SOME THEOREMS OF FIXED POINTS

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1. Preliminaries

Recently fixed point theory in complete metric space has been developed rapidly, such as Boyd and Wong [1], Maki [3], Meir and Keeler [4] and Yen [7], etc. (For a complete references one should see [8]).

Thoughout this paper we assume that X is a topological space whose topology is generated by a family $\{d_\lambda\}_{\lambda\in\Lambda}$ of pseudo-metrics on X (For examples of this topology one may see Cain and Kasriel [2] and Tan [6] also). The results that we prove here are some fixed point theorems in this topological space which relative to theorems of Maki [3], Meir and Keeler [4], Boyd and Wong [1], and Yen [7].

First we give some definitions:

Definition 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $\lambda\in\Lambda$, we say that $\{x_n\}_{n=1}^{\infty}$ is d_λ-Cauchy if and only if for each $\lambda\in\Lambda$, $d_\lambda(x_n,x_m)\to 0$ as $n,m\to\infty$ We say that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, if it is d_λ-Cauchy for all $\lambda\in\Lambda$.

Definition 2. X is complete if and only if every Cauchy sequence in X converges to some elements in X.

Remark. The limit of a Cauchy sequence in X is unique.

2. Theorems

We shall first prove a theorem of Maki [3] in our space X.

Theorem 1. Let X be connected and complete, and let T be a regular mapping of X into itself, that is,

(1) for any $\lambda \epsilon \Lambda$, there is a positive number k_λ, $k_\lambda < 1$, such that for any $\epsilon > 0$, there exists $\delta_\lambda > 0$, satisfying that

$$d_\lambda (T^n x, T^n y) < k_\lambda^n \epsilon, n = 1,2,3,\ldots,$$

whenever $d_\lambda(x,y) < \delta_\lambda$. Then, for any $x \epsilon X$, $\{T^n x\}$ converges to the unique fixed point of T.

Remark 1. We know if $(X, \mathcal{J}_1)$ and $(X, \mathcal{J}_2)$ are topological spaces on X, and $\mathcal{J}_1 \subset \mathcal{J}_2$. If $(X, \mathcal{J}_2)$ is connected then $(X, \mathcal{J}_1)$ is connected too. In particular, if $\mathcal{J}_\lambda$ is a topology on X induced by d_λ, then $(X, \mathcal{J}_\lambda)$ is connected, for all $\lambda \epsilon \Lambda$. We use this fact to prove Theorem 1.

Proof of Theorem 1. For each $\lambda \epsilon \Lambda$, each $x \epsilon X$, define $[X]_\lambda = \{y \epsilon X \mid d_\lambda (T^n x, T^n y) \to 0\}$. We claim that $[X]_\lambda$ is open and closed in $(X, \mathcal{J}_\lambda)$. If $y_n \epsilon [X]_\lambda$, and $y_n \to y$. By (1), for any $\epsilon > o$, there is $\delta_\lambda > 0$ such that $d_\lambda(x,y) < \delta_\lambda$ implies $d_\lambda(T^n x, T^n y) < k_\lambda^n \epsilon$. Since $y_n \to y, \delta_\lambda > 0$, we have $d_\lambda(y_m, y) < \delta_\lambda$, for sufficiently large m, and then $d_\lambda(T^n y_m, T^n y) < k_\lambda^n \epsilon$.

$$d_\lambda (T^n x, T^n y) \leq d_\lambda (T^n x, T^n y_m) + d_\lambda (T^n y_m, T^n y).$$

It is due to the fact that $d_\lambda(T^n x, T^n y_m) \to 0$, $d_\lambda(T^n y_m, T^n y) \to 0$, we get $y \epsilon [X]_\lambda$.

