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On The Fitting Ideals of Finitely Generated Modules

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On The Fitting Ideals of Finitely Generated Modules

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Abstract

In this master thesis, we will show some basic properties of the *Fitting ideals* of finitely generated modules and how to compute these ideals. Let E be a finitely generated R-module with the generators $e = (e_1, e_2, \ldots, e_q)$. If $c_1e_1 + c_2e_2 + \cdots + c_qe_q = 0$, then we call $(c_1, c_2, \ldots, c_q)$ a *relation* between the generators $e_1, e_2, \ldots, e_q$. Suppose M is a matrix rows of which are relations between $e_1, e_2, \ldots, e_q$. For any $0 \leq n \leq q$, the nth Fitting ideal of E is the ideal of R which is generated by all $(q-n) \times (q-n)$ subdeterminants of all such M. We prove that the nth Fitting ideal of E is not dependent on the particular sequence of generators that is used to compute it. In other words, the nth Fitting ideal is an invariant of the module E. So Fitting ideals are also called *Fitting invariants*. When E has a finite presentation, or equivalently is finitely presented, we have another approach to compute these Fitting ideals.

1 Preliminaries

Definition 1.1. Let R be a commutative ring with 1. A *multiplicatively closed subset* of R is a subset S of R such that $1 \in S$ and S is closed under multiplication. In other words, S is a subsemigroup of the multiplicative semigroup of R. We define a relation $\equiv$ on $R \times S$ as follows:

$$(a, s) \equiv (b, t) \Leftrightarrow u(ta - sb) = 0 \text{ for some } u \in S.$$

Then we have an equivalence relation. Let a/s denote the equivalence class of (a, s), and let $S^{-1}R$ denote the set of equivalence classes. We put a ring structure on $S^{-1}R$ by defining addition and multiplication of these fractions a/s as follows:

$$(a/s) + (b/t) = (ta + sb)/st,$$
$$(a/s)(b/t) = ab/st.$$

The commutative ring $S^{-1}R$ with 1 is called the ring of fractions of R with respect to S. Furthermore, $S^{-1}R$ is a non-trivial ring if and only if $0 \notin S$.

If I is an ideal of R, then we define

$$S^{-1}I = \{a/s \mid a \in I, s \in S\},$$

which is an ideal of $S^{-1}R$.

Let $A = (a_{ij})$ be a $p \times q$ matrix with entries in R and let S a multiplicative closed subset of R. We define

$$S^{-1}A = (a_{ij}/1)$$

to be a $p \times q$ matrix with entries in $S^{-1}R$.

The construction of $S^{-1}R$ can be carried through with an R-module E in place of the ring R. We define a relation $\equiv$ on $E \times S$ as follows:

$$(e, s) \equiv (f, t) \Leftrightarrow u(te - sf) = 0 \text{ for some } u \in S.$$

As before, this is an equivalence relation. Let e/s denote the equivalence class of (e, s). Let $S^{-1}E$ denote the set of such fractions and make $S^{-1}E$ into an $S^{-1}R$-module with

$$(e/s) + (f/t) = (te + sf)/st,$$
$$(r/t)(e/s) = re/ts.$$

The following are some properties of modules of fractions. Readers can find the proofs in [9].

Proposition 1.1. *Let R be a commutative ring with 1 and S a multiplicatively closed subset of R.*

1. *Let $g : E \to M$ be a homomorphism of R-modules. Then there is a homomorphism*
$$S^{-1}g : S^{-1}E \to S^{-1}M$$
of $S^{-1}R$-modules such that
$$S^{-1}g(e/s) = g(e)/s.$$

2. *Let* $E \xrightarrow{g} M \xrightarrow{h} N$ *be an exact sequence of* R*-modules. Then*

$$S^{-1}E \xrightarrow{S^{-1}g} S^{-1}M \xrightarrow{S^{-1}h} S^{-1}N$$

is an exact sequence of $S^{-1}R$*-modules.*

3. *Let* E *be an* R*-module generated by the elements of* $\{e_i\}_{i\in I}$*. Then* $S^{-1}E$ *is an* $S^{-1}R$*-module generated by elements of* $\{e_i/1\}_{i\in I}$*.*

4. *Let* F *be a free* R*-module with a basis* $\{e_i\}_{i\in I}$*. Let* $0 \notin S$*. Then* $S^{-1}F$ *is a free* $S^{-1}R$*-module with basis* $\{e_i/1\}_{i\in I}$*. In particular, if* F *is a finite free* R*-module, then* $S^{-1}F$ *is a finite free* $S^{-1}R$*-module.*

Definition 1.2. Let E be a finitely generated R-module with generators $e_1, e_2, \dots, e_q$. A sequence $(a_1, a_2, \dots, a_q) \in R^q$ is called a *relation* between $e_1, e_2, \dots, e_q$ if

$$a_1e_1 + a_2e_2 + \cdots + a_qe_q = 0.$$

Clearly the set of all such relations form an R-submodule of R^q, and it is called the *module of relations* between $e_1, e_2, \dots, e_q$.

Definition 1.3. An R-module E is said to be *finitely presented* if it is possible to find a finite system $e_1, e_2, \dots, e_q$ of generators such that the module of relations between these generators is finitely generated.

Theorem 1.2. *If* E *is finitely presented, then whatever finite system of generators we may choose the corresponding module of relations is also finitely generated.*

Proof. Since E is finitely presented, there is a finite system $x_1, x_2, \dots, x_m$ of generators such that the module of relations between these generators is finitely generated. Suppose $y_1, y_2, \dots, y_n$ is any finite system of generators of E. Let F be a free R-module of finite rank m and $\{f_1, f_2, \dots, f_m\}$ a basis for F. Define an epimorphism $\phi : F \to E$ so that $\phi(f_i) = x_i$ for all $1 \le i \le m$ and put $K = Ker\phi$. We now have an exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} E \longrightarrow 0.$$

Note that if $a_1, a_2, \dots, a_m \in R$ then

$$\begin{aligned} &a_1f_1 + a_2f_2 + \cdots + a_mf_m \in K \\ \Leftrightarrow &(a_1, a_2, \dots, a_m) \text{ is a relation between } x_1, x_2, \dots, x_m. \end{aligned}$$

It follows that K is isomorphic to the module of relations between $x_1, x_2, \ldots, x_m$. In a similar manner we can construct an exact sequence

$$0 \longrightarrow K' \longrightarrow F' \xrightarrow{\phi'} E \longrightarrow 0,$$

where F' is a finite free module with rank n and K' is isomorphic to the module of relations between $y_1, y_2, \ldots, y_n$.

By schanuel's lemma, we have $K' \oplus F \cong K \oplus F'$. Since K is finitely generated, $K' \oplus F \cong K \oplus F'$ are also finitely generated. It follows that K', which is a direct summand of finitely generated R-module, must be finitely generated as well. So the module of relations between $y_1, y_2, \ldots, y_n$ is finitely generated. □

Definition 1.4. A *finite presentation* of an R-module E is an exact sequence $G \longrightarrow F \longrightarrow E \longrightarrow 0$, where F and G are finite free R-modules.

Theorem 1.3. *An R-module E is finitely presented if and only if it has a finite presentation.*

Proof. Suppose E is finitely presented. From the proof of Theorem 1.2, we can define an epimorphism $\phi : F \to E$ such that F is a finite free R-module and $Ker\phi$, which is isomorphic to the module of relations, is finitely generated. Since $Ker\phi$ is finitely generated, we can define an epimorphism $\psi : G \to Ker\phi$ with a finite free R-module G, and hence $Im\psi = Ker\phi$. Thus we have an exact sequence

$$G \xrightarrow{\psi} F \xrightarrow{\phi} E \longrightarrow 0,$$

where G and F are finite free R-modules.

Conversely, suppose $G \xrightarrow{\psi} F \xrightarrow{\phi} E \longrightarrow 0$ is an exact sequence with finite free R-modules F and G. Let $\{f_1, f_2, \ldots, f_m\}$ be a basis for F, then E can be generated by $\phi(f_i), \phi(f_2), \ldots, \phi(f_m)$. Since G is a finite free R-module, $Ker\phi = Im\psi$ is finitely generated. So the module of relations between $\phi(f_1), \phi(f_2), \ldots, \phi(f_m)$ is isomorphic to $Ker\phi$ and hence finitely generated. □

2 Fitting Ideals

In this section, R is denoted to be a commutative ring with 1.

