國立臺灣師範大學數學研究所博士班博士論文

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The Study of Singularities for Two Parabolic Problems

拋物型問題的奇異點研究

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摘要

在本論文中，我們要討論從二個拋物型方程得到的二種不同類型的奇異點問題。本論文分為二個部份，

在第一部份中，我們考慮具有快速擴散項與強吸收非線性項之方程的殆核問題。首先，我們證明解殆核的速度是非自我相似的。接著，在考慮重新縮放的解與殆核最終在單點發生的狀態下，我們得到一些更精確的估計。

在第二部份中，我們探討一個由複數取值的熱方程得到的柯西問題，而其中的非線性項是倒數型的。首先，我們提供了一些解的全局存在性與消失性的判斷準則。接下來，我們證明當初始值為漸近常數時，解是否會在無窮遠處消失或是在任意的時間內全局存在，均依賴於初始值的漸近極限值。

關鍵字：殆核解，非自我相似，複數值熱方程，消失性。

ABSTRACT

In this thesis, we study two different singularities arising from two parabolic problems.

This thesis is divided into two parts. In the first part, we consider the dead-core problem for the fast diffusion equation with a strong absorption. First, we show that the temporal rate of formation of the dead-core is not self-similar. Then we obtain some precise estimates on rescaled solutions and on the single-point final dead-core profile. In the second part, we study the Cauchy problem for a parabolic system which is derived from a complex-valued heat equation with an inverse nonlinearity. We first provide some criteria for the global existence and quenching of solutions. Then we show that, for the initial data which are asymptotically constants, the solution either quenches at space infinity or exists globally in time depending on the asymptotic limits.

Key words：dead-core, non-self-similar, complex-valued heat equation, quenching.

The study of singularities for two parabolic problems

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Chapter 1

Introduction

In this thesis, we study two different singularities arising from two parabolic problems. This thesis is divided into two parts. In the first part, we consider the dead-core problem for the fast diffusion equation with a strong absorption. In the second part, we study the Cauchy problem for a parabolic system which is derived from a complex-valued heat equation with an inverse nonlinearity.

1.1 Non-self-similar dead-core rate

In the first part, we study the following initial boundary value problem:

$$\begin{cases} u_t = (u^m)_{xx} - u^p, & x \in (-1,1), \quad t > 0, \\ u(\pm 1, t) = k, & t > 0, \\ u(x,0) = u_0(x), & x \in [-1,1], \end{cases} \tag{1.1.1}$$

in the parameter range

$$0 < p < m < 1. \tag{1.1.2}$$

It is assumed that $k > 0$ and that the initial data u_0 satisfies

$$u_0 \in C([-1,1]), \quad u_0 > 0 \text{ in } [-1,1], \quad u_0(\pm 1) = k. \tag{1.1.3}$$

Problem (1.1.1) admits a unique, global classical solution $u \geq 0$. We set

$$\Lambda(t) := \min_{|x| \leq 1} u(x,t)$$

and denote

$$T = T(u_0) := \inf\{t > 0;\ \Lambda(t) = 0\} > 0.$$

For suitable initial data, we show that $T(u_0) < \infty$. In this case, we say that the solution develops a dead-core in finite time, and T is called the dead-core time.

The main goal of this part is to study the asymptotic behavior of the solution as $t \to T^-$ when $T < \infty$. For the asymptotic dead-core behavior, we shall assume further that u_0 satisfies the conditions

$$u_0 \in C^2([-1,1]), \qquad (u_0^m)'' \leq u_0^p \quad in \quad [-1,1], \tag{1.1.4}$$

and

$$u_0 \ \ is\ \ even\ \ and\ \ nondecreasing\ \ in\ \ |x|\ \ and\ \ T(u_0) < \infty. \tag{1.1.5}$$

To study the asymptotic behavior, we use the following self-similar variables

$$y = \frac{x}{(T-t)^{\alpha}}, \quad s = -\ln(T-t), \quad z(y,s) := \left[\frac{u(x,t)}{(T-t)^{\beta}}\right]^m, \tag{1.1.6}$$

where the exponents α, β are given by

$$\alpha = \frac{m-p}{2(1-p)}, \quad \beta = \frac{1}{1-p}.$$

Then z satisfies the equation

$$\gamma z^{\gamma-1} z_s = z_{yy} - \alpha\gamma y z^{\gamma-1} z_y + \beta z^{\gamma} - z^q \quad \text{in} \ \ \Omega, \tag{1.1.7}$$

where $\gamma := 1/m$, $q := p/m$, and

$$\Omega := \{(y,s);\ |y| < e^{\alpha s},\ -\ln(T) =: s_0 < s < \infty\}.$$

Also, the boundary and initial conditions are transformed into

$$z(e^{\alpha s}, s) = k^m e^{\beta m s}, \quad s > s_0, \tag{1.1.8}$$

$$z(y, s_0) = z_0(y) := T^{-\beta m} u_0^m(yT^{\alpha}), \quad y \in [-T^{-\alpha}, T^{-\alpha}]. \tag{1.1.9}$$

By deriving some a priori estimates and constructing a Lyapunov function, we show that the dead-core rate is non-self-similar. Also, we obtain some precise estimates on the single-point final dead-core profile near $x = 0$.

This part is organized as follows. In section 2.2, we give sufficient conditions under which the solution of problem (1.1.1) develops a dead-core in finite time. Some a priori estimates for solutions of (1.1.1) will be derived in section 2.3. In section 2.4, we study the stationary equation associated with (1.1.7), from the point of view of uniqueness of global solutions (with suitable growth at infinity) and of backward continuation of local solutions. This is needed, respectively, in the study of omega limits and in the construction of a Lyapunov function. Finally, the temporal rate of formation of the dead-core is given in section 2.5.

This part is taken from a joint work with J.-S. Guo and Ph. Souplet which is published in *Nonlinearity* [27].

1.2 Dynamics for a complex-valued heat equation

In the second part, we are concerned with the Cauchy problem for a parabolic system which is derived from the following complex-valued heat equation with an inverse nonlinearity

$$z_t = z_{xx} - \frac{1}{z}, \tag{1.2.10}$$

where $z = z(x,t)$ is a complex-valued function of the spatial variable $x \in \mathbb{R}$ and the time variable $t \geq 0$. If we set $z(x,t) = u(x,t) + iv(x,t)$, where $i = \sqrt{-1}$ and $u(x,t)$, $v(x,t) \in \mathbb{R}$, then (1.2.10) can be written as a system of parabolic equations

$$\begin{cases} u_t = u_{xx} - u/(u^2+v^2), \\ v_t = v_{xx} + v/(u^2+v^2). \end{cases} \tag{1.2.11}$$

If $z(x,t)$ is real-valued (i.e., $v \equiv 0$), then the system is reduced to the equation

$$u_t = u_{xx} - \frac{1}{u}.$$

The goal of this part is to study the dynamics of solutions of the system (1.2.11) with $v \not\equiv 0$. More precisely, we consider the initial value problem (P) for (1.2.11) with the initial condition

$$(u(\cdot,0), v(\cdot,0)) = (u_0, v_0), \tag{1.2.12}$$

where it is assumed that

$$u_0 > 0, \quad v_0 \geq 0, \quad u_0, v_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \quad \inf_{\mathbb{R}} u_0 + \inf_{\mathbb{R}} v_0 > 0.$$

Then the problem (P) has a unique solution $(u,v) \in (C([0,T); L^\infty(\mathbb{R})))^2$, where $T = T(u_0,v_0) \in (0,\infty]$ is the maximal existence time of the solution. Furthermore, we have either $T = \infty$, or

$$T < \infty \quad \text{and} \quad \liminf_{t \to T} \{\inf_{x \in \mathbb{R}} u(x,t) + \inf_{x \in \mathbb{R}} v(x,t)\} = 0.$$

In the first case, we have the *global* existence. For the second case, we say that the solution of (P) *quenches* in a finite time T in which T is called the *quenching time*. From (1.2.11) it is easy to see that both u and v quench simultaneously whenever quenching occurs. Moreover, we say that $x_Q \in \mathbb{R}$ is a (finite) *quenching point* for (u, v) if there exists a sequence $\{(x_j, t_j)\}$ such that $x_j \to x_Q$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$. We shall investigate the global and non-global existence of solutions of (P).

