8.

Assume that z is rationally visible. Pick t in $T\,(f,z)$ which is periodic. There is a positive integer m such that $\mathcal{R}_t$ is a fixed ray of f^m. It follows that z is a repelling or parabolic fixed point of f^m from Lemma 2.4. This completes the proof of the necessary condition. □

Lemma 5.2. *The rational type of a rationally visible fixed point z of f is a ratation set of S^1.*

Proof. Clearly, there is at least a periodic ray landing at z. It follows that $T(f,z)$ is finite from Lemma 2.2.

In addition, they have common period, say q. Note that all q-cycles of f are rotation cycles with rotation numbers $\frac{p}{q}$ and, from the orbit of smallest, the circle is divided into q sectors, and the proof is completed. □

Lemma 5.3. *Let f be the monic polynomial obtained from Theorem 4.1 and $z_1, z_2, \ldots, z_d$ be the repelling fixed points of f with $z_j \in \mathscr{J}_j$. Then $T\,(f,z_j)$ is the set consisting of all the repelling periodic points of the associated map ϕ_j.*

Proof. From Lemma 4.4 and Theorem 4.1, we may rearrangement the d repelling fixed points such that the itinerary $jjjjj\overline{\cdots}$ of $z_j \in \mathscr{J}_j$. Hence $T\,(f,z_j)$ is the set consisting of all periodic points of ϕ_j on $\overline{L_j}$ which are repelling. □

Theorem 5.1. *Let $\mathscr{P} = \{T_1, T_2, \ldots, T_k\}$ be the fixed point portrait of f, then the following four conditions hold:*

P1 *Every T_j is a rational rotation set.*

P2 *The T_j are disjoint and pairwise unlinked.*

P3 $\bigcup_{\rho(T_j)=0} T_j = \left\{0, \frac{1}{d-1}, \ldots, \frac{d-2}{d-1}\right\}$.

P4 *$i \neq j$ with $\rho(T_i) \neq 0$ and $\rho(T_j) \neq 0$, then there exists a T_l with $\rho(T_l) = 0$ such that T_l seperates T_i, T_j.*

P1 is a direct consequence of Lemma 5.2. It is straightforward that P2 and P3 holds. As for P4, we omit the proof and refer to [GM].

Now, we consider the converse statement with exactly d nonempty subsets of $\mathbb{Q}/\mathbb{Z}$ which are called a candidate of fixed point portrait. Is there a polynomial of degree d whose fixed point portrait is the given one?

We start with the so-called elementary fixed point portrait $\mathscr{P}_0 = \left\{\left\{\frac{0}{d-1}\right\}, \ldots, \left\{\frac{d-2}{d-1}\right\}, T\right\}$ and illustrate how a critical portrait is generated from $\mathscr{P}_0$.

Like in Section 3, say $T = \{t_0 < t_1 < \cdots < t_{kq-1}\}$ with rotation number $\frac{p}{q}$. For those i with $\omega(A_i) > 0$, choose θ_i in A_i such that $\left\{\theta_i + \frac{h}{d} \,|h = 0, 1, \ldots, \omega(A_i)\right\}$ is contained in A_i and θ_i belongs to the backward orbit of a fixed point in A_i. Furthermore, we require $(t_i, \theta_i]$ is disjoint from any $\frac{p}{q}$-rotation cycles and any points of the form $\frac{j}{d-1} - \frac{h}{d}$ where $\frac{j}{d-1} \in A_i$ and $0 \leq h \leq \omega(A_i)$.

List those i whose weight is larger than one and relabel them as follows: $i_1 < i_2 < \cdots < i_m$. Let Θ_j denote the angles obtained from A_{i_j}. We remark that there might be many choices, and we might not have uniqueness.

Lemma 5.4. *The portrait $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_m\}$ as constructed above sastisfies* C1-C5*, and is realized by a unique critically preperiodic polynomial f of degree d by Theorem 4.1.*

Proof. C1 follows since each A_{i_j} is non-overlapping, C2,C3 and C4 follows by the form we pick from A_{i_j}, and C5 follows from the fact that each equivalence class contains precisely one fixed point type of $\mathscr{P}_0$. □

Theorem 5.2. *A collection $\mathscr{P} = \{T_1, T_2, \ldots, T_d\}$ of exactly d nonempty subsets of $\mathbb{Q}/\mathbb{Z}$ is the fixed point portrait of some critically pre-periodic polynomial of degree d if and only if the following four conditions* P1-P4 *in Theorem 5.1 hold.*

Proof. The necessary condition is done by Theorem 5.1.