2.1 Determinantal Ideals

We first recall that if $M = (m_{ij})$ *is a* $n \times n$ *matrix, then the determinant of* M *is*

$$det(M) = \sum_{\sigma \in S_n} \left(sgn(\sigma) \prod_{k=1}^{n} m_{k\sigma(k)} \right), \tag{2.1}$$

where S_n *is the symmetric group of degree* n*, and*

$$sgn(\sigma) = \begin{cases} +1 & \textit{if } \sigma \textit{ is an even permutation} \ ; \\ -1 & \textit{if } \sigma \textit{ is an odd permutation} \ . \end{cases}$$

It is possible to expand a determinant along a row or column. For any row i*,*

$$det(M) = \sum_{j=1}^{n} (-1)^{i+j} m_{ij} det\left(\widetilde{M_{ij}}\right), \tag{2.2}$$

where $\widetilde{M_{ij}}$ *is the* $(n-1) \times (n-1)$ *matrix obtained from* M *by deleting row* i *and column* j*.*

Definition 2.1. Let $A = (a_{ij})$ be a $p \times q$ matrix with entries in R. Let $n \in \mathbb{N}$. We define

$$S_n^p = \{S = (s_1, s_2, \ldots, s_n) \in \mathbb{N}^n \mid 1 \leq s_1 < s_2 < \cdots < s_n \leq p\}.$$

If $n > p$, then $S_n^p = \emptyset$. If $n \leq p$, then S_n^p contains $\binom{p}{n}$ members. Suppose $1 \leq n \leq min(p, q)$, $S = (s_1, s_2, \ldots, s_n) \in S_n^p$ and $T = (t_1, t_2, \ldots, t_n) \in S_n^q$. We define

$$A_{ST}^{(n)} = \begin{vmatrix} a_{s_1t_1} & a_{s_1t_2} & \cdots & a_{s_1t_n} \\ a_{s_2t_1} & a_{s_2t_2} & \cdots & a_{s_2t_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s_nt_1} & a_{s_nt_2} & \cdots & a_{s_nt_n} \end{vmatrix}$$

to be the determinant of the $n \times n$ submatrix of A. With S and T ranging over S_n^p and S_n^q, respectively, we may view $A_{ST}^{(n)}$ as the ST-component of a $S_n^p \times S_n^q$ matrix $A^{(n)}$, which is called the *n-th exterior power* of A. For $n = 0$, we define $A^{(0)}$ to be the 1×1 matrix whose single entry is the identity element of R. We also note that $A^{(1)} = A$.

Lemma 2.1. *Let A be a $p \times q$ matrix and B a $q \times r$ matrix. Let $0 \le n \le min(p, q, r)$. Then*

$$(AB)^{(n)} = A^{(n)} B^{(n)}.$$

Proof. When $n = 0$, by definition, $(AB)^{(0)} = (1)_{1\times 1} = A^{(0)}B^{(0)}$. When $1 \le n \le min(p, q, r)$, let $A = (a_{ij})_{p\times q}$, $B = (b_{kl})_{q\times r}$, and $C = AB = (c_{uv})_{p\times r}$. For any $S = (s_1, s_2, \ldots, s_n) \in S_n^p$ and $T = (t_1, t_2, \ldots, t_n) \in S_n^r$,

$$\begin{aligned} C_{ST}^{(n)} &= \begin{vmatrix} c_{s_1t_1} & c_{s_1t_2} & \cdots & c_{s_1t_n} \\ c_{s_2t_1} & c_{s_2t_2} & \cdots & c_{s_2t_n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s_nt_1} & c_{s_nt_2} & \cdots & c_{s_nt_n} \end{vmatrix} \\ &= \begin{vmatrix} \sum_{\alpha_1=1}^{q} a_{s_1\alpha_1}b_{\alpha_1t_1} & \sum_{\alpha_2=1}^{q} a_{s_1\alpha_2}b_{\alpha_2t_2} & \cdots & \sum_{\alpha_n=1}^{q} a_{s_1\alpha_n}b_{\alpha_nt_n} \\ \sum_{\alpha_1=1}^{q} a_{s_2\alpha_1}b_{\alpha_1t_1} & \sum_{\alpha_2=1}^{q} a_{s_2\alpha_2}b_{\alpha_2t_2} & \cdots & \sum_{\alpha_n=1}^{q} a_{s_2\alpha_n}b_{\alpha_nt_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\alpha_1=1}^{q} a_{s_n\alpha_1}b_{\alpha_1t_1} & \sum_{\alpha_2=1}^{q} a_{s_n\alpha_2}b_{\alpha_2t_2} & \cdots & \sum_{\alpha_n=1}^{q} a_{s_n\alpha_n}b_{\alpha_nt_n} \end{vmatrix}, \end{aligned}$$

whence

$$C_{ST}^{(n)} = \sum_{1\le\alpha_1,\alpha_2,\cdots,\alpha_n\le q} b_{\alpha_1t_1}b_{\alpha_2t_2}\cdots b_{\alpha_nt_n} \begin{vmatrix} a_{s_1\alpha_1} & a_{s_1\alpha_2} & \cdots & a_{s_1\alpha_n} \\ a_{s_2\alpha_1} & a_{s_2\alpha_2} & \cdots & a_{s_2\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s_n\alpha_1} & a_{s_n\alpha_2} & \cdots & a_{s_n\alpha_n} \end{vmatrix}. \qquad (2.3)$$

However, the determinants in (2.3) vanishes, unless $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct. Moreover, for any $K = (k_1, k_2, \ldots, k_n) \in S_n^q$, if $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a permutation of $\{k_1, k_2, \ldots, k_n\}$, i.e. there is a σ in the symmetric group S_n such that $\alpha_i = k_{\sigma(i)}$ for $i = 1, 2, \ldots, n$, then

$$\begin{vmatrix} a_{s_1\alpha_1} & a_{s_1\alpha_2} & \cdots & a_{s_1\alpha_n} \\ a_{s_2\alpha_1} & a_{s_2\alpha_2} & \cdots & a_{s_2\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s_n\alpha_1} & a_{s_n\alpha_2} & \cdots & a_{s_n\alpha_n} \end{vmatrix} = sgn_{\alpha_1\alpha_2\ldots\alpha_n} A_{SK}^{(n)},$$

where

$$
\begin{aligned}
& sgn_{\alpha_1\alpha_2\ldots\alpha_n} \\
= & \begin{cases} +1 & \text{if } \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ is an even permutation of } \{k_1, k_2, \ldots, k_n\}; \\ -1 & \text{if } \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ is an odd permutation of } \{k_1, k_2, \ldots, k_n\}. \end{cases}
\end{aligned}
$$

For this K, the contribution to the sum on the right hand side of (2.3) from all the permutations of $\{k_1, k_2, \ldots, k_n\}$ is equal to $A^{(n)}_{SK}B^{(n)}_{KT}$. Indeed,

$$
\begin{aligned}
C^{(n)}_{ST} &= \sum_{K\in S^q_n}\left(\sum_{\sigma\in S_n}\left(b_{k_{\sigma(1)}t_1}b_{k_{\sigma(2)}t_2}\cdots b_{k_{\sigma(n)}t_n}sgn(\sigma)A^{(n)}_{SK}\right)\right) \\
&= \sum_{K\in S^q_n} A^{(n)}_{SK}\left(\sum_{\sigma\in S_n} sgn(\sigma)\prod_{i=1}^{n} b_{k_{\sigma(i)}t_i}\right) \\
&= \sum_{K\in S^q_n} A^{(n)}_{SK}B^{(n)}_{KT}.
\end{aligned}
$$

and we have

$$(AB)^{(n)} = A^{(n)}B^{(n)}.$$

□

Definition 2.2. Let A be a $p \times q$ matrix and $n \geq 0$. We define the *determinantal ideal* $I_n(A)$ of A to be the ideal generated by all entries in $A^{(n)}$, or equivalently by all $n \times n$ subdeterminants of A . Especially we define

$$I_n(A) = 0 \text{ for } n > min(p, q). \tag{2.4}$$

By definition,

$$
\begin{aligned}
& I_0(A) = R, \\
& I_1(A) = \text{the ideal generated by all entries in } A.
\end{aligned}
$$

Furthermore, by (2.2), we have

$$R = I_0(A) \supseteq I_1(A) \supseteq I_2(A) \supseteq \cdots . \tag{2.5}$$

Let B be a $q \times r$ matrix, then, by Lemma 2.1, we also have

$$I_n(AB) \subseteq I_n(A) \cap I_n(B), \tag{2.6}$$

which extends to a product of any finite number of matrices.

Proposition 2.2. *Let $A = (a_{ij})$ be a $p \times q$ matrix with entries in R and S a multiplicatively closed subset of R. Suppose $S^{-1}A = (a_{ij}/1)$ is a $p \times q$ matrix with entries in $S^{-1}R$. If now $J \in S_n^p$ and $K \in S_n^q$, then*

$$\left(S^{-1}A\right)_{JK}^{(n)} = A_{JK}^{(n)}/1,$$

and hence

$$I_n(S^{-1}A) = S^{-1}I_n(A). \tag{2.7}$$

Definition 2.3. The $p \times q$ matrix A' is said to be *equivalent* to the $p \times q$ matrix A if there exists invertible $p \times p$ matrix U and invertible $q \times q$ matrix V such that $A' = UAV$. Clearly it is an equivalence relation.

Lemma 2.3. *If A and B are equivalent $p \times q$ matrices, then*

$$I_n(A) = I_n(B) \quad \textit{for all } n \geq 0.$$

Proof. Since A and B are equivalent, there exists invertible $p \times p$ matrix and $q \times q$ matrix V such that $A = UBV$ and $B = U^{-1}AV^{-1}$. By (2.6), we have

$$I_n(A) \subseteq I_n(B) \text{ and } I_n(B) \subseteq I_n(A).$$

Thus $I_n(A) = I_n(B)$ for all $n \geq 0$. □

In fact, if R is a PID, then the converse of Lemma 2.3 holds.

