4 Asymptotic behavior

We next study the asymptotic behavior of the globally monotone decreasing solution of (P).

Lemma 4.1 *If $g(y)$ is globally monotone decreasing to zero, then $g'(y) \to 0$ as $y \to \infty$.*

Proof. Define $\rho(y) = \exp\{\alpha k \int_0^y s g^{\frac{1}{m}-1}(s)ds\}$. From (7), it follows that

$$(\rho g')'(y) = -\alpha m \rho(y) g^{\frac{1}{m}}(y)$$

and so

$$g'(y) = \frac{g'(0) - \alpha m \int_0^y g^{\frac{1}{m}}(s)\rho(s)ds}{\rho(y)}.$$

By $[-\alpha m \int_0^y g^{\frac{1}{m}}(s)\rho(s)ds]' = -\alpha m g^{\frac{1}{m}}(y)\rho(y) > 0$, the integral

$$I = -\alpha m \int_0^{\infty} g^{\frac{1}{m}}(s)\rho(s)ds$$

exists and $0 < I \leq \infty$. We claim that $g'(0) + I = 0$. Otherwise $|g'(y)| \to \infty$ as $y \to \infty$, since $\rho(y) \to 0$ as $y \to \infty$. A contradiction. Hence $g'(0) + I = 0$.
By L'Hôpital's Rule, we get that

$$\begin{aligned} \lim_{y\to\infty} g'(y) &= \lim_{y\to\infty} \frac{g'(0) - \alpha m \int_0^y g^{\frac{1}{m}}(s)\rho(s)ds}{\rho(y)} \\ &= \lim_{y\to\infty} \frac{-m}{kyg^{-1}(y)} \\ &= 0. \end{aligned}$$

□

By L'Hôpital's Rule again, we obtain that

$$\begin{aligned} \lim_{y\to\infty} \frac{g'(y)}{g(y)} &= \lim_{y\to\infty} \frac{g'(0) - \alpha m \int_0^y g^{\frac{1}{m}}(s)\rho(s)ds}{\rho(y)g(y)} \\ &= \lim_{y\to\infty} \frac{-\alpha m}{g^{-1/m}g' + \alpha k y} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{y\to\infty} \frac{yg'(y)}{g(y)} &= \lim_{y\to\infty} \frac{g'(0) - \alpha m \int_0^y g^{\frac{1}{m}}(s)\rho(s)ds}{y^{-1}\rho(y)g(y)} \\ &= \lim_{y\to\infty} \frac{-\alpha m}{-y^{-2}g^{-\frac{1}{m}+1} + y^{-1}g^{-1/m}g' + \alpha k} \\ &= -\frac{m}{k}. \end{aligned}$$

Let $z(y) = g'(y)/g(y)$. Then $z(y)$ satisfies the equation

$$z'(y) + \alpha k y g^{\frac{1}{m}-1}(y) z(y) + \alpha m g^{\frac{1}{m}-1}(y) + z^2(y) = 0.$$

It follows that

$$z(y) = \frac{\int_0^y [-\alpha m g^{\frac{1}{m}-1}(s)\rho(s) - z^2(s)\rho(s)]ds + z(0)}{\rho(y)}. \tag{16}$$

where $\rho(y) = \exp\{\alpha k \int_0^y s g^{\frac{1}{m}-1}(s)ds\}$.

The following proof is similar to the one given in [8].

Theorem 3 *The limit*

$$\lim_{y\to\infty} [y^{\frac{m}{k}} g(y)]$$

exists and is positive.

Proof. From (16), we can write

$$[yz(y) + \frac{m}{k}]y^{\lambda} = \frac{\int_0^y [-\alpha m g^{\frac{1}{m}-1}(s)\rho(s) - z^2(s)\rho(s)]ds + z(0) + \frac{m}{k} y^{-1}\rho(y)}{y^{-\lambda-1}\rho(y)},$$

where $\lambda \in (0, 2)$. Note that $\rho(y) \to 0$ and $z(y) \to 0$ as $y \to \infty$. From (16), we get $\int_0^y [-\alpha m g^{\frac{1}{m}-1}(s)\rho(s) - z^2(s)\rho(s)]ds + z(0) \to 0$ as $y \to \infty$.

By L'Hôpital's Rule, we get that

$$\begin{aligned} \lim_{y\to\infty} [yz(y) + \frac{m}{k}]y^{\lambda} &= \lim_{y\to\infty} \frac{-z^2 y^2 g^{1-\frac{1}{m}} - \frac{m}{k} g^{1-\frac{1}{m}}}{g^{1-\frac{1}{m}}(-\lambda - 1)y^{-\lambda} + y^{-\lambda+2}\alpha k} \\ &= 0, \forall \lambda \in (0, 2). \end{aligned}$$

By integration, we obtain

$$g(y) = A y^{-\frac{m}{k}} [1 + o(y^{-\lambda})]$$

as $y \to \infty$, for some positive constant A. Hence the Theorem follows. □

From Theorem 3 and using (4), we have

$$\lim_{t\to T^-} u(x, t) = A^{1/m} |x|^{-1/k}, \ x \neq 0,$$

for any globally monotone decreasing solution of (P).