This shows that $[X]_\lambda$ is closed. If $z \epsilon [X]_\lambda$, $B_\lambda(z, \delta_\lambda) = \{y \epsilon X \mid d_\lambda(z,y) < \delta_\lambda\}$. Let $w \epsilon B_\lambda(z, \delta_\lambda)$, then $d_\lambda(z,w) < \delta_\lambda$, implies $d_\lambda(T^n z, T^n w) \to 0$, and $w \epsilon [X]_\lambda$. Hence $[X]_\lambda$ is open.

For $(X, \mathcal{J}_\lambda)$ is connected and $x \epsilon [X]_\lambda$, so $[X]_\lambda = X$ for each $\lambda \epsilon \Lambda$

Next, we claim that $\{x, Tx, \ldots, T^n x, \ldots\}$ is Cauchy. For each $\lambda \epsilon \Lambda$, $x, Tx \epsilon [X]_\lambda$ implies $d_\lambda(T^n x, T^{n+1} x) \to 0$. So, there exists positive integer n_0, such that $d_\lambda (T^{n_0} x, T^{n_0+1} x) < \delta_\lambda$. This implies $d_\lambda (T^{n_0+n} x, T^{n_0+n+1} x) < k_\lambda^n \epsilon$. If $m > n \geq n_0$, then

$$d_\lambda(T^{n_0+n}x,\ T^{n_0+m}x)$$

$$\leq d_\lambda(T^{n_0+n}x,\ T^{n_0+n+1}x) + d_\lambda(T^{n_0+n+1}x,\ T^{n_0+n+2}x) + \ldots\ldots + d_\lambda(T^{n_0+m-1}x, T^{n_0+m}x)$$

$$< (k_\lambda^n + k_\lambda^{n+1} + \ldots\ldots + k_\lambda^{m-1})\epsilon$$

$$= \frac{k_\lambda^n(1-k_\lambda^{m-n})}{1-k_\lambda}\epsilon < \frac{k_\lambda^n}{1-k_\lambda}\epsilon\ .$$

For $k_\lambda^n \to 0$, we know that $\{x, Tx, \ldots., T^n x, \ldots.\}$ is Cauchy.

By the completeness of X that $T^n x \to y$ for some $y \in X$, and we get $Ty=y$. For if $Ty \neq y$, then there exists $\lambda \in \Lambda$, such that $d_\lambda(Ty, y) > 0$. Taking $\delta' = \min(\delta_\lambda, \epsilon)$, we know $d_\lambda(x,y) < \delta'$ implies $d_\lambda(T^n x, T^n y) < k_\lambda^n \epsilon$. Since $T^n x \to y$, so there is a positive integer n_0, such that $n \geq n_0$, $d_\lambda(T^n x, y) < \delta'$. Hence

$$d_\lambda(Ty, y) \leq d_\lambda(T^{n+1}x,\ Ty) + d_\lambda(T^{n+1}x,\ y) < k_\lambda\epsilon + \delta' \leq (k_\lambda+1)\epsilon$$

which is a contradiction.

Finally, if $Ty=y$ and $Tz=z$, then $y=z$. If not, there exists $\lambda \in \Lambda$, such that $d_\lambda(y,z) > 0$. $d_\lambda(y,z) = d_\lambda(T^n y,\ T^n z) \to 0$ which is a contradiction. This shows that the fixed point of T is unique.

Theorem 2. Let T be a mapping of a complete space X into itself with the the property

(2) For each $\lambda \in \Lambda$, given $\epsilon > 0$, there exists $\delta_\lambda > 0$, such that

$$d_\lambda(x,y) < \epsilon + \delta_\lambda \quad \text{implies} \quad d_\lambda(Tx, Ty) < \epsilon$$

Then, T has a unique fixed point y.