Corollary 2.4. *Let the matrix A' be obtained by means of elementary row or column operations. Then A' and A are equivalent and hence they have the same determinantal ideals.*

Lemma 2.5. *Let A be a $p \times q$ matrix, B a $q \times p$ matrix, and $ABA = A$. Then there exists an idempotent γ such that $I_1(A) = R\gamma$.*

Proof. Let $A = (a_{ij})$, $C = BA = (c_{kl})$, and $I_1(C)$ be the ideal generated by all c_{kl}. We first prove that there exists $\gamma \in I_1(C)$ such that $(1-\gamma)I_1(A) = 0$. Since $AC = A$, we have $A(I - C) = 0$, where I is the $q \times q$ identity matrix. Hence $(a_{i1}, a_{i2}, \ldots, a_{iq})(I - C) = 0$ for $i = 1, 2, \ldots, p$. For each $i = 1, 2, \ldots, p$, let D_{ij} be the matrix obtained by taking the $q \times q$ identity matrix and replacing its jth row by $(a_{i1}, a_{i2}, \ldots, a_{iq})$. Then $D_{ij}(I - C)$ has a row of zeros and therefore $det\,(D_{ij}(I - C)) = 0$. Hence

$$a_{ij}det(I - C) = det(D_{ij})det(I - C) = det\left(D_{ij}\left(I - C\right)\right) = 0 \quad \text{for all } i, j,$$

and $det(I-C)I_1(A)=0$. Since $det(I-C)=1-\gamma$ for a suitable $\gamma \in I_1(C)$, we have $(1-\gamma)I_1(A)=0$.

Moreover, $C=BA$ implies $I_1(C) \subseteq I_1(A)$, so $\gamma \in I_1(A)$ and therefore $(1-\gamma)\gamma=0$. Thus γ is an idempotent and $R\gamma \subseteq I_1(A)$. For any $x \in I_1(A)$, we have $(1-\gamma)x=0$, so $x=\gamma x$ and $x \in R\gamma$. Thus $I_1(A)=R\gamma$. □

Theorem 2.6. *Let A be a $p \times q$ matrix, B a $q \times p$ matrix, and $ABA=A$. Then for any $n \geq 0$, $I_n(A)$ is generated by an idempotent. Hence if R has no nontrivial idempotents, then either $I_n(A)=0$ or $I_n(A)=R$.*

Proof. When $n=0$, $I_n(A)=R$ is generated by 1, which is an idempotent. For $n > min(p,q)$, $I_n(A)=0$ is generated by 0, which is an idempotent. For $1 \leq n \leq min(p,q)$, we have $A^{(n)}B^{(n)}A^{(n)}=A^{(n)}$ by Lemma 2.1. Since $I_n(A)$ is the ideal generated by all entries in $A^{(n)}$, the desired result therefore follows from Lemma 2.5. □

In fact, if R has no non-trivial idempotents and A is a $p \times q$ matrix, then the following two statements are equivalent:

1. *There is a $p \times q$ matrix B such that $ABA=A$.*

2. *For each $n \geq 0$, $I_n(A)$ is either zero or the ring R itself.*

Readers can find the proof on page 6 in [9].

Definition 2.4. Let E be a finitely generated R-module. Suppose $e_1, e_2, \ldots, e_q$ are elements which generate E. We use e to denote the sequence $(e_1, e_2, \ldots, e_q)$ and say that E has the generators $e=(e_1, e_2, \ldots, e_q)$. Now let $n \geq 0$ be an integer. We define the *determinantal ideal* of e to be

$$I_n(e|E)=\sum_A I_n(A), \tag{2.8}$$

where

1. A ,which is called a matrix of relations between $e_1, e_2, \ldots, e_q$, stands for an arbitrary (finite) matrix with q columns each row of which is a relation between $e_1, e_2, \ldots, e_q$, and

2. $I_n(A)$ denotes the nth determinantal ideal of A.

Accordingly $I_n(e|E)$ is the ideal generated by all $n \times n$ subdeterminants of all matrices of relations between $e_1, e_2, \ldots, e_q$. Note that, by (2.4) and (2.5), we have

$$R = I_0(e|E) \supseteq I_1(e|E) \supseteq I_2(e|E) \supseteq \cdots , \tag{2.9}$$

and

$$I_n(e|E) = 0 \quad \text{if } n > q. \tag{2.10}$$

There may be infinitely many matrices A which contribute to the sum in (2.8). The following will show that how we can reduce the number.

Lemma 2.7. *Let E be a finitely presented R-module with the generators $e = (e_1, e_2, \ldots, e_q)$. There is a matrix M such that rows of it can generate the module of relations between $e_1, e_2, \ldots, e_q$, and*

$$I_n(e|E) = \sum_A I_n(A) = I_n(M) \quad \textit{for all } n \geq 0.$$

Proof. Since the module of relations between $e_1, e_2, \ldots, e_q$ is finitely generated, we define M to be the matrix rows of which are the finite many generators. It is clear that $I_n(M) \subseteq \sum_A I_n(A) = I_n(e|E)$.

On the other hand, for any matrix A of relations between $e_1, e_2, \ldots, e_q$, each row of A is an R-linear combination of the rows of A. Hence we can find a matrix P such that $A = PM$. By (2.6), we have $I_n(A) \subseteq I_n(M)$ for all A. Thus $\sum_A I_n(A) \subseteq I_n(M)$ and therefore $I_n(e|E) = \sum_A I_n(A) = I_n(M)$. □

Corollary 2.8. *Let R be a Noetherian ring and E a finitely generated R-module with generators $e = (e_1, e_2, \ldots, e_q)$. Then there is a matrix M rows of which can generate the module of relations between $e_1, e_2, \ldots, e_q$ and*

$$I_n(e|E) = I_n(M) \quad \textit{for all } n \geq 0.$$

Proof. Since R is a Noetherian ring and R^q is an R-module, we know that R^q is a Noetherian R-module. Hence the module of relations between $e_1, e_2, \ldots, e_q$, which is a submodule of R^q, is also finitely generated, and therefore E is a finitely presented R-module. Thus, by Lemma 2.7, we have the result. □

Lemma 2.9. *Let E be a finitely generated R-module with the generators $e = (e_1, e_2, \ldots, e_q)$. Let S be a multiplicatively closed subset of R and let $0 \notin S$. Suppose $e/1 = (e_1/1, e_2/1, \ldots, e_q/1)$. Then*

$$I_n\left(e/1 \,\middle|\, S^{-1}E\right) = S^{-1} I_n\left(e \,|E\right) \quad \textit{for all } n \geq 0.$$

Proof. We can construct an exact sequence

$$G \xrightarrow{\psi} F \xrightarrow{\phi} E \longrightarrow 0, \tag{2.11}$$

where

1. F is a free R-module with a basis $\{f_1, f_2, \dots, f_q\}$,
2. for $1 \leq k \leq n$, we have $\phi(f_k) = e_k$,
3. G is a free R-module with a basis $\{g_j\}_{j\in J}$.

Let $\psi(g_j) = a_{j1}f_1 + a_{j2}f_2 + \cdots + a_{jq}f_q$ for suitable $a_{jk} \in R$. Since

$$\phi\left(\psi(g_j)\right) = a_{j1}e_1 + a_{j2}e_2 + \cdots + a_{jq}e_q = 0,$$

$(a_{j1}, a_{j2}, \dots, a_{jq})$ belongs to the module of relations between $e_1, e_2, \dots, e_q$. Furthermore, for any $(c_1, c_2, \dots, c_q)$ belongs to the module of relations between $e_1, e_2, \dots, e_q$, we have $\sum_{i=1}^{q} c_i e_i = 0$, and hence $\sum_{k=1}^{q} c_k f_k \in Ker\phi = Im\psi$. So there exists $g \in G$ such that $\psi(g) = \sum_{k=1}^{q} c_k f_k$. Suppose $g = \sum_{j=1}^{n} r_j g_j$, then

$$\begin{aligned}\sum_{k=1}^{q} c_k f_k &= \psi(g) = \sum_{j=1}^{n} r_j \psi(g_j) \\ &= \sum_{j=1}^{n} r_j \left(\sum_{k=1}^{q} a_{jk} f_k\right) = \sum_{k=1}^{q} \left(\sum_{j=1}^{n} r_j a_{jk}\right) f_k.\end{aligned}$$

So $c_k = \sum_{j=1}^{n} r_j a_{jk}$, and hence

$$(c_1, c_2, \dots, c_q) = \sum_{j=1}^{n} r_j (a_{j1}, a_{j2}, \dots, a_{jq}).$$

Therefore, $\{(a_{j1}, a_{j2}, \dots, a_{jq})\}_{j\in J}$ is a family of relations between $e_1, e_2, \dots, e_q$ which generates the module of relations between $e_1, e_2, \dots, e_q$. Let Ω be the set of all matrices A having the property that each row of A is one of the relations $(a_{j1}, a_{j2}, \dots, a_{jq})$. Then, as we see in Lemma 2.7,

$$I_n(e\,|E) = \sum_{A\in\Omega} I_n(A) \quad \text{for all } n \geq 0, \tag{2.12}$$

and hence

$$S^{-1}I_n\left(e\,|E\right) = S^{-1}\left(\sum_{A\in\Omega} I_n(A)\right) = \sum_{A\in\Omega}\left(S^{-1}I_n(A)\right). \qquad (2.13)$$