First, by using an invariant set argument, we prove the global existence and (time) asymptotic behavior of solution of the problem (P) for certain initial data. Next, to find solutions quenching in finite time, we consider the case when the initial data are *asymptotically constants*. Namely, we impose the following conditions on initial data:

$$u_0, v_0 \in C^1(\mathbb{R}), \quad u_0 \geq M, \quad u_0 \not\equiv M, \quad v_0 \geq 0, \quad v_0 \not\equiv 0, \tag{1.2.13}$$

$$\lim_{|x|\to\infty} u_0(x) = M, \quad \lim_{|x|\to\infty} v_0(x) = N \tag{1.2.14}$$

for some constants $M > 0$ and $N \geq 0$.

We show that the solution of (1.2.11) with initial data satisfying (1.2.13) and (1.2.14) with $N > 0$ exists globally and behaves like the solution the ODE system

$$\begin{cases} U_t = -U/(U^2 + V^2), \\ V_t = V/(U^2 + V^2). \end{cases}$$

with $(U(0), V(0)) = (M, N)$. On the other hand, if the initial data of the solution of (1.2.11) satisfy (1.2.13) and (1.2.14) with $N = 0$, then the solution quenches only at space infinity. Namely, there are no (finite) quenching points, while there exists a sequence $\{(x_j, t_j)\}$ such that $|x_j| \to \infty$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$.

This part is organized as follows. In section 3.2, we provide a sufficient condition for the existence of global solutions and study the asymptotic behavior of solutions as $t \to \infty$. In section 3.3, we study the solution of (1.2.11) with asymptotically constant initial data.

Chapter 2

Non-self-similar dead-core rate

2.1 Introduction

We study the following initial boundary value problem:

$$\begin{cases} u_t = (u^m)_{xx} - u^p, & x \in (-1,1), \quad t > 0, \\ u(\pm 1, t) = k, & t > 0, \\ u(x,0) = u_0(x), & x \in [-1,1], \end{cases} \tag{2.1.1}$$

in the parameter range

$$0 < p < m < 1. \tag{2.1.2}$$

It is assumed that $k > 0$ and that the initial data u_0 satisfies

$$u_0 \in C([-1,1]), \quad u_0 > 0 \ \text{ in } [-1,1], \quad u_0(\pm 1) = k. \tag{2.1.3}$$

Also, throughout this chapter, we denote

$$\beta = \frac{1}{1-p}.$$

Problem (2.1.1) admits a unique, global classical solution $u \geq 0$. We set

$$\Lambda(t) := \min_{|x| \leq 1} u(x,t)$$

and denote

$$T = T(u_0) := \inf \{t > 0;\ \Lambda(t) = 0\} > 0.$$

For suitable initial data, we shall show that $T(u_0) < \infty$ (see Theorem 2.1.1 below). We say that the solution develops a dead-core in finite time, and T is called the dead-core time. The main goal of this chapter is to study the asymptotic behavior of the solution as $t \to T^-$ when $T < \infty$.

In the semilinear case $0 < p < m = 1$, the question of temporal dead-core rates has been studied in [29, 30, 50]. We refer to [3, 53, 4] for earlier work on the semilinear dead-core problem and to [58, 6, 15, 54, 7] for studies of the regularity and behavior of interfaces for this problem. It was shown in [29] that the rate is *not* self-similar, i.e. its order is not the same as for the corresponding ODE $y' = -y^p$. Namely, it was found that

$$\lim_{t \to T} (T-t)^{-\beta} \Lambda(t) = 0. \tag{2.1.4}$$

It is clear that such phenomenon is due to a tight interaction between absorption and reaction near the level $u = 0$, since diffusion dominance would prevent the appearance of a dead-core and absorption dominance would lead to an ODE rate. The corresponding Cauchy problem was further investigated in [30], in which they constructed some special solutions with different dead-core rates following the idea of Herrero and Velázquez [34, 35]. Recently, this construction for general higher spatial dimension was carried out by Seki [50].

It was also observed in [29] that the non-self-similar behavior (2.1.4) strongly departs from the related extinction problem for the same equation on the whole real line: starting from compactly supported initial data, the solutions becomes identically zero after a finite time and the rate of extinction is self-similar (see [12, 46, 32, 33]). Other singularity formation mechanisms in related reaction-diffusion equations, such as blow-up and quenching, also exhibit self-similar behaviors, at least in one space dimension (see [55, 13, 18, 19, 42] for blow-up and [22, 11, 23, 39] for quenching). However, another exception is the phenomenon of boundary gradient blow-up for the equation $u_t - u_{xx} = |u_x|^p$ $(p > 2)$ under Dirichlet boundary conditions, for which the rate was recently found to be non-self-similar (see [26] and cf. also [36, 40] for recent related results).

The goal of this chapter is to study the dead-core rate in the presence of fast diffusion $(m < 1)$. In view of the above observation concerning the interaction of diffusion and absorption, this question is of interest since the effect of fast diffusion, as compared with linear diffusion, is much stronger near the level $u = 0$. It will turn out that again the rate is non-self-similar. Although our strategy of proof is close to that in [29], the proof is technically much more difficult due to the presence of a nonlinear diffusion operator.

The fast diffusion equation with strong absorption in (2.1.1) was studied before from the point of view of decay and extinction [47, 10, 8, 48, 57], and of regularity and behavior of interfaces [15, 14, 1]. On the other hand, the porous medium or slow diffusion case $m > 1 > p$ was also studied from the latter point of view [6, 15, 16, 17, 14] and that the existence of finite-time dead-core was proved in [2]. The dead-core rate was studied by Chen-Guo-Hu [5]. In [5], they obtained the self-similar singularity of dead-core rate for certain class of initial data in the slow diffusion case.

Our first result gives sufficient conditions under which the solution of problem (2.1.1) develops a dead-core in finite time. To formulate this, let us first recall some well-known facts: (2.1.1) admits a unique steady state $U_k \in C^2([-1,1])$ for each given $k > 0$. Moreover, U_k is an even and nondecreasing function of $|x|$ and it is a nondecreasing function of k. Furthermore, there exists $k_0 = k_0(m,p) > 0$ such that: if $k \in (0, k_0)$ then U_k vanishes on an interval of positive length, if $k = k_0$ then U_k vanishes only at $x = 0$, and if $k > k_0$ then U_k is positive.

Theorem 2.1.1 *Assume* (2.1.2) *and* (2.1.3)*.*

(i) Let $0 < k < k_0$. Then $T(u_0) < \infty$ for any u_0.

(ii) Let $k \geq k_0$. For any $\eta, M > 0$ there exists $\delta = \delta(\eta, M) > 0$ such that $T(u_0) < \infty$ whenever $\|u_0\|_\infty \leq M$ and $u_0 \leq \delta$ on a subinterval of $[-1,1]$ of length η.

Remark 2.1.1 We note that the assumption $k < k_0$ in Theorem 2.1.1(i) is essentially optimal due to existence of a positive steady state for $k > k_0$. The assumption (2.1.2) is also necessary throughout this chapter, since the dead-core phenomenon never occurs if $p \geq m$ (indeed $\underline{u}(x) = [\varepsilon(1+x^2)]^{1/m}$ is then a positive subsolution to (2.1.1) for sufficiently small $\varepsilon > 0$).

For our main results on the asymptotic dead-core behavior, we shall assume that u_0 satisfies the conditions

$$(2.1.5) \qquad u_0 \in C^2([-1,1]), \qquad (u_0^m)'' \leq u_0^p \quad in \quad [-1,1],$$

and

$$(2.1.6) \qquad u_0 \;\; is \;\; even \;\; and \;\; nondecreasing \;\; in \;\; |x| \;\; and \;\; T(u_0) < \infty.$$

It then follows from the strong maximum principle that $u_t < 0$ in $Q_T := (-1,1) \times (0,T)$, $u(-x,t) = u(x,t)$ for $(x,t) \in Q_T$ and $u_x > 0$ in $(0,1) \times (0,T)$.