Remark 2. We first observe that (2) implies that for $x, y \in X$

$$d_\lambda(Tx, Ty) \leq d_\lambda(x,y).$$

If $d_\lambda(x,y)=0$, for any $\epsilon>0$, there exists $\delta_\lambda>0$, such that $d_\lambda(x,y)<\epsilon+\delta_\lambda$ implies $d_\lambda(Tx,Ty)<\epsilon$. Hence $d_\lambda(Tx,Ty)=0$, we get $d_\lambda(Tx,Ty)=d_\lambda(x,y)$, if $d_\lambda(x,y)>0$, taking $\epsilon=d_\lambda(x,y)$, there is $\delta_\lambda>0$, such that $d_\lambda(x,y)<\epsilon+\delta_\lambda$ implies $d_\lambda(Tx,Ty)<\epsilon=d_\lambda(x,y)$.

Proof of Theorem 2. For fixed $\lambda\in\Lambda$, and $x\in X$, define $c_n=d_\lambda(T^n x,\ T^{n-1}x)$, $n=1,2,3,\ldots\ldots.$ Then, it follows from Remark 2 that $\{c_n\}$ is a decreasing sequence of nonnegative numbers. Say its limit c. We claim that $c=0$. If $c>0$, then there exists a positive integer n_0, such that $c\leq c_{n_0}<c+\delta_\lambda$. By (2),

$$d_\lambda(T(T^{n_0}x),\ T(T^{n_0-1}x))<c \quad \text{implies} \quad c_{n_0+1}<c$$

which contradicts to that $\{c_n\}$ decreases to c.

Now, we claim that $\{x, Tx, \ldots\ldots, T^n x, \ldots\ldots\}$ is Cauchy. If not, there exist $2\epsilon>0$, and $\lambda\in\Lambda$, such that

$$\overline{\lim}\ d_\lambda(T^m x,\ T^n x)>2\epsilon .$$

By (2), there is $\delta_\lambda>0$, such that

$$(3) \qquad d_\lambda(x,y)<\epsilon+\delta_\lambda \text{ implies } d_\lambda(Tx,Ty)<\epsilon .$$

(3) will remain true with δ_λ replaced by $\delta'_\lambda=\min(\delta_\lambda,\epsilon)$. Since $c_n\to 0$, so we can find a positive integer n_0, such that $n\geq n_0$, $c_n<\frac{\delta'_\lambda}{3}$. Pick $n_0\leq m<n$, so that $d_\lambda(T^m x,\ T^n x)>2\epsilon$. For $m\leq j\leq n$

$$|d_\lambda(T^m x,\ T^j x)-d_\lambda(T^m x,\ T^{j+1}x)|<c_j<\frac{\delta'_\lambda}{3}.$$

Since $d_\lambda(T^m x,\ T^{m+1}x)<\epsilon$ and $d_\lambda(T^m x,\ T^n x)>\epsilon+\delta'_\lambda$, so that there exists j, $m<j<n$ with

$$(4) \qquad \epsilon+\frac{2\delta'_\lambda}{3}<d_\lambda(T^m x,\ T^j x)<\epsilon+\delta'_\lambda .$$

However, for all m and j.

$$d_\lambda(T^m x, T^j x) \leq d_\lambda(T^m x, T^{m+1}x) + d_\lambda(T^{m+1}x, T^{j+1}x) + d_\lambda(T^{j+1}x, T^j x).$$

Therefore, by (3) and (4)

$$d_\lambda(T^m x, T^j x) \leq c_{m+1} + \epsilon + c_{j+1} < \frac{\delta'_\lambda}{3} + \epsilon + \frac{\delta'_\lambda}{3}$$

which contradicts to (4). Hence, $\{x, Tx, \ldots, T^n x, \ldots\}$ is a Cauchy sequence, and $T^n x \to y$ for some $y \in X$.

Next, we show that $Ty = y$. If not, there is $\lambda \in \Lambda$, such that $d_\lambda(Ty, y) > 0$. Since

$$d_\lambda(Ty, y) \leq d_\lambda(Ty, T^{n+1}x) + d_\lambda(T^{n+1}x, y) \leq d_\lambda(y, T^n x) + d_\lambda(T^{n+1}x, y),$$

for $d_\lambda(T^n x, y) \to 0$, $d_\lambda(T^{n+1}x, y) \to 0$. Which is a contradiction.