On the other hand, by proposition 1.1, (2.11) can induce an exact sequence

$$S^{-1}G \xrightarrow{S^{-1}\psi} S^{-1}F \xrightarrow{S^{-1}\phi} S^{-1}E \longrightarrow 0,$$

where $S^{-1}F$ and $S^{-1}G$ are free $S^{-1}R$-module with bases $\{f_k/1\}_{1\leq k\leq g}$ and $\{g_j/1\}_{j\in J}$, respectively. Also $S^{-1}\phi\,(f_k/1) = \phi(f_k)/1 = e_k/1$ and

$$S^{-1}\psi(g_j/1) = (a_{j1}/1)(g_1/1) + (a_{j2}/1)(g_2/1) + \cdots + (a_{jn}/1)(g_n/1).$$

For any matrix $A = (a_{ij})$, if we define $S^{-1}A = (a_{ij}/1)$ with entries in $S^{-1}R$, then $\{S^{-1}A\}_{A\in\Omega}$ is the set of all matrices rows of which belong to the family $\{(a_{j1}/1, a_{j2}/1, \ldots, a_{jq}/1)\}_{j\in J}$. Accordingly, the argument which yielded (2.12) can be used again this time to show that

$$I_n\left(e/1\,\middle|S^{-1}E\right) = \sum_{A\in\Omega} I_n(S^{-1}A).$$

So, by (2.7) and (2.13),

$$I_n\left(e/1\,\middle|S^{-1}E\right) = \sum_{A\in\Omega}\left(S^{-1}I_n(A)\right) = S^{-1}I_n\left(e\,|E\right).$$

The proof is now complete. □

Lemma 2.10. *Let E be a finitely generated R-module and let $e_1, e_2, \ldots, e_q$ be elements which generate E. Suppose $g_1, g_2, \ldots, g_r$ are elements of E. If we denote $e = (e_1, e_2, \ldots, e_q)$, $g = (g_1, g_2, \ldots, g_r)$, and $eg = (e_1, e_2, \ldots, e_q, g_1, g_2, \ldots, g_r)$, then:*

1. *$I_t\left(eg\,|E\right) = R$ for $0 \leq t \leq r$;*
2. *$I_s\left(e\,|E\right) = I_{s+r}\left(eg\,|E\right)$ for all $s \geq 0$.*

Proof. For each k with $1 \leq k \leq r$, since $g_k \in Re_1 + Re_2 + \cdots + Re_q$, there are $c_{k1}, c_{k2}, \ldots, c_{kq} \in R$ such that

$$c_{k_1}e_1 + c_{k_2}e_2 + \cdots + c_{k_q}e_q + g_k = 0.$$

Now let $A = (a_{ij})$ be a $p \times q$ matrix each row of which is a relation between $e_1, e_2, \ldots, e_q$ and let

$$B = \begin{pmatrix} a_{11} & \cdots & a_{1q} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pq} & 0 & \cdots & 0 \\ c_{11} & \cdots & c_{1q} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{r1} & \cdots & c_{rq} & 0 & \cdots & 1 \end{pmatrix}.$$

Then the rows of B are relations between $(e_1, e_2, \ldots, e_q, g_1, g_2, \ldots, g_r)$. Since $I_t(B) = R$ for $0 \leq t \leq r$, this shows that

$$I_t\left(eg\,|E\right) = R \text{ for all } 0 \leq t \leq r.$$

By elementary column operations, we can change B to a matrix

$$B' = \begin{pmatrix} A & 0 \\ 0 & I_{r\times r} \end{pmatrix}$$

and such operations do not change the determinant ideals by Corollary2.4. Then we conclude that for all $s \geq 0$ we have

$$I_s(A) = I_{s+r}(B') = I_{s+r}(B) \subseteq I_{s+r}\left(eg\,|E\right),$$

and therefore

$$I_s\left(e\,|E\right) \subseteq I_{s+r}\left(eg\,|E\right).$$

Conversely, let D be a matrix with $q + r$ columns each row of which is a relation between $e_1, e_2, \ldots, e_q, g_1, g_2, \ldots, g_r$. Let

$$Q = \begin{pmatrix} & & & D & & \\ c_{11} & \cdots & c_{1q} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{r1} & \cdots & c_{rq} & 0 & \cdots & 1 \end{pmatrix}.$$

By elementary row operations, we can bring Q to the form

$$Q' = \begin{pmatrix} & A' & & 0 \\ c_{11} & \cdots & c_{1q} & \\ \vdots & \ddots & \vdots & I_{r\times r} \\ c_{r1} & \cdots & c_{rq} & \end{pmatrix}.$$

Since the rows of Q are relations between $e_1, e_2, \ldots, e_q, g_1, g_2, \ldots, g_r$, so do the rows of Q'. It follows that the rows of A' are relations between $e_1, e_2, \ldots, e_q$ and therefore $I_s(A') \subseteq I_s(e\,|E)$ for all $s \geq 0$. But

$$I_{s+r}(D) \subseteq I_{s+r}(Q) = I_{s+r}(Q') = I_s(A')$$

shows that

$$I_{s+r}(D) \subseteq I_s(e\,|E)\,.$$

Finally by allowing D to vary, we conclude that

$$I_{s+r}(eg\,|E) \subseteq I_s(e\,|E)$$

and with this the proof is complete. □

2.2 Fitting Ideals

In this section, we will define the Fitting ideals of a finitely generated R-module and show some basic properties of them.

Definition 2.5. Let E be a finitely generated R-module with the generators $e = (e_1, e_2, \ldots, e_q)$. Then we define

$$\mathfrak{F}_n(E) = \begin{cases} I_{q-n}(e\,|E) & \text{if } 0 \leq n \leq q; \\ R & \text{if } n > q, \end{cases}$$

and call $\mathfrak{F}_n(E)$ the *nth Fitting ideal* (after H. Fitting, who investigated their properties) or the *nth Fitting invariant* of E.

The Fitting ideal $\mathfrak{F}_0(E)$ is called the *zero-th Fitting ideal* or *initial Fitting ideal*. In some applications it is the only one which comes up, in which case it is called *the Fitting ideal* $\mathfrak{F}(E)$ of E. It is the ideal generated by all $q \times q$ subdeterminants of all matrices each row of which is a relation between the q generators of the module E.

In fact, these Fitting ideals are not depend on the particular sequence of generators $e = (e_1, e_2, \ldots, e_q)$ that is used to compute them. We write this to be a theorem.

Theorem 2.11. *Let E be a finitely generated R-module with the generators $e = (e_1, e_2, \ldots, e_q)$ and $g = (g_1, g_2, \ldots, g_r)$. Then for each n, the two Fitting ideals defined by e and g respectively are the same.*

Proof. Let $\mathfrak{F}_n^{(e)}(E)$ and $\mathfrak{F}_n^{(g)}(E)$ denote the nth Fitting ideals defined by e and g, respectively. Without loss of generality, we assume that $q \le r$. Since e and g both can generate E, we can apply Lemma 2.10. For $0 \le n \le q$, by Lemma 2.10, we have

$$I_{q-n}(e\,|E) = I_{q-n+r}(eg\,|E) = I_{r-n+q}(ge\,|E) = I_{r-n}(g\,|E),$$

and hence

$$\mathfrak{F}_n^{(e)}(E) = I_{q-n}(e\,|E) = I_{r-n}(g\,|E) = \mathfrak{F}_n^{(g)}(E).$$

For $q \le n \le r$, we have

$$I_{r-n}(g\,|E) = I_{r-n+q}(eg\,|E) = R,$$

and hence

$$\mathfrak{F}_n^{(e)}(E) = R = I_{r-n}(g\,|E) = \mathfrak{F}_n^{(g)}(E).$$

For $q \le r < n$, by definition we have

$$\mathfrak{F}_n^{(e)}(E) = R = \mathfrak{F}_n^{(g)}(E).$$

Thus we complete the proof. □

We now can restate some of our earlier conclusions in terms of Fitting ideals.

Corollary 2.12. *Let E be a finitely generated R-module.*

1. *We have an increasing sequence*

$$\mathfrak{F}_0(E) \subseteq \mathfrak{F}_1(E) \subseteq \mathfrak{F}_2(E) \subseteq \cdots$$

of ideals. Furthermore, if E can be generated by q elements, then

$$\mathfrak{F}_q(E) = R.$$

2. *If E is finitely presented, then $\mathfrak{F}_n(E)$ is a finitely generated ideal for all n. Suppose E can be generated by q elements and M is a matrix rows of which can generate the module of relations between the q generators of E. Then*

$$\mathfrak{F}_n(E) = \begin{cases} I_{q-n}(M) & \text{if } 0 \le n < q; \\ R & \text{if } n \ge q. \end{cases}$$

3. *Suppose S is a mutiplicatively closed subset of R with $0 \notin S$. Then for every integer $n \geq 0$ we have*

$$S^{-1}\mathfrak{F}_n(E) = \mathfrak{F}_n^{(S^{-1}R)}\left(S^{-1}E\right),$$

where $\mathfrak{F}_n^{(S^{-1}R)}(S^{-1}E)$ denotes the nth Fitting ideal of the $S^{-1}R$-module $S^{-1}E$.