Theorem 2.1.2 *Let $k > 0$ and assume* (2.1.2), (2.1.3), (2.1.5) *and* (2.1.6)*. Then*

$$\lim_{t \to T^-} (T-t)^{-\beta} \Lambda(t) = 0.$$

As a basic ingredient in the proof of Theorem 2.1.2, we use the following self-similar variables

$$(2.1.7) \qquad y = \frac{x}{(T-t)^\alpha}, \quad s = -\ln(T-t), \quad z(y,s) := \left[\frac{u(x,t)}{(T-t)^\beta}\right]^m,$$

where the exponent α is given by

$$\alpha = \frac{m-p}{2(1-p)}.$$

Then z satisfies the equation

$$\gamma z^{\gamma-1} z_s = z_{yy} - \alpha\gamma y z^{\gamma-1} z_y + \beta z^\gamma - z^q \quad \text{in } \Omega, \tag{2.1.8}$$

where $\gamma := 1/m$, $q := p/m$, and

$$\Omega := \{(y,s);\ |y| < e^{\alpha s},\ -\ln(T) =: s_0 < s < \infty\}.$$

The boundary and initial conditions are transformed into

$$z(e^{\alpha s}, s) = k^m e^{\beta m s}, \quad s > s_0, \tag{2.1.9}$$

$$z(y, s_0) = z_0(y) := T^{-\beta m} u_0^m(yT^\alpha), \quad y \in [-T^{-\alpha}, T^{-\alpha}]. \tag{2.1.10}$$

Theorem 2.1.2 will actually be a consequence of the following more precise result:

Theorem 2.1.3 *Under the assumptions of Theorem 2.1.2, the corresponding global solution z of* (2.1.8)-(2.1.10) *satisfies*

$$\lim_{s\to\infty} z(y,s) = V_1(y) := k_{p,m}|y|^{\frac{2m}{m-p}},$$

where $k_{p,m} = (\frac{(m-p)^2}{2m(m+p)})^{\frac{m}{m-p}}$, uniformly on $\{|y| < R\}$, for any $R > 0$ fixed.

The key ingredients to prove Theorem 2.1.3 are the following:

1. derive some a priori estimates of z from above and below;
2. construct a Lyapunov function by the method of Zelenyak [59] (cf. also [25]). Note that, unlike for $m = 1$, the standard energy argument does not work here;
3. classify the possible steady states for z on the whole real line.

In the course of the proof, we shall obtain the following precise estimates on the single-point final dead-core profile near $x = 0$.

Theorem 2.1.4 *Under the assumptions of Theorem 2.1.2, there exist $c_1, c_2 > 0$ (depending on u) such that*

$$c_1|x|^{2/(m-p)} \le u(x,T) \le c_2|x|^{2/(m-p)}, \qquad |x| \le 1.$$

This chapter is organized as follows. Section 2.2 is devoted to the proof of Theorem 2.1.1. Some a priori estimates will be derived in section 2.3. As a by-product, this also gives a proof of Theorem 2.1.4. In section 2.4, we study the stationary equation associated with (2.1.8), from the point of view of uniqueness of global solutions (with suitable growth at infinity) and of backward continuation of local solutions. This is needed, respectively, in the study of omega limits and in the construction of a Lyapunov function. Finally, the proof of Theorem 2.1.3 is given in section 2.5.

2.2 Proof of Theorem 2.1.1

Step 1. We look for a supersolution $\overline{u}$ of $u_t - (u^m)_{xx} + u^p = 0$ in $Q_T := (-1,1) \times (0,T)$, which develops a dead-core at time T. Namely, motivated by an idea from [52], for any $T \in (0, T_0)$ we shall construct $\overline{u}$ under the following self-similar form:

$$\overline{u}(x,t) = \varepsilon(T-t)^{\beta} V(y), \qquad y = x(T-t)^{-\gamma}, \qquad V(y) = (1+y^2)^{\nu},$$

where

$$0 < \gamma < \frac{\beta(m-p)}{2} = \frac{m-p}{2(1-p)}$$

and $\nu, \varepsilon, T_0 > 0$ will be determined. Note that $\overline{u}(0,T) = 0$. Simple computations yield

$$\begin{aligned} P\overline{u} &:= \overline{u}_t - (\overline{u}^m)_{xx} + \overline{u}^p \\ &= \varepsilon(T-t)^{\beta-1}(-\beta V + \gamma y V') - \varepsilon^m (T-t)^{\beta m - 2\gamma}(V^m)'' + \varepsilon^p (T-t)^{\beta p} V^p \\ &= \varepsilon(T-t)^{\beta p}\big\{\varepsilon^{p-1} V^p - \beta V + \gamma y V' - \varepsilon^{m-1}(T-t)^{\theta}(V^m)''\big\} \end{aligned}$$

for $(x,t) \in Q_T$, where $\theta = \beta(m-p) - 2\gamma > 0$. Assuming $T \le T_0(\varepsilon) := \varepsilon^{(1-m)/\theta}$, we see that

$$P\overline{u} \ge \varepsilon(T-t)^{\beta p}\big\{\varepsilon^{p-1} - h(y)\big\}, \qquad \text{where } h(y) = \beta V - \gamma y V' + |(V^m)''|.$$

Next taking $\nu > \beta/(2\gamma)$ and using $|(V^m)''| \sim C|y|^{2m\nu - 2}$ as $|y| \to \infty$, we observe that

$$h(y) \sim (\beta - 2\gamma\nu)|y|^{2\nu} \to -\infty, \qquad \text{as } |y| \to \infty.$$

It follows that $\sup_{y\in\mathbb{R}} h(y) < \infty$ and choosing $\varepsilon = \varepsilon(m,p,\gamma,\nu) > 0$ sufficiently small, we conclude that $P\overline{u} \ge 0$ in Q_T. For further reference we also note that

$$\overline{u}(x,t) \ge \varepsilon|x|^{2\nu} T^{-\mu} \ \text{ in } Q_T, \qquad \text{where } \mu = 2\gamma\nu - \beta > 0. \tag{2.2.1}$$

Step 2. We prove assertion (ii). Fix $\eta, M > 0$ and $x_0 \in [-1+\eta/2, 1-\eta/2]$. Let $\overline{u}$, T_0 be as in Step 1 and set $\overline{v}(x,t) = \overline{u}(x - x_0, t)$. Taking $T \le \min(T_0, T_1)$, where $T_1 = T_1(\eta, M) > 0$

is sufficiently small, and using (2.2.1), we see that $\overline{v}(x,t) \geq M$ for $|x - x_0| \geq \eta/2$ and $t \in [0,T)$, hence in particular $\overline{v}(\pm 1, t) \geq k$. Next put $\delta := \min_{|x-x_0|\leq \eta/2} \overline{v}(x,0)$. Then assuming $\|u_0\|_\infty \leq M$ and $u_0 \leq \delta$ for $|x - x_0| < \eta/2$, we get $u_0 \leq \overline{v}(\cdot, 0)$ and it follows from the comparison principle that $u \leq \overline{v}$ in Q_T, hence $T(u_0) \leq T < \infty$. This proves assertion (ii).

Step 3. We prove assertion (i). First observe that assertion (ii) is actually true for any $k > 0$ in view of Step 2. On the other hand, by standard energy arguments, one can show that $u(\cdot, t)$ converges to U_k in $L^\infty(-1,1)$ as $t \to \infty$. Since $U_k = 0$ on $[-\eta/2, \eta/2]$ for some $\eta > 0$, it follows that for t_0 large, the new initial data $\tilde{u}_0 := u(\cdot, t_0)$ satisfies the assumptions of part (ii) with $M = k + 1$. The conclusion follows. $\square$

2.3 Some a priori estimates

In this section, we shall derive some a priori estimates for solutions of (2.1.1). Let u be a solution of (2.1.1). Since $u_t < 0$ in Q_T and $u_x > 0$ in $(0,1) \times (0,T)$, we have $(u^m)_{xx} < u^p$. Let $v = u^m$. Then from $v_x = mu^{m-1}u_x$, $0 < u \leq k$ in Q_T and

$$v_{xx}v_x < v^q v_x \quad \text{in } (0,1) \times (0,T),$$

it follows that v_x (and so u_x) is bounded in Q_T. We have the following two lemmas (which in particular imply Theorem 2.1.4).