Finally, if $Ty = y$ and $Tx = z$, then $y = z$. If $y \neq z$, then there is $\lambda \in \Lambda$, such that $d_\lambda(y, z) > 0$. By (2), taking $\epsilon = d_\lambda(y, z)$, then

$$d_\lambda(y, z) = d_\lambda(Ty, Tz) < \epsilon = d_\lambda(y, z)$$

which is a contradiction.

Remark 3. The property (2) can not be weaken as that
(5) for each $\lambda \in \Lambda$, given $\epsilon > 0$, there exists $\delta_\lambda > 0$, such that

$$\epsilon \leq d_\lambda(x, y) < \epsilon + \delta_\lambda \text{ implies } d_\lambda(Tx, Ty) < \epsilon .$$

For example. If $X = \{a, b, c\}$ with the pseudo-metrics $\{d_i\}_{i=1}^{3}$ defined by

$$\begin{cases} d_1(a,b) = d_1(b,a) = 1 \\ d_1(x,y) = 0 \text{ otherwise,} \end{cases} \quad \begin{cases} d_2(b,c) = d_2(c,b) = 1 \\ d_2(x,y) = 0 \text{ otherwise,} \end{cases} \quad \begin{cases} d_3(c,a) = d_3(a,c) = 1 \\ d_3(x,y) = 0 \text{ otherwise.} \end{cases}$$

If T is defined by

$$T(a) = b, \quad T(b) = c, \quad T(c) = a.$$

Then, T satisfies the condition (5). But T has no fiexed point in X.

Theorem 3. Let T be a mapping of a complete space X into itself, with the property

(6) for each $\lambda \in \Lambda$, there is a function f_λ, such that

$$d_\lambda(Tx, Ty) \leq f_\lambda(d_\lambda(x,y))$$

where $f_\lambda : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function such that

$$f_\lambda(t) < t \quad \text{if} \quad t > 0, \text{ and } f_\lambda(0) = 0.$$

Then, T has a unique fixed point y.

Proof. Let $x \in X$, and $\lambda \in \Lambda$, we define $c_n = d_\lambda(T^n x, T^{n-1}x)$, $n = 1, 2, 3, \ldots$ Then, $\{c_n\}$ decreases to zero. For

$$c_{n+1} = d_\lambda(T^{n+1}x, T^n x) \leq f_\lambda(d_\lambda(T^n x, T^{n-1}x)) = f_\lambda(c_n) \leq c_n,$$

then $\{c_n\}$ converages to a limit c. If $c > 0$, since $c_{n+1} \leq f_\lambda(c_n)$ and f_λ is upper semicontinuous, we have

$$\overline{\lim_{n\to\infty}}\, c_{n+1} \leq \overline{\lim_{n\to\infty}}\, f_\lambda(c_n) \leq f_\lambda(c).$$

Hence, $c \leq f_\lambda(c)$, which is a contradiction.

Now, we show that for each $x \in X$, $\{x, Tx, \ldots, T^n x, \ldots\}$ is Cauchy. Suppose not, there exist $\epsilon > 0$, $\lambda \in \Lambda$, $n_1 < m_1 < n_2 < m_2 < \ldots$, such that

$$d_\lambda(T^{n_i}x, T^{m_i}x) \geq \epsilon \quad \text{and} \quad d_\lambda(T^{n_i}x, T^{m_{i-1}}x) < \epsilon, \; i = 1, 2, 3, \ldots$$

$$\epsilon \leq d_k = d_\lambda(T^{n_k}x, T^{m_k}x) \leq d_\lambda(T^{n_k}x, T^{m_k-1}x) + d_\lambda(T^{m_k-1}x, T^{m_k}x) < \epsilon + c_{m_k}.$$

Since $c_{m_k} \to 0$, so $d_k \to \epsilon$.