In the elementary case, assume $\mathscr{P} = \left\{\left\{\frac{0}{d-1}\right\}, \ldots, \left\{\frac{d-2}{d-1}\right\}, T\right\}$, we have a critically pre-periodic polynomial f of degree d in Lemma 5.4. It turns to examine the fixed point portrait is indeed the given one. Again, it follows the fixed point portrait $\mathscr{P}' = \left\{\left\{\frac{0}{d-1}\right\}, \ldots, \left\{\frac{d-2}{d-1}\right\}, T'\right\}$ where $T' \supset T$ from the fact each equivalence class contains precisely an element of $\mathscr{P}$.

Assume $T' \neq T$, we may have $t' \in T' \setminus T$, then let A_i' denote the rightmost arc of the arc A_i generated by T with $\ell(A_i') < \frac{1}{d}$. Note that $f_d^q(A_i') = A_i'$ and $\ell\left(f_d^q(A_i')\right) > \ell(A_i')$, then this makes a contradiction.

For the general case, say $\mathscr{P} = \{T_1, T_2, \ldots, T_d\}$, we still define the equivalence relation $\sim$ on $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{i=1}^{d} T_i$ where $t \sim s$ means $\forall i = 1, 2, \ldots, d, \quad t, s$ belong to the same component of $\mathbb{R}/\mathbb{Z} \setminus T_i$. Assume $U_1, U_2, \ldots, U_m$ are equivalent classes of $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{i=1}^{d} T_i$.

From P4, there is at most one T_i with nonzero rotation number intersecting the boundary of each U_j, and there is at least one or otherwise, we have fixed point types less than d.

From the discussion above, we see that each U_j is an arc of $\mathbb{R}/\mathbb{Z} \setminus T_i$ or the remaining set removed a finite number of intervals with endpoints fixed points from that arc for corresponding i, including degenerated intervals.

Also, define the weight of U_j, denoted by $\omega(U_j)$, to be the number of missing intervals. As the proof in the elementary case, we construct a critical portrait $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_s\}$ from these i where $\omega(U_j) > 0$. We see that Θ satisfies C1-C5 and we get a critically pre-periodic polynomial f to realize Θ.

Each U_j has a well-defined rotation number, and the fixed point portrait $\mathscr{P}' = \{T_1', T_2', \ldots, T_d'\}$ has the property $T_i \subseteq T_i'$. As the proof in the elementary case, we have $T_i = T_i'$. $\square$

6 Classification of Degree 3 Fixed Point Portraits

We consider what fixed point portraits generated from critical portraits with C1-C5 as $d = 3$.

Since $\sum_{j=1}^{s} (\#\Theta_j - 1) = 2$, then there are two possible cases that either Θ has only one element or Θ has two elements.

The former case: Say, $\Theta = \left\{\theta, \theta + \frac{1}{3}, \theta + \frac{2}{3}\right\}$ with $\theta \in \left(0, \frac{1}{3}\right)$.

The resulted fixed point portrait will be $T = \left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}, (s_1, q)\right)\right\}$ where $\frac{p}{q}$ is rational and strictly between 0 and 1. Moreover, its deployment sequence is either $(0, q)$ or (q, q).

θ	$\mathscr{P}$
$0 < \theta < \frac{1}{6}$	$\left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}, (q, q)\right)\right\}$
$\frac{1}{6} < \theta < \frac{1}{3}$	$\left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}, (0, q)\right)\right\}$

The latter case: Say, $\Theta = \{\Theta_1, \Theta_2\}$.

$A = \left\{\alpha, \alpha + \frac{1}{3}\right\}$ ($\alpha \in \left(0, \frac{1}{3}\right)$), $B = \left\{\beta, \beta + \frac{1}{3}\right\}$ ($\beta \in \left(\frac{1}{3}, \frac{2}{3}\right)$), $C = \left\{\gamma, \gamma - \frac{2}{3}\right\}$ ($\gamma \in \left(\frac{2}{3}, 1\right)$).

Θ_1 and Θ_2 can be composed of two of A, B, C.

$\Theta = \{A, B\}$ (Implicitly, $0 < \alpha < \alpha + \frac{1}{3} < \beta < \beta + \frac{1}{3} < \frac{2}{3}$.)

If $0 < \alpha < \frac{1}{6}$ and $\frac{1}{2} < \beta < \frac{2}{3}$, then $T = \left\{\left\{0, \frac{1}{2}\right\}, T\left(\frac{p}{q}; (q, q)\right), T\left(\frac{p'}{q'}; (0, q')\right)\right\}$.