Proof. This proposition is directly obtained by (2.9), Lemma 2.7, and Lemma 2.9, respectively. □

Example 2.1. Let E be a free R-module of rank q. Then

$$\mathfrak{F}_n(E) = \begin{cases} 0 & \text{if } 0 \leq n < q; \\ R & \text{if } n \geq q. \end{cases}$$

This is immediate from the definitions of nth Fitting ideals and the fact that the only relation of a basis for E is the trivial one.

Lemma 2.13. *Let E be a finitely generated R-module. If there exists $f : E \longrightarrow E'$ which is a surjection of R-modules, then*

$$\mathfrak{F}_n(E) \subseteq \mathfrak{F}_n(E') \quad \text{for all } n \geq 0.$$

Proof. Let E can be generated by $e_1, e_2, \ldots, e_q$. Since f is a surjection, E' can be generated by $f(e_1), f(e_2), \ldots, f(e_q)$. For any relation $(a_1, a_2, \ldots, a_q)$ between $e_1, e_2, \ldots, e_q$,

$$\sum_{i=1}^{q} a_i f(e_i) = f\left(\sum_{i=1}^{q} a_i e_i\right) = f(0) = 0,$$

so $(a_1, a_2, \ldots, a_q)$ is also a relation between $f(e_1), f(e_2), \ldots, f(e_q)$. Therefore, if $0 \leq n \leq q$,

$$\mathfrak{F}_n(E) = \sum_{A} I_{q-n}(A) \subseteq \sum_{A'} I_{q-n}(A') = \mathfrak{F}_n(E'),$$

where A and A' are matrices of relations between $e_1, e_2, \ldots, e_q$ and $f(e_1), f(e_2), \ldots, f(e_q)$, respectively. If $n > q$, then

$$\mathfrak{F}_n(E) = \mathfrak{F}_n(E') = R.$$

□

It is clear that any two isomorphic finitely generated modules have the same nth Fitting ideals. We note that if E is an R-submodule of E' then we do not necessarily have an inclusion of the two nth Fitting ideals. For an example where R is a polynomial ring in three variables over a field, see Ex.10(g) of chapter VII in [4].

We recall that for any R-module E we denote the annihilator of E in R to be

$$Ann_R(E) = \{a \in R \,|\, aE = 0\}$$

We also recall that if I and I' are ideals of R, we denote

$$II' = \left\{ \sum_{i=1}^{n} r_i r_i' \,|\, \textit{for some } n \in \mathbb{N}, r_i \in I, r_i' \in I' \right\}.$$

Theorem 2.14. *Let E be an R-module which can be generated by q elements. Then*

$$(Ann_R(E))^q \subseteq \mathfrak{F}(E) \subseteq Ann_R(E).$$

In particular, if E can be generated by a single element, then

$$\mathfrak{F}(E) = Ann_R(E).$$

Proof. Let $E = Re_1 + Re_2 + \cdots + Re_q$ and $c_1, c_2, \ldots, e_q \in Ann_R(E)$. Then each row of the $q \times q$ diagonal matrix

$$C = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_q \end{pmatrix}$$

is a relation between $e_1, e_2, \ldots, e_q$ and therefore

$$c_1 c_2 \cdots c_q = det(C) \in I_q(C) \subseteq \mathfrak{F}(E).$$

This shows that $(Ann_R(E))^q \subseteq \mathfrak{F}(E)$.

Now, let A be a $q \times q$ matrix of relations between $e_1, e_2, \ldots, e_q$. Then $det(A)e_i = 0$ for $i = 1, 2, \ldots, q$, as may be seen by multiplying the equation

$$A \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_q \end{pmatrix} = 0$$

by the adjugate of A. Thus $det(A) \in Ann_R(E)$. Since $\mathfrak{F}(E)$ is generated by elements of the form $det(A)$, it follows that $\mathfrak{F}(E) \subseteq Ann_R(E)$. □

Corollary 2.15. *Let $E = R/I$ for some ideal I of R. Then $\mathfrak{F}(E) = I$.*

Proof. The module R/I can be generated by a single element $\overline{1}$, which is the image of 1 in R/I. Consequently, by Theorem 2.14,

$$\mathfrak{F}(E) = Ann_R(R/I) = I.$$

□

Lemma 2.16. *Let $0 \longrightarrow E' \xrightarrow{\phi_1} E \xrightarrow{\phi_2} E'' \longrightarrow 0$ be an exact sequence of finitely generated R-modules. For integers $m, n \geq 0$ we have*

$$\mathfrak{F}_m(E')\,\mathfrak{F}_n(E'') \subseteq \mathfrak{F}_{m+n}(E).$$

In particular,

$$\mathfrak{F}(E')\,\mathfrak{F}(E'') \subseteq \mathfrak{F}(E).$$

Proof. Let $e_1, e_2, \ldots, e_p$ be elements in E such that their images $e''_1, e''_2, \ldots, e''_p$ in E'' generate E''. Let $e'_1, e'_2, \ldots, e'_q$ generate E' and $\phi_1(e'_i) = e_{p+i}$ for $i = 1, 2, \ldots, q$. Then $e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}$ is a system of generators for E. We first consider the case that $0 \leq m \leq q$ and $0 \leq n \leq p$. Let A be a matrix with p columns rows of which are relations between $e''_1, e''_2, \ldots, e''_p$. If $(a_1, a_2, \ldots, a_p)$ is such a relation, then

$$\phi_2\left(\sum_{i=1}^{p} a_i e_i\right) = \sum_{i=1}^{p} a_i \phi_2(e_i) = \sum_{i=1}^{p} a_i e''_i = 0.$$

We have $\sum_{i=1}^{p} a_i e_i \in Ker\phi_2 = Im\phi_1$, and hence there exists $b_1, b_2, \ldots, b_q \in R$ such that

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=1}^{q} b_j e_{p+j} = 0.$$

Thus we can find a matrix B, with q columns and the same number of rows as A, such that the rows of the matrix

$$(A\,|B\,)$$

are relations between $e_1, \ldots, e_{p+q}$. Let C be a matrix with q columns rows of which are relations between $e'_1, e'_2, \ldots, e'_q$. Then

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

is a matrix of relations between $e_1, \ldots, e_{p+q}$. If d' is a $(q-m) \times (q-m)$ subdeterminant of C and d'' a $(p-n) \times (p-n)$ subdeterminant of A, then $d'd''$ is a $(p+q-m-n) \times (p+q-m-n)$ subdeterminant of D and hence $d'd'' \in \mathfrak{F}_{m+n}(E)$. Since $\mathfrak{F}_m(E')$ is generated by determinants like d' and $\mathfrak{F}_n(E'')$ is generated by determinants like d'', we prove

$$\mathfrak{F}_m(E')\,\mathfrak{F}_n(E'') \subseteq \mathfrak{F}_{m+n}(E)$$

in this case.

If $m > q$ and $n > p$, then $\mathfrak{F}_m(E') = \mathfrak{F}_n(E'') = \mathfrak{F}_{m+n}(E) = R$ and hence the lemma is trivial in this case.

If $0 \leq m \leq q$ and $n > p$, then $\mathfrak{F}_n(E'') = R = \mathfrak{F}_p(E'')$ and therefore

$$\mathfrak{F}_m(E')\,\mathfrak{F}_n(E'') = \mathfrak{F}_m(E')\,\mathfrak{F}_p(E'') \subseteq \mathfrak{F}_{m+p}(E) \subseteq \mathfrak{F}_{m+n}(E)\,.$$

A similar argument proves the remaining case with $m > q$ and $0 \leq n \leq p$. Thus we have

$$\mathfrak{F}_m(E')\,\mathfrak{F}_n(E'') \subseteq \mathfrak{F}_{m+n}(E)\,.$$

By taking $m = n = 0$, we have

$$\mathfrak{F}(E')\,\mathfrak{F}(E'') \subseteq \mathfrak{F}(E)\,.$$

□

Theorem 2.17. *Let E' and E'' be finitely generated R-modules. For any integer $n \geq 0$ we heve*

$$\mathfrak{F}_n(E' \oplus E'') = \sum_{r+s=n} \mathfrak{F}_r(E')\,\mathfrak{F}_s(E'')\,.$$

In particular,

$$\mathfrak{F}(E' \oplus E'') = \mathfrak{F}(E')\,\mathfrak{F}(E'')\,.$$

Proof. Let $e'_1, e'_2, \ldots, e'_p$ generate E' and $e''_1, e''_2, \ldots, e''_q$ generate E''. Then $e'_1, e'_2, \ldots, e'_p, e''_1, e''_2, \ldots, e''_q$ generate $E' \oplus E''$. By Lemma 2.16, we know that

$$\sum_{r+s=n} \mathfrak{F}_r(E') \mathfrak{F}_s(E'') \subseteq \mathfrak{F}_n(E' \oplus E''),$$

so we have to prove the converse.

If $n \geq p+q$, then we can choose $r \geq p$ and $s \geq q$ such that $s+r=n$. So we have

$$\mathfrak{F}_n(E' \oplus E'') = \mathfrak{F}_r(E') = \mathfrak{F}_s(E'') = R,$$

and obtain the opposite inclusion.