$$\begin{aligned} d_k &= d_\lambda(T^{n_k}x, T^{m_k}x) \\ &\leq d_\lambda(T^{m_k}x, T^{m_{k+1}}x) + d_\lambda(T^{m_{k+1}}x, T^{n_{k+1}}x) + d_\lambda(T^{n_{k+1}}x, T^{n_k}x) \\ &\leq c_{m_{k+1}} + f_\lambda(d_k) + c_{n_{k+1}}. \end{aligned}$$

We get $\epsilon = \overline{\lim}\, d_k \leq \overline{\lim}\, f_\lambda\ (d_k) \leq f_\lambda\ (\epsilon)$, which is a contradiction. Hence, $\{x,\ Tx,\ \ldots,\ T^n x, \ldots\ldots\}$ is Cauchy.

By the completeness of X, we know $T^{n_k} \longrightarrow y$ for some $y \epsilon x$ and $Ty = y$. If $Ty \neq y$, then there is $\lambda\ \epsilon\ \Lambda$, such that $d_\lambda\ (Ty, y) > 0$, Since $T^n x \rightarrow y$, so for any $\epsilon > 0$, there exists a positive integer n_0, such that $n \geq n_0$ implies $d_\lambda (T^n x, y) < \epsilon$.

$$d_\lambda (Ty, y) \leq d_\lambda\ (Ty,\ T^{n+1} x) + d_\lambda\ (T^{n+1} x,\ y)$$

$$\leq f_\lambda\ (d_\lambda\ (T^n x, y)) + d_\lambda\ (T^{n+1} x, y)$$

$$\leq d_\lambda\ (T^n x, y) + d_\lambda\ (T^{n+1} x, y) < 2\epsilon$$

which is a contradiction.

Finally, if $Ty = y$, $Tz = z$, then $y = z$. If not, there exists $\lambda\ \epsilon\ \Lambda$, such that $d_\lambda\ (y, z) > 0$.

$$d_\lambda\ (y, z) \leq d_\lambda (Ty, y) + d_\lambda\ (Ty, Tz) + d_\lambda\ (Tz, z)$$

$$\leq f_\lambda\ (d_\lambda\ (y, z)) < d_\lambda\ (y, z),$$

which is a contradiction. This completes the proof of the theorem.

The above theorem is due to theorem of Boyd and Wong [1], and the following theorem is a theorem of Yen [7] in our space X.

It is easy to prove that let S_1 and S_2 be two mappings of a complete space X into itself. If (S_1, S_2) satisfies $C\ (a_\lambda, b_\lambda, c_\lambda)$, that is

(7) for each $\lambda\ \epsilon\ \Lambda$, $a_\lambda \geq 0$, $b_\lambda \geq 0$, $c_\lambda \geq 0$ and $a_\lambda + b_\lambda + c_\lambda < 1$,

$$d_\lambda\ (S_1 x,\ S_2 y) \leq a_\lambda\, d_\lambda (S_1 x, x) + b_\lambda\, d_\lambda\ (S_2 y, y) + c_\lambda\, d_\lambda\ (x, y).$$

Then, for all x in X,

$$d_\lambda (S_1 S_2 x,\ S_2 x) \leq \frac{b_\lambda + c_\lambda}{1 - a_\lambda}\, d_\lambda\ (S_2 x, x)$$

$$d_\lambda\ (S_2 S_1 x,\ S_1 x) \leq \frac{a_\lambda + c_\lambda}{1 - b_\lambda}\, d_\lambda\ (S_1 x, x)\ .$$

Theorem 4. Let T_1 and T_2 be two mappings of a complete space X into itself. If (T_1^p, T_2^q) satisfies $C(a_\lambda, b_\lambda, c_\lambda)$ for some positive integers p, q, then T_1, T_2 have a unique and common fixed point.

We show our theorem from the following proposition

Proposition. Let S_1 and S_2 be two mappings of a complete space X into itself. If (S_1, S_2) satisfies $C(a_\lambda, b_\lambda, c_\lambda)$. Then, S_1 and S_2 have a unique and common fixed point.