Lemma 2.3.1 *Let the assumptions of Theorem 2.1.2 be in force and fix $t_0 \in (0,T)$. Then there exists $\varepsilon > 0$ such that the auxiliary function*

$$J := (u^m)_x - \varepsilon x u^p$$

satisfies $J \geq 0$ in $[0,1] \times (t_0, T)$. In particular, there exists $c > 0$ such that

$$u(x,t) \geq c|x|^{2/(m-p)}, \quad x \in (-1,1),\ t_0 < t < T. \tag{2.3.1}$$

Proof. Notice that the differential equation in (2.1.1) can be written under the form

$$u_t - au_{xx} = m(m-1)u^{m-2}(u_x)^2 - u^p,$$

with

$$a = mu^{m-1}.$$

For $(x,t)\in(0,1)\times(0,T)$, we compute

$$\begin{aligned}(xu^p)_t &= pxu^{p-1}u_t,\\ (xu^p)_x &= u^p + pxu^{p-1}u_x,\\ (xu^p)_{xx} &= 2pu^{p-1}u_x + p(p-1)xu^{p-2}(u_x)^2 + pxu^{p-1}u_{xx}.\end{aligned}$$

Therefore

$$\begin{aligned}(xu^p)_t - a(xu^p)_{xx} &= pxu^{p-1}\big(u_t - au_{xx}\big) - a\big(2pu^{p-1}u_x + p(p-1)xu^{p-2}(u_x)^2\big)\\ &= -pxu^{2p-1} - 2mpu^{p+m-2}u_x - mp(p-m)xu^{p+m-3}(u_x)^2\\ &= -pxu^{2p-1} - 2pu^{p-1}(u^m)_x - m^{-1}p(p-m)xu^{p-m-1}((u^m)_x)^2.\end{aligned}$$

Using

$$(u^m)_x = J + \varepsilon xu^p,$$

we deduce that

$$(xu^p)_t - a(xu^p)_{xx} = b_1J - pxu^{2p-1}\big\{1 + 2\varepsilon + m^{-1}(p-m)\varepsilon^2x^2u^{p-m}\big\}. \tag{2.3.2}$$

Here and below, $b_1, b_2, \ldots$ denote functions which are smooth in $[-1,1]\times(0,T)$. On the other hand, we have

$$\begin{aligned}(u^m)_{xt} - a(u^m)_{xxx} &= (au_x)_t - a(u^m)_{xxx} = a(u_t - (u^m)_{xx})_x + a_tu_x\\ &= -pau^{p-1}u_x + m(m-1)u^{m-2}u_tu_x\\ &= [-p + (m-1)u^{-p}u_t]u^{p-1}(u^m)_x\\ &= b_2J + \varepsilon xu^{2p-1}[-p + (m-1)u^{-p}u_t]\end{aligned}$$

hence

$$(u^m)_{xt} - a(u^m)_{xxx} = b_2J + \varepsilon xu^{2p-1}[1 - m - p + (m-1)u^{-p}(u^m)_{xx}].$$

Since

$$\begin{aligned}(u^m)_{xx} &= (J + \varepsilon xu^p)_x = J_x + \varepsilon(u^p + pxu^{p-1}u_x)\\ &= J_x + \varepsilon u^p[1 + pm^{-1}xu^{-m}(u^m)_x]\\ &= J_x + b_3J + \varepsilon u^p[1 + pm^{-1}\varepsilon x^2u^{p-m}],\end{aligned}$$

it follows that

$$(u^m)_{xt} - a(u^m)_{xxx} = b_4J + b_5J_x + \varepsilon xu^{2p-1}\big\{1 - m - p + (m-1)\varepsilon[1 + pm^{-1}\varepsilon x^2u^{p-m}]\big\}. \tag{2.3.3}$$

Combining (2.3.2) and (2.3.3), we obtain

$$\begin{aligned}J_t - aJ_{xx} - b_6J - b_7J_x &= \varepsilon xu^{2p-1}\big\{1 - m - p + (m-1)\varepsilon[1 + pm^{-1}\varepsilon x^2u^{p-m}]\big\}\\ &\qquad + p\varepsilon xu^{2p-1}\big\{1 + 2\varepsilon + m^{-1}(p-m)\varepsilon^2x^2u^{p-m}\big\}\\ &= \varepsilon xu^{2p-1}\big\{1 - m + (m-1)\varepsilon[1 + pm^{-1}\varepsilon x^2u^{p-m}]\\ &\qquad + p\varepsilon[2 + m^{-1}(p-m)\varepsilon x^2u^{p-m}]\big\}.\end{aligned}$$

Since $m < 1$, by choosing $0 < \varepsilon \le \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(m,p) > 0$ small enough, it follows that

$$\begin{aligned} J_t - aJ_{xx} - b_6 J - b_7 J_x &\ge \varepsilon x u^{2p-1}\Big\{\frac{1-m}{2} - \frac{p(1-p)}{m}\varepsilon^2 x^2 u^{p-m}\Big\} \\ &= \frac{1-m}{2}\varepsilon x u^{3p-m-1}\big\{u^{m-p} - \lambda\varepsilon^2 x^2\big\}, \end{aligned}$$

where $\lambda := 2p(1-p)/[m(1-m)] > 0$. Now observe that

$$\begin{aligned} \Big[u^{m-p} - \frac{m-p}{2m}\varepsilon x^2\Big]_x &= (m-p)[u^{m-p-1}u_x - m^{-1}\varepsilon x] \\ &= \frac{m-p}{m}u^{-p}\big[(u^m)_x - \varepsilon x u^p\big] = \frac{m-p}{m}u^{-p}J, \end{aligned}$$

hence

$$u^{m-p} - \frac{m-p}{2m}\varepsilon x^2 \ge \frac{m-p}{m}\int_0^x u^{-p}J(y,t)\,dy.$$

Thus, taking ε_0 possibly smaller, we get

$$J_t - aJ_{xx} - b_6 J - b_7 J_x \ge \frac{1-m}{2}\varepsilon x u^{3p-m-1}\big\{u^{m-p} - \frac{m-p}{2m}\varepsilon x^2\big\}$$

hence

$$J_t - aJ_{xx} - b_6 J - b_7 J_x \ge C\varepsilon x u^{3p-m-1}\int_0^x u^{-p}J(y,t)\,dy,$$

with $C = (1-m)(m-p)/2m > 0$. Now, for any $0 < t_0 < t_1 < T$, it follows from the maximum principle that J attains its minimum in $Q = [0,1] \times [t_0,t_1]$ on the parabolic boundary of Q (see p.659 of [29] for details in a similar situation).

It is thus sufficient to check that $J \ge 0$ on the parabolic boundary of Q for ε small. Clearly $J = 0$ for $x = 0$. Since u_x is bounded on Q_T, $u(x,t) \ge \eta > 0$ in $[1-\delta,1]\times(t_0,T)$ for some small constant $\delta > 0$. Therefore u extends to a classical solution on $[1-\delta,1]\times(t_0,T]$ and Hopf's Lemma implies that $u_x(1,t) \ge \tilde{\delta} > 0$ for $t_0 < t < T$, hence $J(1,t) \ge 0$ for $t_0 < t < T$ if ε is chosen small enough. Moreover, also as a consequence of Hopf's Lemma, we have $u_x(x,t_0) \ge cx$ in $[0,1]$ for some $c > 0$. Again decreasing ε if necessary, we deduce that $J(x,t_0) \ge 0$ in $[0,1]$. The lemma follows. $\square$

From (2.3.1) and using similarity variables, we have for any fixed $t_0 \in (0,T)$,

(2.3.4) $$z(y,s) \ge D^*|y|^\delta \quad \text{in } \ \Omega_0 := \{(y,s);\ |y| < e^{\alpha s},\ -\ln(T-t_0) < s < \infty\},$$

where $\delta := \frac{2m}{m-p} = \frac{2}{1-q}$ and $D^* > 0$. Next we shall prove a local uniform bound for z.

Lemma 2.3.2 *Under the assumptions of Theorem 2.1.2, the corresponding global solution z of* (2.1.8) *satisfies*

(2.3.5) $$z(y,s) \le c(1+|y|)^{\frac{2}{1-q}},$$

(2.3.6) $$|z_y(y,s)| \le c|y|^{\frac{1+q}{1-q}}, \quad \textit{if } \ |y| \ge 1,$$

(2.3.7) $$|z_y(y,s)| \le c|y|, \quad \textit{if } \ |y| \le 1,$$

for some constant $c > 0$ in $\{(y,s); |y| < e^{\alpha s},\ -\ln(T-t_0) < s < \infty\}$.