Now we assume $0 \leq n < p+q$. In what follows, Q will be used to denote a matrix of the form

$$Q = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix},$$

where the rows of A' are relations between $e'_1, e'_2, \ldots, e'_p$ and the rows of A'' are relations between $e''_1, e''_2, \ldots, e''_q$. Note that, for any $a'_1, a'_2, \ldots, a'_q, a''_1, a''_2 \ldots, a''_q \in R$, if

$$a'_1 e'_1 + \cdots + a'_p e'_p + a''_1 e''_1 + \cdots + a''_q e''_q = 0,$$

then $(a'_1, a'_2, \ldots, a'_q)$ is a relations between $e'_1, e'_2, \ldots, e'_p$ and $(a''_1, a''_2 \ldots, a''_q)$ a relations between $e''_1, e''_2, \ldots, e'_q$. Hence the rows of matrices such as Q generate the module of relations between $e'_1, e'_2, \ldots, e'_p, e''_1, e''_2, \ldots, e''_q$ and therefore

$$\mathfrak{F}_n(E' \oplus E'') = \sum_Q I_{p+q-n}(Q).$$

Let d be a $(p+q-n) \times (p+q-n)$ subdeterminant of Q. Then d is the determinant of the submatrix D of Q which has the form

$$D = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix},$$

where B' is a $k' \times (p-r)$ matrix and B'' a $k'' \times (q-s)$ matrix with some nonneqtive integers k', k'', r, s satisfying

$$k' + k'' = p+q-n \text{ and } r+s=n.$$

However, $d = 0$ unless $k' = p-r, k'' = q-s$, and B' is a submatrix of A', B'' a submatrix of A''. In that case,

$$d = det(B')det(B'') \subseteq \mathfrak{F}_r(E') \mathfrak{F}_s(E'').$$

Since $\mathfrak{F}_n(E' \oplus E'')$ is generated by elements such as d,

$$\mathfrak{F}_n(E' \oplus E'') \subseteq \sum_{r+s=n} \mathfrak{F}_r(E') \mathfrak{F}_s(E'') .$$

This proof is complete. □

Corollary 2.18. *Let*

$$E = R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_t,$$

where $I_1, I_2, \ldots, I_t$ are ideals of R. Then

$$\mathfrak{F}(E) = I_1 I_2 \cdots I_t.$$

Proof. Since $I_1, I_2, \cdots, I_t$ are ideals of R, by Corollary 2.15, we have

$$\mathfrak{F}(R/I_k) = I_k \quad \text{for } k = 1, 2, \ldots, t.$$

Thus, by Theorem 2.17,

$$\mathfrak{F}(E) = \mathfrak{F}(R/I_1) \mathfrak{F}(R/I_2) \cdots \mathfrak{F}(R/I_t) = I_1 I_2 \ldots I_t.$$

□

Lemma 2.19. *Let E be a finitely generated R-module and I an ideal of R. Then*

1. $\mathfrak{F}_n(E/IE) \subseteq \mathfrak{F}_n(E) + I \subseteq R$;
2. $\mathfrak{F}_n^{(R/I)}(E/IE)$ *is the image of* $\mathfrak{F}_n(E)$ *in* R/I, *where* $\mathfrak{F}_n^{(R/I)}(E/IE)$ *denotes the nth Fitting ideal of the R/I-module E/IE.*

Proof. In this proof, we use $\overline{r}$ to denote the image of r in R/I, and use $\overline{e}$ to denote the image of e in E/IE. Suppose E can be generated by $e_1, e_2, \ldots, e_q$.

(1) We know that E/IE is an R-module given by $r\overline{e} = \overline{re}$, for any $r \in R$, $\overline{e} \in E/IE$. So E/IE is a finitely generated R-module with generators $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$. Let A be a $p \times q$ matrix of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$ and $(a_{i1}, a_{i2}, \ldots, a_{iq})$ a row of A. Since $\sum_{j=1}^q a_{ij}\overline{e_j} = 0$, we have $\sum_{j=1}^q a_{ij}e_j \in IE$. There are suitable $d_{i1}, d_{i2}, \ldots, d_{iq} \in I$ such that

$$\sum_{j=1}^q a_{ij}e_j = \sum_{j=1}^q d_{ij}e_j.$$

So we have $\sum_{j=1}^{q}(a_{ij}-d_{ij})e_j = 0$ and hence

$$B = \begin{pmatrix} a_{11}-d_{11} & a_{12}-d_{12} & \cdots & a_{1q}-d_{1q} \\ a_{21}-d_{21} & a_{22}-d_{22} & \cdots & a_{2q}-d_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}-d_{p1} & a_{p2}-d_{p2} & \cdots & a_{pq}-d_{pq} \end{pmatrix}$$

is a $p \times q$ matrix of relations between $e_1, e_2, \ldots, e_q$. Suppose A' is an $r \times r$ submatrix of A for some $r \in \mathbb{N}$, then there are an $r \times r$ submatrix B' of B and an $r \times r$ matrix D' with entries in I such that

$$A' = B' + D'.$$

So we have

$$det(A') = det(B') + d', \text{ for some } d' \in I.$$

Hence for $0 \leq n \leq q$,

$$\mathfrak{F}_n(E/IE) = \sum_{C_1} I_{q-n}(C_1) \subseteq \sum_{C_2} I_{q-n}(C_2) + I = \mathfrak{F}_n(E) + I \subseteq R,$$

where C_1 is a matrix of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$ and C_2 is a matrix of relations between $e_1, e_2, \ldots, e_q$. For $n > q$, we have

$$\mathfrak{F}_n(E/IE) = R = \mathfrak{F}_n(E)$$

and obtain the result.

(2)We have that E/IE is a R/I-module given by $\overline{r}\,\overline{e} = \overline{re}$, for any $\overline{r} \in R/I, \overline{e} \in E/IE$. So E/IE is a finitely generated R/I-module with generators $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$. In this proof, if $M = (m_{ij})$ is a matrix over R, then we define $\overline{M} = (\overline{m_{ij}})$ with $\overline{m_{ij}}$ being the image of m_{ij} in R/I. We first claim that there is a correspondence with the matrix of relations between $e_1, e_2, \ldots, e_q$ and between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$. Let $A = (a_{ij})$ be a $p \times q$ matrix of relations between $e_1, e_2, \ldots, e_q$ and $(a_{i1}, a_{i2}, \ldots, a_{iq})$ a row of A. We have $\sum_{j=1}^{q} a_{ij}e_j = 0$ and hence $\sum_{j=1}^{q} \overline{a_{ij}}\,\overline{e_j} = 0$, where $\overline{a_{ij}}$ is the image of a_{ij} in R/I. So $\overline{A}$ is a matrix of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$.

Conversely, let $\overline{C} = (\overline{c_{ij}})$ be a $p \times q$ matrix of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_q}$ and $(\overline{c_{i1}}, \overline{c_{i2}}, \ldots, \overline{c_{iq}})$ a row of C. We have $\sum_{j=1}^{q} \overline{c_{ij}}\,\overline{e_j} = 0$ and hence $\sum_{j=1}^{q} c_{ij}e_j \in IE$. There are suitable $d_{i1}, d_{i2}, \ldots, d_{iq} \in I$ such that

$$\sum_{j=1}^{q} c_{ij}e_j = \sum_{j=1}^{q} d_{ij}e_j.$$

So we have $\sum_{j=1}^{q}(c_{ij}-d_{ij})e_j=0$ and hence

$$Q=\begin{pmatrix} c_{11}-d_{11} & c_{12}-d_{12} & \cdots & c_{1q}-d_{1q} \\ c_{21}-d_{21} & c_{22}-d_{22} & \cdots & c_{2q}-d_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1}-d_{p1} & c_{p2}-d_{p2} & \cdots & c_{pq}-d_{pq} \end{pmatrix}$$

is a matrix of relations between $e_1, e_2, \ldots, e_q$. Moreover, we have

$$\begin{aligned} \overline{Q} &= \begin{pmatrix} \overline{c_{11}-d_{11}} & \overline{c_{12}-d_{12}} & \cdots & \overline{c_{1q}-d_{1q}} \\ \overline{c_{21}-d_{21}} & \overline{c_{22}-d_{22}} & \cdots & \overline{c_{2q}-d_{2q}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{c_{p1}-d_{p1}} & \overline{c_{p2}-d_{p2}} & \cdots & \overline{c_{pq}-d_{pq}} \end{pmatrix} \\ &= \begin{pmatrix} \overline{c_{11}} & \overline{c_{12}} & \cdots & \overline{c_{1q}} \\ \overline{c_{21}} & \overline{c_{22}} & \cdots & \overline{c_{2q}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{c_{p1}} & \overline{c_{p2}} & \cdots & \overline{c_{pq}} \end{pmatrix} = \overline{C}, \end{aligned}$$

Now, for $0 \le n \le q$,

$$\begin{aligned} \mathfrak{F}_n^{(R/I)}(E/IE) &= \sum_B I_{q-n}\left(\overline{B}\right) \\ &= \text{the image of } \sum_B I_{q-n}(B) \text{ in } R/I \\ &= \text{the image of } \mathfrak{F}_n(E) \text{ in } R/I, \end{aligned}$$

where B is a matrix of relations between $e_1, e_2, \ldots, e_q$. □

By 2 of Corollary 2.12, if E is finitely presented, or equivalently E has a finite presentation, then there is a matrix M such that $\mathfrak{F}_n(E)$ is the ideal generated by all $(q-n)\times(q-n)$ subdeterminants of M, where q is the number of generators of E and $n<q$. But how to find the matrix M? The proof of Theorem 1.2 gives us some information. We write this to be a lemma.