Proof. For $x \in X$, let us consider the sequence $\{x_n\}_{n=1}^{\infty}$ in X as follows:

$$x_1 = x$$

$$x_2 = S_1 x$$

$$\vdots$$

$$x_{2n} = S_1 x_{2n-1}$$

$$x_{2n+1} = S_2 x_{2n}$$

$$\vdots$$

We claim that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. For fixed $\lambda \in \Lambda$, assume

$k_\lambda = \max\left\{\frac{b_\lambda + c_\lambda}{1-a_\lambda}, \frac{a_\lambda + c_\lambda}{1-b_\lambda}\right\} < 1$, then $d_\lambda(x_{n+1}, x_n) \leq k_\lambda d_\lambda(x_n, x_{n+1})$. For, if $n = 2m$, then $d_\lambda(x_{n+1}, x_n) = d_\lambda(S_2 S_1 x_{2m-1}, S_1 x_{2m-1})$

$$\leq k_\lambda d_\lambda(S_1 x_{2m-1}, x_{2m-1}) = k_\lambda d_\lambda(x_n, x_{n-1}).$$

if $n = 2m+1$, then $d_\lambda(x_{n+1}, x_n) = d_\lambda(S_1 S_2 x_{2m}, S_2 x_{2m})$

$$\leq k_\lambda d_\lambda(S_2 x_{2m}, x_{2m}) = k_\lambda d_\lambda(x_n, x_{n-1}).$$

Hence, $d_\lambda(x_{n+1}, x_n) \leq k_\lambda^{n-1} d_\lambda(x_2, x_1)$, it implies that $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Therefore, $x_n \to y$ for some $y \in X$.

Now, we show that $S_1y=y$. If not, there exists $\lambda \in \Lambda$,such that $d_\lambda(S_1y,y)>0$.

$$d_\lambda (S_1y,y) \leq d_\lambda (S_1y, x_{2n+1}) + d_\lambda (x_{2n+1},y)$$

$$\leq a_\lambda d_\lambda (S_1y,y)+b_\lambda d_\lambda (x_{2n+1}, x_{2n})+c_\lambda d_\lambda (y,x_{2n})+d_\lambda (x_{2n+1},y)$$

$$d_\lambda (S_1y,y) \leq \frac{1}{1-a_\lambda} (b_\lambda d_\lambda (x_{2n+1},x_{2n})+c_\lambda d_\lambda (y,x_{2n})+d_\lambda (x_{2n+1},y)).$$

Since $d_\lambda (x_{2n+1}, x_{2n})\to 0$, $d_\lambda (y,x_{2n})\to 0$, $d_\lambda (x_{2n+1},y)\to 0$,

which is a contradiction. Hence $S_1y = y$. Similarly, $S_2y=y$.

Finally, if $S_1z = z$, then for every $\lambda \in \Lambda$,

$$d_\lambda (y,z) = d_\lambda (S_2y,S_1z) \leq a_\lambda d_\lambda (S_1z,z)+b_\lambda d_\lambda (S_2y,y)+c_\lambda d_\lambda (y,z)$$

$$= c_\lambda d_\lambda (y,z)$$

since $c_\lambda < 1$, then $d_\lambda(y,z) =0$ for any $\lambda \in \Lambda$. We get $y=z$. Similarly, if $S_2z = z$, then $z=y$. This shows that y is the unique and common fixed point of S_1 and S_2.

Proof of Theorem 4. Let $T_1^p = S_1$, $T_2^q = S_2$, by the proposition we have that there is a unique and common fixed point y of T_1^p and T_2^q. Since $T_1^p(T_1y) = T_1y$ and $T_2^q(T_2y) = T_2y$, we have $T_1y = y = T_2y$. This shows that y is a common fixed point of T_1 and T_2.

Finally, if $T_1z = z = T_2z$ then $T_1^pz = z$, $T_2^qz = z$. It implies that $x = y$. This shows that the common fixed point of T_1 and T_2 is unique.