Proof. Without loss of generality, we may consider the case $y > 0$. Since $x = 0$ is the unique minimum point for each t, we have $u_t(0,t) \geq -u^p(0,t)$. By an integration we obtain

$$z(0,s) \leq \kappa := (1-p)^{\frac{m}{1-p}} \quad \text{for any } s \geq s_0.$$

Since $u_t < 0$ in Q_T and (2.1.7), we obtain $\gamma z^{\gamma-1} z_s + \alpha\gamma y z^{\gamma-1} z_y - \beta z^\gamma < 0$ in Ω and so

$$z_{yy} - z^q < 0 \quad \text{in } \Omega. \tag{2.3.8}$$

Multiplying (2.3.8) by z_y and integrating it over $[0,y]$ for each s, we obtain

$$\frac{1}{2} z_y^2(y,s) \leq \frac{1}{q+1}(z^{q+1}(y,s) - z^{q+1}(0,s)) < \frac{1}{q+1} z^{q+1}(y,s). \tag{2.3.9}$$

Then (2.3.5) follows by an integration of (2.3.9) from 0 to y. Clearly, (2.3.6) follows from (2.3.5) and (2.3.9), whereas (2.3.7) follows from (2.3.5), (2.3.8) and $z_y(0,s) = 0$. □

2.4 The associated ordinary differential equation

This section is devoted to the study of the ordinary differential equation associated with (2.1.8). We first consider the equation on the whole real line $\mathbb{R}$.

$$V'' - \alpha\gamma y V^{\gamma-1} V' + \beta V^\gamma - V^q = 0, \quad y \in \mathbb{R}. \tag{2.4.1}$$

Recall that

$$\alpha = \frac{m-p}{2(1-p)}, \quad \beta = \frac{1}{1-p}, \quad \gamma = \frac{1}{m}, \quad q = \frac{p}{m}, \quad 0 < p < m < 1.$$

Proposition 2.4.1 *Let $V \in C^2(\mathbb{R})$ be a solution of (2.4.1) such that*

$$V = V(|y|), \quad \textit{with} \quad V' \geq 0, \quad V > 0 \quad \textit{for all } y > 0,$$

and such that V is polynomially bounded. Then

$$V = V_1 := k_{p,m}|y|^{\frac{2m}{m-p}} \quad \textit{or} \quad V = V_2 := \kappa$$

where $k_{p,m} = (\frac{(m-p)^2}{2m(m+p)})^{\frac{m}{m-p}}$ and $\kappa = (1-p)^{\frac{m}{1-p}}$.

Proof. Let $W := V^{\frac{m-p}{m}}$ and denote $' = d/dy$. At any point $y \in \mathbb{R}$ such that $W(y) > 0$, the equation for W is:

$$W'' - \frac{m-p}{2m(1-p)} y W^{\frac{1-m}{m-p}} W' + \frac{p}{m-p}\frac{(W')^2}{W} + \frac{m-p}{m(1-p)} W^{\frac{1-p}{m-p}} = \frac{m-p}{m}. \tag{2.4.2}$$

By differentiating (2.4.2), we obtain that

$$W''' - \frac{m-p}{2m(1-p)} yW^{\frac{1-m}{m-p}} W'' + \frac{2-m-p}{2m(1-p)} W^{\frac{1-m}{m-p}} W' - \frac{1-m}{2m(1-p)} yW^{\frac{1-2m+p}{m-p}} (W')^2 = -\frac{p}{m-p} \frac{W'(2W''W - (W')^2)}{W^2}. \tag{2.4.3}$$

Set $H := W - yW'/2$. Then

$$H' = \frac{1}{2}W' - \frac{y}{2}W'', \quad H'' = -\frac{y}{2}W'''.$$

Define

$$D := \{y \in \mathbb{R};\ W(y) > 0 \text{ and } H(y) \neq 0\}.$$

The function $Z := |H|^{\frac{m}{m-p}}$ is smooth in D and we can compute

$$Z' = \frac{m}{m-p}|H|^{\frac{2p-m}{m-p}} HH', \quad Z'' = \frac{m}{m-p}|H|^{\frac{2p-m}{m-p}} \Big(HH'' + \frac{p}{m-p}H'^2\Big).$$

Hence

$$Z'' - \frac{m-p}{2m(1-p)} yW^{\frac{1-m}{m-p}} Z' = \frac{m}{m-p}|H|^{\frac{2p-m}{m-p}} F, \quad y \in D,$$

where

$$F := \frac{p}{m-p}H'^2 + H\Big(H'' - \frac{m-p}{2m(1-p)} yW^{\frac{1-m}{m-p}} H'\Big).$$

From the definition of H, we have

$$F = \frac{p}{m-p}\Big(\frac{1}{2}W' - \frac{y}{2}W''\Big)^2 - \frac{y}{2}(W - \frac{y}{2}W')\Big[W''' + \frac{m-p}{2m(1-p)} W^{\frac{1-m}{m-p}} (W' - yW'')\Big].$$

Using (2.4.3), it follows that

$$\begin{aligned} F &= \frac{p}{m-p}\Big(\frac{1}{2}W' - \frac{y}{2}W''\Big)^2 - \frac{y}{2}(W - \frac{y}{2}W')\Big[\frac{-2+2m}{2m(1-p)} W^{\frac{1-m}{m-p}} W' \\ &\quad + \frac{1-m}{2m(1-p)} yW^{\frac{1-2m+p}{m-p}} W'^2 - \frac{p}{m-p} \frac{W'(2W''W - W'^2)}{W^2}\Big] \\ &= \frac{p}{4(m-p)}\Big\{(W' - yW'')^2 - y(2W - yW')\Big[-\frac{(1-m)(m-p)}{mp(1-p)} W^{\frac{1-m}{m-p}} W' \\ &\quad + \frac{(1-m)(m-p)}{2mp(1-p)} yW^{\frac{1-2m+p}{m-p}} W'^2 - \frac{W'(2W''W - W'^2)}{W^2}\Big]\Big\} \\ &= \frac{p}{4(m-p)}\Big\{\Big[(W' - yW'')^2 + y(2W - yW')\frac{W'(2W''W - W'^2)}{W^2}\Big] \\ &\quad + y(2W - yW')\frac{(1-m)(m-p)}{mp(1-p)} W^{\frac{1-2m+p}{m-p}} \Big(WW' - \frac{y}{2}W'^2\Big)\Big\}. \end{aligned}$$

Then we have

$$\begin{aligned} F &= \frac{p}{4(m-p)}\Big\{\Big[W'+y\Big(\frac{WW''-W'^2}{W}\Big)\Big]^2 \\ &\qquad + y(2W-yW')\frac{(1-m)(m-p)}{mp(1-p)}W^{\frac{1-2m+p}{m-p}}\Big(WW'-\frac{y}{2}W'^2\Big)\Big\} \\ &= \frac{p}{4(m-p)}\Big\{\Big[W'+y\Big(\frac{WW''-W'^2}{W}\Big)\Big]^2 \\ &\qquad + \frac{(1-m)(m-p)}{mp(1-p)}yW^{\frac{1-2m+p}{m-p}}(2W-yW')W'\Big(W-\frac{y}{2}W'\Big)\Big\} \\ &= \frac{p}{4(m-p)}\Big\{\Big[W'+y\Big(\frac{WW''-W'^2}{W}\Big)\Big]^2 \\ &\qquad + \frac{2(1-m)(m-p)}{mp(1-p)}yW^{\frac{1-2m+p}{m-p}}\Big(W-\frac{y}{2}W'\Big)^2W'\Big\} \geq 0 \end{aligned}$$

for all $y \in D$, and hence

$$(e^{-\rho(y)}Z')' \geq 0 \quad \text{in} \quad D, \tag{2.4.4}$$

where

$$\rho(y) := \int_0^y \frac{m-p}{2m(1-p)} sW^{\frac{1-m}{m-p}}(s)ds.$$

Next, we claim that the function Z is nonincreasing in $[0,\infty)$. Otherwise, we can find a $y_0 > 0$ such that $Z(y_0) > 0$ and $Z'(y_0) > 0$. Hence $e^{-\rho(y)}Z'(y) \geq e^{-\rho(y_0)}Z'(y_0)$ for $y \geq y_0$ by (2.4.4). Noting that $W(y_0) > 0$ and $W' \geq 0$ for all $y > 0$, it follows that for $y \geq y_0$

$$\rho(y) \geq \frac{m-p}{2m(1-p)}\int_{y_0}^y sW(y_0)^{\frac{1-m}{m-p}}ds \geq cy^2$$

for some $c > 0$. Therefore, we get $Z'(y) \geq e^{-\rho(y_0)}Z'(y_0)e^{cy^2} = \widehat{C}e^{cy^2}$ for $y \geq y_0$, where $\widehat{C} = e^{-\rho(y_0)}Z'(y_0) > 0$. From $|(y^{-2}W)'| = 2y^{-3}Z^{1-(p/m)}$, we would get $W \geq e^{\hat{\eta}y^2}$ as $y \to \infty$ for some $\hat{\eta} > 0$, contradicting the polynomially bounded assumption on V. Hence Z is nonincreasing in $[0,\infty)$.