Lemma 2.20. *Let E be an R-module with a finite presentation*

$$G \xrightarrow{h} F \xrightarrow{\phi} E \longrightarrow 0,$$

where G and F are finite free R-modules with free bases β and γ, respectively. Let $M = \left([h]_{\beta}^{\gamma}\right)^t$ be the transport of the matrix representation of h in ordered bases β and γ. If $|\gamma| = q$, then, for $0 \leq n < q$, $\mathfrak{F}_n(E)$ is the ideal of R generated by all $(q-n) \times (q-n)$ subdeterminants of M.

Proof. Suppose $\beta = \{g_1, g_2, \ldots, g_p\}$, $\gamma = \{f_1, f_2, \ldots, f_q\}$, and $M = \left([h]_{\beta}^{\gamma}\right)^t = (m_{ij})$. Since ϕ is surjective, $\phi(f_1), \phi(f_2), \ldots, \phi(f_q)$ generate E. For any relation $(a_1, a_2, \ldots, a_q)$ between $\phi(f_1), \phi(f_2), \ldots, \phi(f_q)$,

$$\phi\left(\sum_{i=1}^{q} a_i f_i\right) = \sum_{i=1}^{q} a_i \phi(f_i) = 0,$$

so $\sum_{i=1}^{q} a_i f_i \in Ker\phi = Imh$, which is generated by $h(g_1), h(g_2), \ldots, h(g_p)$. Since $h(g_i) = \sum_{j=1}^{q} m_{ij} f_j$ and $\{f_1, f_2, \ldots, f_q\}$ is a basis for F, $(a_1, a_2, \ldots, a_q)$ is an R-linear combination of rows of M. Conversely, let $(m_{i1}, m_{i2}, \ldots, m_{iq})$ be a row of M, then

$$\sum_{j=1}^{q} m_{ij} \phi(f_j) = \phi\left(\sum_{j=1}^{q} m_{ij} f_j\right) = \phi\left(h(g_i)\right) = 0.$$

Hence the module of relations between $\phi(f_1), \phi(f_2), \ldots, \phi(f_q)$ can be generated by rows of M. By 2 of Corollary 2.12, for $0 \leq n < q$, we have

$$\mathfrak{F}_n(E) = I_{q-n}(M),$$

which is the ideal of R generated by all $(q-n) \times (q-n)$ subdeterminants of M. □

In fact, any nth Fitting ideal of E is not depend on the choices of the finite presentation and the free bases.

Now we can use Lemma 2.20 to prove Lemma 2.19 in another way.

Lemma 2.21. *Let E be an R-module with a finite presentation and I an ideal of R.*

1. *$\mathfrak{F}_n^{(R/I)}(E/IE) =$ the image of $\mathfrak{F}_n(E)$ in R/I.*
2. *If I is a finitely generated ideal of R, then*

$$\mathfrak{F}_n(E/IE) \subseteq \mathfrak{F}_n(E) + I \subseteq R.$$

Proof. In this proof, we use $\overline{r}$ to denote the image of r in R/I, and use $\overline{e}$ to denote the image of e in E/IE. If $A = (a_{ij})$ is a matrix over R, then we denote $\overline{A} = (\overline{a_{ij}})$ with $\overline{a_{ij}}$ is the image of a_{ij} in R/I.

(1) Suppose

$$R^p \xrightarrow{h} R^q \xrightarrow{\phi} E \longrightarrow 0$$

is a finite presentation of E. Then we can construct a finite R/I-presentation of E/IE as follows:

$$(R/I)^p \xrightarrow{\overline{h}} (R/I)^q \xrightarrow{\overline{\phi}} E/IE \longrightarrow 0,$$

where

$$\overline{h}\,(\overline{r_1}, \overline{r_2}, \ldots, \overline{r_p}) = (\overline{s_1}, \overline{s_2}, \ldots, \overline{s_q}) \text{ if } h(r_1, r_2, \ldots, r_p) = (s_1, s_2, \ldots, s_q),$$
$$\overline{\phi}\,(\overline{s_1}, \overline{s_2}, \ldots, \overline{s_q}) = \overline{e} \text{ if } \phi(s_1, s_2, \ldots, s_q) = e.$$

Indeed, for $d_1, d_2, \ldots, d_p \in I$, each component of

$$\begin{aligned} & h(d_1, d_2, \ldots, d_p) \\ = \; & d_1 h(1, 0, \ldots, 0) + d_2 h(0, 1, 0, \ldots, 0) + \cdots + d_p h(0, \ldots, 0, 1) \end{aligned}$$

is in I, so $\overline{\phi}$ is well-defined. For $d_1', d_2', \ldots, d_q' \in I$,

$$\begin{aligned} & \phi\left(d_1', d_2', \ldots, d_q'\right) \\ = \; & d_1'\phi(1, 0, \ldots, 0) + d_2'\phi(0, 1, 0, \ldots, 0) + \cdots + d_q'\phi(0, \ldots, 0, 1) \in IE, \end{aligned}$$

so $\overline{\phi}$ is well-defined. It is easy to see that $\overline{\phi}$ is surjective. For any $(\overline{r_1}, \overline{r_2}, \ldots, \overline{r_p}) \in (R/I)^p$,

$$\overline{\phi}\,\overline{h}\,(\overline{r_1}, \overline{r_2}, \ldots, \overline{r_p}) = \overline{\phi h\,(r_1, r_2, \ldots, r_p)} = 0$$

in E/IE, so $Im\overline{h} \subseteq Ker\overline{\phi}$. Conversely, for any $\left(\overline{t_1}, \overline{t_2}, \ldots, \overline{t_q}\right) \in Ker\overline{\phi}$, since

$$\overline{\phi(t_1, t_2, \ldots, t_q)} = \overline{\phi}\left(\overline{t_1}, \overline{t_2}, \ldots, \overline{t_q}\right) = 0$$

in E/IE,

$$\phi(t_1, t_2, \ldots, t_q) = \sum_{i=1}^{k} c_i e_i \quad \text{for some } c_i \in I, e_i \in E.$$

Suppose $e_i = \phi(v_{i1}, v_{i2}, \ldots, v_{iq})$ for some $(v_{i1}, v_{i2}, \ldots, v_{iq}) \in R^q$, then

$$\begin{aligned} & \phi(t_1, t_2, \ldots, t_q) \\ = & \sum_{i=1}^{k} c_i \phi(v_{i1}, v_{i2}, \ldots, v_{iq}) \\ = & \phi\left(\sum_{i=1}^{k} c_i v_{i1}, \sum_{i=1}^{k} c_i v_{i2}, \ldots, \sum_{i=1}^{k} c_i v_{iq}\right). \end{aligned}$$

Hence

$$(t_1, t_2, \ldots, t_q) - \left(\sum_{i=1}^{k} c_i v_{i1}, \sum_{i=1}^{k} c_i v_{i2}, \ldots, \sum_{i=1}^{k} c_i v_{iq}\right) \in Ker\phi = Imh.$$

Therefore,

$$\begin{aligned} & (t_1, t_2, \ldots, t_q) - \left(\sum_{i=1}^{k} c_i v_{i1}, \sum_{i=1}^{k} c_i v_{i2}, \ldots, \sum_{i=1}^{k} c_i v_{iq}\right) \\ = & \; h(u_1, u_2, \ldots, u_p) \end{aligned}$$

for some $(u_1, u_2, \ldots, u_p) \in R^p$. Since, for $j = 1, 2, \ldots, q$, $\sum_{i=1}^{k} c_i v_{ij} \in I$,we have

$$\begin{aligned} \overline{h}\left(\overline{u_1}, \overline{u_2}, \ldots, \overline{u_p}\right) &= \left(\overline{t_1 - \sum_{i=1}^{k} c_i v_{i1}}, \overline{t_2 - \sum_{i=1}^{k} c_i v_{i2}}, \ldots, \overline{t_q - \sum_{i=1}^{k} c_i v_{iq}}\right) \\ &= \left(\overline{t_1}, \overline{t_2}, \ldots, \overline{t_q}\right). \end{aligned}$$

Thus $Ker\overline{\phi} \subseteq Im\overline{h}$ and we really have a finite R/I-representation of E/IE.