If $0 < \alpha < \frac{1}{6}$ and $\alpha + \frac{1}{3} < \beta < \frac{1}{2}$, then $T = \left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}; (q, q)\right)\right\}$.

If $\frac{1}{6} < \alpha < \frac{1}{3}$, then $T = \left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}; (0, q)\right)\right\}$.

α	β	$\mathscr{P}$
$0 < \alpha < \frac{1}{6}$	$\frac{1}{2} < \beta < \frac{2}{3}$	$\left\{\left\{0, \frac{1}{2}\right\}, T\left(\frac{p}{q}, (q, q)\right) T\left(\frac{p'}{q'}, (0, q')\right)\right\}$
$0 < \alpha < \frac{1}{6}$	$\alpha + \frac{1}{3} < \beta < \frac{1}{2}$	$\left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}, (q, q)\right)\right\}$
$\frac{1}{6} < \alpha < \frac{1}{3}$	$\alpha + \frac{1}{3} < \beta < \frac{2}{3}$	$\left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}, (0, q)\right)\right\}$

$\Theta = \{A, C\}$ (Implicitly, $0 < \gamma - \frac{2}{3} < \alpha < \alpha + \frac{1}{3} < \frac{2}{3} < \gamma < 1$.)

If $0 < \alpha < \frac{1}{6}$, then $T = \left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}; (q, q)\right)\right\}$.

If $\frac{1}{6} < \alpha < \frac{1}{3}$, then $T = \left\{\{0\}, \left\{\frac{1}{2}\right\}, T\left(\frac{p}{q}; (s_1, kq)\right)\right\}$ where $0 \le s_1 \le q$, as $k = 1$; $0 \le s_1 \le 2q$ and s_1 is odd as $k = 2$.

α	γ	$\mathscr{P}$
$0 < \alpha < \frac{1}{6}$	Independent	$\left\{\{0\}, \{\frac{1}{2}\}, T\left(\frac{p}{q}, (q, q)\right)\right\}$
$\frac{1}{6} < \alpha < \frac{1}{3}$	Independent	$\left\{\{0\}, \{\frac{1}{2}\}, T\left(\frac{p}{q}, (s_1, kq)\right)\right\}$

where $0 \leq s_1 \leq q$, as $k = 1$; $0 \leq s_1 \leq 2q$ and s_1 is odd as $k = 2$.

$\Theta = \{B, C\}$ (Implicitly, $0 < \gamma - \frac{2}{3} < \frac{1}{3} < \beta < \beta + \frac{1}{3} < \gamma < 1$.)

If $\frac{1}{2} < \beta < \frac{2}{3}$, then $T = \left\{\{0\}, \{\frac{1}{2}\}, T\left(\frac{p}{q}; (0, q)\right)\right\}$.

If $\frac{1}{3} < \beta < \frac{1}{2}$, then $T = \left\{\{0\}, \{\frac{1}{2}\}, T\left(\frac{p}{q}; (s_1, kq)\right)\right\}$ where $0 \leq s_1 \leq q$, as $k = 1$; $0 \leq s_1 \leq 2q$ and s_1 is odd as $k = 2$.

β	γ	$\mathscr{P}$
$\frac{1}{3} < \beta < \frac{1}{2}$	Independent	$\left\{\{0\}, \{\frac{1}{2}\}, T\left(\frac{p}{q}, (s_1, kq)\right)\right\}$
$\frac{1}{2} < \beta < \frac{2}{3}$	Independent	$\left\{\{0\}, \{\frac{1}{2}\}, T\left(\frac{p}{q}, (0, q)\right)\right\}$

where $0 \leq s_1 \leq q$, as $k = 1$; $0 \leq s_1 \leq 2q$ and s_1 is odd as $k = 2$.

Now we will demonstrate how to use Maple to obtain $\frac{p}{q}$-rotation cycles. We provide an algorithm as follows:

Input: Set degree d,period q, $c = d^q - 1$, arrays v, w with length c, q, resp..
Output: Classify All possible $\frac{p}{q}$-rotation cycles with periods.
Let $f : \theta \longmapsto d\theta \mod c$, $x = \frac{c}{2}$,and v be the zero vector in $\mathbb{R}^c$.
For each i
Let w be an undefined vector in $\mathbb{R}^q$ with first coordinate i.
if $v[i] = 0$ **then**
  set $j = 1, v[i] = 1, w[1] = i$
  For each j
  $w[j] = f(w[j-1])$
  $v[w[j]] = 1$
  if $w[1] < w[2] < \cdots < w[q]$ **then**
    set $p = 1$
  end
  elif $w[1] < w[\frac{q+3}{2}] < \cdots < w[\frac{q+1}{2}]$ **then**
    set $p = 2$
  end
  $\vdots$
  elif $w[1] < w[q] < \cdots < w[2]$ **then**
    set $p = q - 1$
  end
  else set $p =$None, **end**
  print w, p
  $a = [w[1], w[2], \ldots, w[q]]$;$z =$sort(a);
  if $p =$*None* **then**
    print p
  end
  elif $z[1] < x, z[2] < x, \ldots, z[q] < x$ **then**
    set $s = q$
  end
  elif $z[1] < x, z[2] < x, \ldots, z[q-1] < x, z[q] > x$ **then**
    set $s = q - 1$
  end
  $\vdots$
  elif $z[1] < x, z[2] > x, \ldots, z[q] > x$ **then**
    set $s = 1$
  end
  else set $s = 0$, end   print z, s
end

Algorithm 6.1: The algorithm for degree d $\frac{p}{q}$-rotation cycles with Maple

The following code is executed in Maple.

Program 6.1 A program for degree 3 $\frac{p}{5}$-rotation cycles with Maple

```
d := 3; q := 5; c := d^q-1; x:=c/2;
f := proc (x) options operator, arrow; `mod`(d*x, c) end proc;
v := array(1 .. c); v[1] := 0; for i from 2 to c do
v[i] := v[i-1] end do;
for i to c-1 do w := array(1 .. q);
if v[i] = 0 then j := 1;w[1] := i; v[i] := 1;
for j from 2 to q do w[j] := f(w[j-1]); v[w[j]] := 1 end do;
if w[1] < w[2] < w[3] < w[4] < w[5] ,then p := 1;
elif w[1] < w[4] < w[2] < w[5] < w[3],then p := 2;
elif w[1] < w[3] < w[5] < w[2] < w[4],then p := 3;
elif w[1] < w[5] < w[4] < w[3] < w[2],then p := 4;
elif w[1] = w[2 = w[3]= w[4]= w[5],then p := 5;
else p := None;
end if: print(w,'p'=p);
a:=[ w[1], w[2], w[3], w[4], w[5]]; z:=sort(a);
if p=None then print(None);
elif z[1]<x and z[2]<x and z[3]<x and z[4]<x and z[5]<x, then s := 5;
elif z[1]<x and z[2]<x and z[3]<x and z[4]<x and z[5]>x, then s := 4;
elif z[1]<x and z[2]<x and z[3]<x and z[4]>x and z[5]>x, then s := 3;
elif z[1]<x and z[2]<x and z[3]>x and z[4]>x and z[5]>x, then s := 2;
elif z[1]<x and z[2]>x and z[3]>x and z[4]>x and z[5]>x, then s := 1;
else s := 0;
end if: print(z,'deployment sequence'=(s,5));
end if end do
```

Appendix

This is the table of degree 3 $\frac{p}{5}$-rotation cyles.

RN\ DS	(0,5)	(1,5)	(2,5)
$1/5$	$\{122, 124, 130, 148, 202\}$	$\{41, 123, 127, 139, 175\}$	$\{14, 42, 126, 136, 166\}$
$2/5$	$\{131, 149, 151, 205, 211\}$	$\{50, 140, 150, 178, 208\}$	$\{47, 59, 141, 177, 181\}$
$3/5$	$\{152, 158, 212, 214, 232\}$	$\{71, 155, 185, 213, 223\}$	$\{62, 74, 182, 186, 222\}$
$4/5$	$\{161, 215, 233, 239, 241\}$	$\{80, 188, 224, 236, 240\}$	$\{79, 107, 197, 227, 237\}$
RN\ DS	(3,5)	(4,5)	(5,5)
$1/5$	$\{5, 15, 45, 135, 163\}$	$\{2, 6, 18, 54, 162\}$	$\{1, 3, 9, 27, 81\}$
$2/5$	$\{20, 56, 60, 168, 180\}$	$\{19, 29, 57, 87, 171\}$	$\{10, 28, 30, 84, 90\}$
$3/5$	$\{61, 65, 101, 183, 195\}$	$\{34, 64, 92, 102, 192\}$	$\{31, 37, 91, 93, 111\}$
$4/5$	$\{76, 106, 116, 200, 228\}$	$\{67, 103, 115, 119, 201\}$	$\{40, 94, 112, 118, 120\}$

RN: Rotation Number; DS: Deployment Sequence.

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