Now, we claim that Z is constant on $(0,\infty)$. For contradiction, we assume that there is $R > 0$ such that $Z(0) > Z(R)$. Then we can choose $\epsilon > 0$ small enough such that $f(0) > Z(0) > f(R)$, where $f := Z + \epsilon e^{2\rho(y)}$. Hence f has a local maximum at some point y_1 in $(-R,R)$. Note that $Z(0) > 0$. Hence $W(0) > 0$ and so we have $D = \{y \in \mathbb{R};\ H(y) \neq 0\}$. Also, it follows from

$$\begin{aligned} &Z(y_1) + \epsilon e^{2\rho(R)} \geq Z(y_1) + \epsilon e^{2\rho(y_1)} = f(y_1) \geq f(0) > Z(0), \\ &Z(0) > f(R) = Z(R) + \epsilon e^{2\rho(R)}, \end{aligned}$$

that $Z(y_1) > Z(R)$. Hence $Z(y_1) > 0$ and so $y_1 \in D$.

Denoting $a = \frac{1-m}{m-p} > 0$ and $c = \frac{m-p}{2m(1-p)} > 0$, we compute, for $y \in D$,

$$\rho'' = c\big[W^a + y(W^a)'\big] \geq cW^a(0) > 0,$$
$$f' = Z' + 2\epsilon\rho' e^{2\rho(y)},$$
$$f'' = Z'' + \epsilon(2\rho')^2 e^{2\rho(y)} + 2\epsilon\rho'' e^{2\rho(y)} > Z'' + 4\epsilon{\rho'}^2 e^{2\rho(y)}.$$

Therefore, at $y = y_1$, using the fact that f has a local maximum and (2.4.4), we obtain

$$0 \leq \rho' f' - f'' < \rho'(Z' + 2\varepsilon\rho' e^{2\rho}) - (Z'' + 4\varepsilon{\rho'}^2 e^{2\rho}) = (\rho' Z' - Z'') - 2\varepsilon{\rho'}^2 e^{2\rho} \leq 0,$$

a contradiction. We conclude that $W - yW'/2 = C$ on $(0,\infty)$ for some constant C. By an integration, we get $W = A + By^2$ for some constants A and B. Putting this into (2.4.2), the conclusion follows. □

Remark 2.4.1 Proposition 2.4.1 generalizes [29, Proposition 3.3] (for $m = 1$). Actually there was a small inaccuracy in the proof of [29, Proposition 3.3], since D was defined there as $\{y > 0;\ H(y) \neq 0\}$ and we cannot rule out the possibility that $y_1 = 0$. This is easily overcome by the current definition of D (the present proof works for $m = 1$ as well).

Next, in order to construct a Lyapunov function, we need to study the backward continuation of solution to the ODE associated with (2.1.8). For this, we take a smooth and nonincreasing function ζ on $\mathbb{R}$ such that

$$\zeta(\varrho) = 0,\ \varrho \geq 2, \quad \zeta(\varrho) = 1,\ \varrho \leq 1, \quad 0 \leq \zeta(\varrho) \leq 1, \quad \varrho \in (1,2).$$

Let $g(v) := \beta v^\gamma - v^q$. Following [25], we define

$$\widehat{g}(\xi, v) = g(v)\left[1 - \zeta\left(\frac{2v}{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]}\right)\right] - v^{-1}\zeta\left(\frac{2v}{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]}\right). \tag{2.4.5}$$

Without loss of generality, we may assume that $D^* < \kappa/(2^\delta)$. Note that $\widehat{g}(\xi, v) = g(v)$ for all ξ whenever $v \geq \kappa$.

Let $\psi(\xi; y, v, w)$ be the solution of the problem:

$$\psi_{\xi\xi} - \alpha\gamma\xi\psi^{\gamma-1}\psi_\xi + \widehat{g}(\xi,\psi) = 0, \quad \xi < y, \tag{2.4.6}$$

$$\psi(y; y, v, w) = v, \quad \psi_\xi(y; y, v, w) = w, \tag{2.4.7}$$

where $v > 0$, $w \in \mathbb{R}$, and the subscript ξ denotes the derivative with respect to ξ. By the standard ODE theory, $\psi(\xi; y, v, w)$ is defined in a neighborhood of $\xi = y$. Clearly, the solution $\psi(\xi; y, v, w)$ can be extended as long as $\psi > 0$ and ψ, ψ_ξ remain bounded.

Now, we assume that $(\hat{y}, y]$, $0 \le \hat{y} < y$, is the maximal existence interval for the solution ψ of (2.4.6)-(2.4.7) in $[0, y]$. We shall prove that $\hat{y} = 0$, i.e., the solution ψ can be continued backward to $\xi = 0$. For this, multiplying (2.4.6) by ψ_ξ and integrating in ξ, we have

$$\tag{2.4.8} \begin{aligned} &\frac{1}{2}\psi_\xi^2(\xi; y, v, w) + \alpha\gamma \int_\xi^y \tau\psi^{\gamma-1}(\tau; y, v, w)\psi_\xi^2(\tau; y, v, w)d\tau + \int_\kappa^{\psi(\xi;y,v,w)} \widehat{g}(\xi, \mu)d\mu \\ &\quad = \frac{1}{2}w^2 + \int_\kappa^v \widehat{g}(y, \mu)d\mu - \int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau. \end{aligned}$$

Next, we have to estimate the last term in the above equation. Clearly, we have

$$\widehat{g}_\xi(\xi, v) = 2v[g(v) + v^{-1}]\zeta'\left(\frac{2v}{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]}\right)\frac{\{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\}'}{\{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\}^2}.$$

Hence

$$\tag{2.4.9} \widehat{g}_\xi(\xi, v) = 0 \ \text{ in } \{v > D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\} \cup \{v < \{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\}/2\},$$

and so $|\widehat{g}_\xi(\xi, v)|$ is uniformly bounded for $\xi > 1$ for all $v > 0$. Therefore, the integral

$$-\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau$$

is bounded if $1 < \xi < y$. This bound may depend on y, but it is independent of ξ, v, w.

If $\xi \le 1$, then we have

$$\widehat{g}_\xi(\xi, v) = 2v[g(v) + v^{-1}]\zeta'\left(\frac{2v}{D^*\xi^\delta}\right)\frac{\delta}{D^*\xi^{\delta+1}}.$$

Note that $D^* < \kappa$. Hence

$$g(v) + v^{-1} \ge 0 \quad \text{for} \quad 0 < v < D^*\xi^\delta, \quad 0 < \xi \le 1.$$

Since $\zeta' \le 0$, we obtain

$$\tag{2.4.10} \widehat{g}_\xi(\xi, v) \le 0 \quad \text{for} \quad 0 < v < D^*\xi^\delta, \quad 0 < \xi \le 1.$$

For $0 \le \xi \le 1$, we write

$$\begin{aligned} -\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau \quad &= \quad -\int_\xi^{\min\{1,y\}} \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau \\ &\quad -\int_{\min\{1,y\}}^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau. \end{aligned}$$

Since the last integral in the above equation is bounded, from (2.4.9) and (2.4.10) it follows that

$$-\int_{\xi}^{y}\int_{\kappa}^{\psi(\tau;y,v,w)}\widehat{g}_{\xi}(\tau,\mu)d\mu d\tau \le C+\int_{\xi}^{\min\{1,y\}}\int_{\psi(\tau;y,v,w)}^{\kappa}\widehat{g}_{\xi}(\tau,\mu)d\mu d\tau \le C, \tag{2.4.11}$$

where the constant C is independent of w, v, and ξ (it may depend on y). Hence, by (2.4.8), we obtain $|\psi_\xi|$ and ψ are bounded from above. On the other hand, we have $\int_0^\kappa \widehat{g}(\xi,\mu)d\mu = -\infty$ for $\xi > 0$. By (2.4.8) again, we also have $\psi(\xi;y,v,w) > 0$ if $\xi > 0$. Thus the solution ψ can be extended beyond $\hat{y}$ if $\hat{y} > 0$, and therefore we must have $\hat{y} = 0$.