Let $\beta = \{\beta_1, \beta_2, \ldots, \beta_p\}$ and $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_q\}$ be standard ordered bases for R^p and R^q, respectively. Let $\overline{\beta} = \left\{\overline{\beta_1}, \overline{\beta_2}, \ldots, \overline{\beta_p}\right\}$ and $\overline{\gamma} = \{\overline{\gamma_1}, \overline{\gamma_2}, \ldots, \overline{\gamma_p}\}$ be standard ordered bases for $(R/I)^p$ and $(R/I)^q$, respectively. Then we have

$$\left[\overline{h}\right]_{\overline{\beta}}^{\overline{\gamma}} = \overline{[h]_{\beta}^{\gamma}}.$$

By Lemma 2.20, we know that

$$\mathfrak{F}_n^{(R/I)}\left(E/IE\right) = \text{the image of } \mathfrak{F}_n\left(E\right) \text{ in } R/I.$$

(2) Suppose

$$R^p \xrightarrow{h} R^q \xrightarrow{\phi} E \longrightarrow 0$$

is a finite presentation of E, and $I = c_1R + c_2R + \cdots + c_rR \subseteq R$. Then we can construct a finite R-presentation of E/IE as follows:

$$R^p \oplus \underbrace{R^q \oplus R^q \oplus \cdots \oplus R^q}_{r} \xrightarrow{h^*} R^q \xrightarrow{\overline{\phi}} E/IE \longrightarrow 0,$$

where

$$\begin{array}{l} \overline{\phi}(s_1, s_2, \ldots, s_q) = \overline{\phi(s_1, s_2, \ldots, s_q)}, \\ h^* = h \oplus h_1 \oplus h_2 \oplus \cdots \oplus h_r \text{ with} \\ h_i : R^q \to R^q \text{ given by } (r_1, r_2, \ldots, r_q) \mapsto (c_ir_1, c_ir_2, \ldots, c_ir_q), \end{array}$$

Indeed, since

$$\begin{array}{rl} & \phi h^* \left(R^p \oplus R^q \oplus \cdots \oplus R^q\right) \\ = & \phi \left(h(R^p) + c_1R^q + c_2R^q + \cdots + c_rR^q\right) \\ = & \phi h(R^p) + c_1\phi(R^q) + c_2\phi(R^q) + \cdots + c_r\phi(R^q) \\ \subseteq & IE. \end{array}$$

we have $Imh^* \subseteq Ker\overline{\phi}$.

Conversely, for any $y \in Ker\overline{\phi}$, $\overline{\phi}(y) = 0$ implies $\phi(y) \in IE$. So we have

$$\phi(y) = \sum_{i=1}^{k} c_ie_i \quad \text{for some } c_i \in I, e_i \in E.$$

Suppose $e_i = \phi(s_{i1}, s_{i2}, \ldots, s_{iq})$ for some $(s_{i1}, s_{i2}, \ldots, s_{iq}) \in R^q$, then

$$\phi(y) = \sum_{i=1}^{k} c_i\phi(s_{i1}, s_{i2}, \ldots, s_{iq}) = \phi\left(\sum_{i=1}^{k} c_is_{i1}, \sum_{i=1}^{k} c_is_{i2}, \ldots, \sum_{i=1}^{k} c_is_{iq}\right).$$

Hence

$$y - \left(\sum_{i=1}^{k} c_is_{i1}, \sum_{i=1}^{k} c_is_{i2}, \ldots, \sum_{i=1}^{k} c_is_{iq}\right) \in Ker\phi = Imh.$$

Therefore

$$y - \left(\sum_{i=1}^{k} c_i s_{i1}, \sum_{i=1}^{k} c_i s_{i2}, \ldots, \sum_{i=1}^{k} c_i s_{iq} \right) = h(x) \text{ for some } x \in R^p.$$

Since

$$\begin{aligned} y &= h(x) + \left(\sum_{i=1}^{k} c_i s_{i1}, \sum_{i=1}^{k} c_i s_{i2}, \ldots, \sum_{i=1}^{k} c_i s_{iq} \right) \\ &= h(x) + \sum_{i=1}^{q} c_i (s_{i1}, s_{i2}, \ldots, s_{iq}), \end{aligned}$$

we have $y \in Imh^*$. Thus $Ker\overline{\phi} \subseteq Imh^*$.

Suppose β, γ, δ are the standard ordered bases for $R^p, R^q, (R^p \oplus R^q \oplus \cdots \oplus R^q)$, respectively. Then

$$([h^*]_\delta^\gamma)^t = \begin{pmatrix} & \left([h]_\beta^\gamma\right)^t & \\ c_1 & & \\ & \ddots & \\ & & c_1 \\ & \vdots & \\ c_r & & \\ & \ddots & \\ & & c_r \end{pmatrix}$$

Thus any $(q-n)\times(q-n)$ subdeterminants of $([h^*]_\delta^\gamma)^t$ is either a $(q-n)\times(q-n)$ subdeterminants of $\left([h]_\beta^\gamma\right)^t$, or a multiple of one of the c_i's. □

3 Some Examples

Example 3.1. Let G be a finitely generated abelian group. We know that either G is free abelian or there is a unique list of positive integers $m_1, m_2, \ldots, m_t$ such that $m_1 > 1, m_1|m_2|\ldots|m_t$ and

$$G \cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t} \oplus F$$

with F free abelian.

If G is finite, then we can find generators $g = (g_1, g_2, \ldots, g_t)$ such that rows of the matrix

$$M = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_t \end{pmatrix}$$

generate the module of relations between $g_1, g_2, \ldots, g_t$. Hence, for $0 \le n < t$,

$$\mathfrak{F}_n(E) = I_{t-n}(M) = \prod_{i=1}^{t-n} m_i \mathbb{Z}.$$

Especially,

$$\mathfrak{F}(G) = |G|\mathbb{Z},$$

which also can obtained from Corollary 2.18.

If G is not finite, $\mathfrak{F}(G) = 0$.

Example 3.2. Let A be an $n \times n$ matrix with entries in a field F, and E the $F[x]$-module obtained from F^n with x acting as A. By this we mean that the module structure of E is given by

$$f(x)\,e = f(A)\,e \quad \text{for any } f(x) \in F[x], e \in E.$$

Let $\beta = \{e_1, e_2, \ldots, e_n\}$ be the standard ordered $F[x]$-basis of $(F[x])^n$, and $\gamma = (\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n})$ the standard ordered F-basis of E. The action of x on γ is given by the products $A\overline{e_1}, A\overline{e_2}, \ldots, A\overline{e_n}$, which on expansion read

$$\begin{gathered} x\overline{e_1} = A\overline{e_1} = a_{11}\overline{e_1} + a_{21}\overline{e_2} + \cdots a_{n1}\overline{e_n}, \\ x\overline{e_2} = A\overline{e_2} = a_{12}\overline{e_1} + a_{22}\overline{e_2} + \cdots a_{n2}\overline{e_n}, \\ \vdots \\ x\overline{e_n} = A\overline{e_n} = a_{1n}\overline{e_1} + a_{2n}\overline{e_2} + \cdots a_{nn}\overline{e_n}, \end{gathered}$$

from which we see that the module of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$ can be generated by

$$\begin{aligned} \rho_1 &= (x - a_{11})e_1 - \cdots - a_{j1}e_j - \cdots - a_{n1}e_n, \\ &\vdots \\ \rho_j &= -a_{1j}e_1 - \cdots + (x - a_{jj})e_j - \cdots - a_{nj}e_n, \\ &\vdots \\ \rho_n &= -a_{1n}e_1 - \cdots - a_{jn}e_j - \cdots + (x - a_{nn})e_n. \end{aligned}$$

Indeed, it is clear that, for each i, ρ_i is a relation between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$. And for any relation $\rho = (f_1, f_2, \ldots, f_n)$ between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$, we will claim that ρ is an $F[x]$-linear combination of $\rho_1, \rho_2, \ldots, \rho_n$.

Suppose $deg(f_m) \geq deg(f_i)$ for all i. If $f_m = (x - a_{mm})q_{m1} + r_{m1}$ for some $r_{m1} \in F, q_{m1} \in F[x]$ with $deg(q_{m1}) = deg(f_m) - 1$ or $q_{m1} = 0$. We define

$$f_{i1} = \begin{cases} f_m - q_{m1}(x - a_{mm}) = r_{m1} & \text{if } i = m; \\ f_i - q_{m1}a_{im} & \text{if } i \neq m. \end{cases}$$

So we have

$$(f_1, f_2, \ldots, f_n) - q_{m1}\rho_m = (f_{11}, f_{21}, \ldots, f_{n1}),$$

and

$$deg(f_{i1}) \leq max\,\{deg(f_i), deg(q_{m1})\} \quad \text{for all } i.$$

Continue this process, we can find $q_i \in F[x], r_i \in F$ such that

$$(f_1, f_2, \ldots, f_n) = \sum_{i=1}^{n} q_i\rho_i + (r_1, r_2, \ldots, r_n).$$

Since $(f_1, f_2, \ldots, f_n)$ and $\sum_{i=1}^{n} q_i\rho_i$ are in the module of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$, so is $(r_1, r_2, \ldots, r_n)$. However, $\sum_{i=1}^{n} r_i e_i = \sum_{i=1}^{n} r_i\overline{e_i} = 0$ implies $r_1 = r_2 = \cdots = r_n = 0$. Thus $(f_1, f_2, \ldots, f_n)$ is an $F[x]$-linear combination of $\rho_1, \rho_2, \ldots, \rho_n$, and the module of relations between $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$ can be generated by $\rho_1, \rho_2, \ldots, \rho_n$. So the Fitting ideal of E is the ideal of $F[x]$ generated by the characteristic polynomial of A.

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