3. Examples

Probabilistic metric spaces were introduced by Menger in [5]. In such spaces, the notion of distance between two points x and y is replaced by a distribution function F_{xy}. Thus one thinks of the distance between points as being probabilistic with $F_{xy}(t)$ representing the probability that the distance between x and y is less than t. Menger spaces (X, F, Δ) were shown by Cain and Kasriel in [2] that they are special cases of spaces generated by a collection of psudo-metrics. We use our results to obtain some fixed point theorems for probabilistic metric spaces. The notion that we use can be found in [2].

Example 1. Assume (X, F, Δ) is connected and complete, and suppose there is a δ, $0 < \delta < 1$, so that for any $\lambda \in (0, \delta)$, there is a k_λ, $0 < k_\lambda < 1$, such that for any $\epsilon > 0$, there is $\delta_\lambda > 0$, so that for $x, y \in X$.

$$F_{T^n x T^n y}(k_\lambda^n \epsilon) > 1 - \lambda, \quad n = 1, 2, 3, \ldots.$$

whenever $F_{xy}(\delta_\lambda) > 1 - \lambda$.

Then, $\{T^n x\}$ converges to the unique fixed point of T.

Proof. Let $\{d_\lambda\}$ be the family of pseudo-metrics associated with F. Let $\lambda \in (0, \delta)$, there is a k_λ, $0 < k_\lambda < 1$, such that for any $\epsilon > 0$, there is $\delta_\lambda > 0$, so that for $x, y \in X$

$$d_\lambda(x, y) < \delta_\lambda \text{ implies } d_\lambda(T^n x, T^n y) < k_\lambda^n \epsilon, \quad n = 1, 2, 3, \ldots.$$

For $d_\lambda(x, y) < \delta_\lambda$ if and only if $F_{xy}(\delta_\lambda) > 1 - \lambda$ and

$$d_\lambda(T^n x, T^n y) < k_\lambda^n \epsilon \text{ if and only if } F_{T^n x T^n y}(k_\lambda^n \epsilon) > 1 - \lambda.$$

It follows from Theorem 1 that we get our conclusion.

Example 2. Assume (X, F, Δ) is complete and suppose that there is a δ, $0 < \delta < 1$, so that for each $\lambda \in (0, \delta)$, given $\epsilon > 0$, there is a $\delta_\lambda > 0$

$$F_{TxTy}(\epsilon) > 1 - \lambda$$

whenever $F_{xy}(\epsilon + \delta_\lambda) > 1 - \lambda$. Then T has a unique fixed point.

Proof. Let $\{d_\lambda\}$ be the family of pseudo-metrics associated with F. $x, y \in X$.

For each $\lambda \in (0, \delta)$, there is a $\delta_\lambda > 0$, such that

$$d_\lambda(x,y) < \epsilon + \delta_\lambda \text{ implies } d_\lambda(Tx,Ty) < \epsilon.$$

Since $d_\lambda(x,y) < \epsilon + \delta_\lambda$ if and only if $F_{xy}(\epsilon + \delta_\lambda) > 1 - \lambda$, and

$$d_\lambda(Tx,Ty) < \epsilon \text{ if and only if } F_{TxTy}(\epsilon) > 1 - \lambda.$$

It follows from Theorem 2 that we get our conclusion.

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〔中文摘要〕

幾個定點定理

Some Theorems of Fixed Points

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在完備度量空間上，定點理論的發展在最近極為迅速。（參考〔8〕中所列最近定點理論的論文）。本文將討論由一族擬度量（A family of pseudo-metrics）$\{d_\lambda\}_{\lambda\in\Lambda}$所衍生的一個拓撲空間 X 上與 Maki〔3〕，Meir and Keeler〔4〕，Boyd and Wong〔1〕，及 Yen〔7〕有關的定點定理。並舉例說明這些定點定理在概率度量空間的應用。

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