Finally, in this section, we shall derive the following identity

$$\psi_y(\xi;y,v,w) = -w\psi_v(\xi;y,v,w) - \left[\alpha\gamma y v^{\gamma-1}w - \widehat{g}(y,v)\right]\psi_w(y;y,v,w). \tag{2.4.12}$$

Differentiating (2.4.7) in y, we obtain

$$\psi_y(y;y,v,w) = -\psi_\xi(y;y,v,w) = -w; \quad \psi_{\xi\xi}(y;y,v,w) + \psi_{\xi y}(y;y,v,w) = 0.$$

Then we have

$$\begin{aligned}
\psi_{\xi y}(y;y,v,w) &= -\psi_{\xi\xi}(y;y,v,w) \\
&= -\alpha\gamma y\psi^{\gamma-1}(y;y,v,w)\psi_\xi(y;y,v,w) + \widehat{g}(y,\psi(y;y,v,w)) \\
&= -\alpha\gamma y v^{\gamma-1}w + \widehat{g}(y,v).
\end{aligned}$$

Furthermore, differentiating (2.4.7) in v and w respectively, we obtain

$$\psi_v(y;y,v,w) = 1, \quad \psi_{v\xi}(y;y,v,w) = 0, \quad \psi_w(y;y,v,w) = 0, \quad \psi_{w\xi}(y;y,v,w) = 1.$$

Hence

$$\psi_y(y;y,v,w) = -w\psi_v(y;y,v,w) - \left[\alpha\gamma y v^{\gamma-1}w - \widehat{g}(y,v)\right]\psi_w(y;y,v,w), \tag{2.4.13}$$

$$\psi_{y\xi}(y;y,v,w) = -w\psi_{v\xi}(y;y,v,w) - \left[\alpha\gamma y v^{\gamma-1}w - \widehat{g}(y,v)\right]\psi_{w\xi}(y;y,v,w). \tag{2.4.14}$$

Differentiating (2.4.6) in y, v, and w respectively, we obtain that both functions

$$\psi_y(\xi;y,v,w), \quad -w\psi_v(\xi;y,v,w) - \left[\alpha\gamma y v^{\gamma-1}w - \widehat{g}(y,v)\right]\psi_w(y;y,v,w)$$

satisfy the following ODE:

$$\theta_{\xi\xi} - \alpha\gamma\xi\psi^{\gamma-1}\theta_\xi + [\widehat{g}_\psi(\xi,\psi) - \alpha\gamma(\gamma-1)\xi\psi^{\gamma-2}\psi_\xi]\theta = 0$$

for unknown θ. Then, using (2.4.13)-(2.4.14) and the uniqueness of the solution to the initial value problem, we obtain (2.4.12).

2.5 Proof of Theorem 2.1.3

This section is devoted to the proof of the main result, Theorem 2.1.3. We first construct a suitable Lyapunov function using a method of Zelenyak [59] (see also [25]). Define

$$E_{R(s)}[z](s) = \int_0^{R(s)} \Phi(y, z(y,s), z_y(y,s))dy \tag{2.5.1}$$

where $R(s) = e^{\alpha s}$ and $\Phi = \Phi(y, v, w)$ will be chosen below. Since $z(y,s) \geq D^*|y|^\delta$ in Ω_0, z also satisfies the equation

$$\gamma z^{\gamma-1} z_s = z_{yy} - \alpha\gamma y z^{\gamma-1} z_y + \widehat{g}(y, z) \quad \text{in } \Omega_0. \tag{2.5.2}$$

Then, integrating by parts and using (2.5.2), we have

$$\frac{d}{ds} E_{R(s)}[z](s) = J_0 + J_1 + J_2,$$

where

$$\begin{aligned}
J_0 &= -\int_0^{R(s)} \gamma\Phi_{ww}(y, z(y,s), z_y(y,s)) z^{\gamma-1}|z_s|^2 dy,\\
J_1 &= \Phi_w(R(s), z(R(s),s), z_y(R(s),s)) z_s(R(s),s) - \Phi_w(0, z(0,s), 0) z_s(0,s)\\
&\quad + \Phi(R(s), z(R(s),s), z_y(R(s),s)) R'(s),\\
J_2 &= \int_0^{R(s)} K(y, z(y,s), z_y(y,s)) z_s(y,s) dy,\\
K(y,v,w) &:= \Phi_v - \Phi_{wy} - \Phi_{wv} w - \Phi_{ww}\left[\alpha\gamma y v^{\gamma-1} w - \widehat{g}(y,v)\right],
\end{aligned}$$

where $\widehat{g}$ is defined in (2.4.5).

Let

$$\Phi(y,v,w) := \int_0^w (w-\sigma) P(y,v,\sigma) d\sigma - \int_\kappa^v \widehat{g}(y,\mu) P(y,\mu,0) d\mu.$$

Then we have

$$\begin{aligned}
\Phi_w(y,v,w) &= \int_0^w P(y,v,\sigma) d\sigma,\\
\Phi_w(y,v,0) &= 0, \quad \Phi_{ww}(y,v,w) = P(y,v,w),\\
K(y,v,w) &= \int_0^w \Big\{ -\sigma \frac{\partial P}{\partial v}(y,v,\sigma) - \frac{\partial P}{\partial y}(y,v,\sigma)\\
&\qquad + \frac{\partial}{\partial\sigma}\left[P(y,v,\sigma)\left(\widehat{g}(y,v) - \alpha\gamma y v^{\gamma-1}\sigma\right)\right]\Big\} d\sigma.
\end{aligned}$$

Moreover, let

$$P(y,v,w) := \exp\left\{-\alpha\gamma \int_0^y \xi\psi(\xi; y, v, w)^{\gamma-1} d\xi\right\},$$

where ψ is defined in (2.4.6)-(2.4.7). Then using (2.4.12) and by a simple computation we obtain that $K(y, v, w) \equiv 0$, and therefore $J_2 = 0$.

Next, we derive some estimates for large v. Recall that $\widehat{g}(y, v) = g(v)$ for $v \geq \kappa$. From (2.4.8) it follows that

$$|\psi_\xi(\bar{\xi}; y, v, w)|^2 \leq w^2 + 2\int_\kappa^v g(\mu)d\mu := A(v, w) \quad \text{for all } \bar{\xi} \in [\xi, y].$$

Obviously, we have

$$\psi(\xi; y, v, w) \geq v - \max_{\xi \leq \bar{\xi} \leq y} |\psi_\xi(\bar{\xi}; y, v, w)|(y - \xi) \geq v - A(v, w)^{1/2}(y - \xi) \geq v/2$$

for any ξ satisfying $0 < y - \xi \leq vA(v, w)^{-1/2}/2$. Let

$$\widehat{\xi} := \max\left\{y/2,\ y - vA(v, w)^{-1/2}/2\right\}.$$

Then

$$P(y, v, w) \leq \exp\left\{-\alpha\gamma\left(\frac{v}{2}\right)^{\gamma-1}\int_{\widehat{\xi}}^y \xi d\xi\right\} \leq \exp\{-B(y, v, w)\}, \tag{2.5.3}$$

where

$$B(y, v, w) := \min\left\{\frac{\alpha\gamma}{4}\left(\frac{v}{2}\right)^{\gamma-1} y^2,\ \ \frac{\alpha\gamma}{2}\left(\frac{v}{2}\right)^{\gamma} yA(v, w)^{-1/2}\right\}.$$

Therefore, we have

$$\Phi(y, v, w) \leq \int_0^w (w - \sigma)P(y, v, \sigma)d\sigma \leq w^2 \exp\{-B(y, v, w)\}, \tag{2.5.4}$$

$$|\Phi_w(y, v, w)| = \left|\int_0^w P(y, v, \sigma)d\sigma\right| \leq |w| \exp\{-B(y, v, w)\}. \tag{2.5.5}$$

We shall estimate J_1 from above. Here we substitute $y = R(s)(= e^{\alpha s})$, $z(R(s), s) = v$, and $w = z_y(R(s), s)$ into the above estimates. Since $z(R(s), s) = k^m e^{\beta m s}$, $z(R(s), s) \to \infty$ as $s \to \infty$. Note that

$$\int_\kappa^v g(\mu)d\mu \sim \frac{\beta}{\gamma + 1}v^{\gamma+1} \ \text{ for } \quad v \gg 1.$$

Hence it follows from (2.5.4) and (2.5.5) that

$$\Phi(y, z(R(s), s), z_y(R(s), s)) \leq |z_y(R(s), s)|^2 \exp\left\{-c^* z(R(s), s)^{(\gamma-1)/2} R(s)\right\},$$
$$|\Phi_w(y, z(R(s), s), z_y(R(s), s))| \leq |z_y(R(s), s)| \exp\left\{-c^* z(R(s), s)^{(\gamma-1)/2} R(s)\right\}$$

for large s for some constant $c^* > 0$.

Next, we shall follow an idea from [24, p.54] to obtain an estimate of $u_x(1,t)$. Since $u_t < 0$ in Q_T and $u_x > 0$ in $(0,1] \times (0,T)$, we have

$$(u^m)_{xx} < u^p \quad \text{in } \; Q_T, \quad (u^m)_{xx}(u^m)_x < u^p (u^m)_x \quad \text{in } \; (0,1]\times(0,T).$$

An integrating of the last inequality from 0 to $x \in (0,1]$ gives

$$0 \le m(u^{m-1}u_x)(x,t) = (u^m)_x(x,t) \le \sqrt{\frac{2m}{p+m}} u^{(p+m)/2}(x,t)$$

for $(x,t) \in (0,1] \times (0,T)$. Therefore, by (2.1.1), we have $0 \le u_x(1,t) \le c^{**}$ for all $t \in (0,T)$ for some positive constant c^{**}. Then

$$0 \le z_y(R(s),s) \le c^{**} m k^{m-1} \exp[(\beta m - \alpha)s]. \tag{2.5.6}$$

This implies that

$$\begin{aligned} &\Phi(y, z(R(s),s), z_y(R(s),s))R'(s) \\ &\quad \le \alpha (c^{**})^2 m^2 k^{2(m-1)} \exp\left[(2\beta m - \alpha)s\right] \exp\left\{-c^* k^{(1-m)/2} e^{s/2}\right\}. \end{aligned} \tag{2.5.7}$$

Similarly, using $z(R(s),s) = k^m e^{\beta m s}$ and (2.5.6), we obtain

$$(\beta m k^m - \alpha m c^{**} k^{m-1}) e^{\beta m s} \le z_s(R(s),s) \le \beta m k^m e^{\beta m s}.$$

Thus we have

$$\begin{aligned} &|\Phi_w(y, z(R(s),s), z_y(R(s),s))||z_s(R(s),s)| \\ &\quad \le \bar{c} \exp\left[(2\beta m - \alpha)s\right] \exp\left\{-c^* k^{(1-m)/2} e^{s/2}\right\} \end{aligned} \tag{2.5.8}$$

for some positive constant $\bar{c}$. Since $\Phi_w(0, z(0,s), 0) = 0$, together with (2.5.7) and (2.5.8), we obtain that J_1 is bounded from above by a function that decays exponentially fast.

Next, we prove that J_1 is bounded below. By (2.5.3) and noting that $\widehat{g}(y,\mu) = g(\mu)$ for $\mu \ge \kappa$, we have

$$P(y,v,0) \le \exp\left\{-c^* v^{(\gamma-1)/2} y\right\} \quad for \;\; v \ge \kappa.$$

Furthermore, we have

$$\begin{aligned} &\Phi(y, z(R(s),s), z_y(R(s),s))R'(s) \\ \ge &\; -R'(s) \int_\kappa^{z(R(s),s)} g(\mu) P(R(s),\mu,0)\, d\mu \\ \ge &\; -\frac{\beta}{\gamma+1} R'(s) z(R(s),s)^{\gamma+1} \exp\left\{-c^* \kappa^{(\gamma-1)/2} R(s)\right\} \\ \ge &\; -\frac{\alpha\beta}{\gamma+1} k^{m(\gamma+1)} \exp\left\{\frac{(3m-p+2)s}{2(1-p)}\right\} \exp\left\{-c^* \kappa^{(\gamma-1)/2} \exp(\alpha s)\right\}. \end{aligned} \tag{2.5.9}$$

Hence by (2.5.8) and (2.5.9) we obtain that J_1 is bounded from below by a function that decays exponentially fast. Thus we have proved:

Lemma 2.5.1 *We have*

$$\frac{d}{ds}E_{R(s)}[z](s) = -\gamma \int_0^{R(s)} P(y, z(y,s), z_y(y,s)) z^{\gamma-1}(y,s)|z_s(y,s)|^2 dy + J_1(s),$$

where J_1 *satisfies the property* $\int_{s_0}^{\infty} |J_1(s)|ds < \infty$.

From this lemma, we obtain

$$\frac{d}{ds}\left[E_{R(s)}[z](s) - \int_{s_0}^{s} J_1(\tau)d\tau\right] \leq 0,$$

and therefore for any $s > s_0$,

$$E_{R(s)}[z](s) \leq E_{R(s)}[z](s_0) + \int_{s_0}^{s} J_1(\tau)d\tau \leq E_{R(s)}[z](s_0) + \int_{s_0}^{\infty} |J_1(\tau)|d\tau \equiv \overline{C}.$$

Proof of Theorem 2.1.3. Let s_j be a sequence with $s_j \to \infty$ as $j \to \infty$. We define $z_j(y,s) = z(y, s+s_j)$ for all $j \in \mathbb{N}$ and $(y,s) \in \Omega$. Define $\tilde{\Omega} := \{(y,s) \in \mathbb{R}^2;\ y \neq 0\}$ and observe that, for any given $\omega \subset\subset \tilde{\Omega}$, z_j is defined on ω for j large enough. Using (2.3.4)-(2.3.7), applying parabolic L^p estimates to (2.1.8) and to the equation satisfied by z_y, and then parabolic Hölder estimates, it follows that the sequence $\{z_j\}_{j\in\mathbb{N}}$ is compact in $C^{2,1}(\omega)$ for any $\omega \subset\subset \tilde{\Omega}$. Therefore, using a diagonal process, there exists a subsequence $\{j_l\}$ and a function $z_\infty \in C^{2,1}(\tilde{\Omega})$, such that $z_{j_l}(y,s) \to z_\infty(y,s)$ as $l \to \infty$, locally uniformly in $C^{2,1}(\tilde{\Omega})$. Moreover, z_∞ satisfies (2.1.8) in $\tilde{\Omega}$ and, due to (2.3.7),

$$|\partial_y z_\infty(y,s)| \leq c|y|, \quad 0 < |y| \leq 1,\ s \in \mathbb{R}. \tag{2.5.10}$$

Consider the Lyapunov function $E_{R(s)}[z](s)$ defined in (2.5.1). Then we have

$$E_{R(s)}[z](s) \leq E_{R(s)}[z](s_0) + \int_{s_0}^{\infty} |J_1(\tau)|d\tau \equiv \overline{C} < \infty \quad \text{for } s > s_0.$$

On the other hand, since

$$P(y, v, 0) \leq \exp\left\{-c^* v^{(\gamma-1)/2} y\right\}$$

for $v \geq \kappa$ and $z(y,s) \geq D^*|y|^\delta \geq D^*$ for any $y \geq 1$, we have

$$\begin{aligned} &\int_0^{R(s)} \int_\kappa^{z(y,s)} \widehat{g}(y,\mu) P(y,\mu,0) d\mu dy \\ &\leq \int_0^1 \int_0^{c^*} |g(\mu)| P(y,\mu,0) d\mu dy + \int_1^\infty \int_{D^*}^\infty |g(\mu)| P(y,\mu,0) d\mu dy \leq \widetilde{C} \end{aligned} \tag{2.5.11}$$

where c^* is the upper bound for $z(y,s)$ over $y \in [0,1]$. From (2.5.11) we obtain

$$E_{R(s)}[z](s) \geq -\int_0^{R(s)} \int_\kappa^{z(y,s)} \widehat{g}(y,\mu)P(y,\mu,0)d\mu dy \geq -\widetilde{C},$$

and hence by using Lemma 2.5.1, we have

$$\gamma \int_{s_0}^\infty \int_0^{R(s)} P(y,z(y,s),z_y(y,s))z^{\gamma-1}(y,s)|z_s(y,s)|^2 dyds \leq \widetilde{C}. \tag{2.5.12}$$

Note that $P(y,v,w)$ is bounded below away from 0 for y,v,w in bounded sets. Also, z and z_y are bounded for y bounded, $z(y,s) \geq D^*|y|^\delta$ in Ω_0 and $\gamma > 1$. For each $0 < \varepsilon < 1$ and $M > e^{\alpha s_0}$, putting $s_M = \alpha^{-1} \ln M$, it follows from (2.5.12) that

$$\int_{s_M}^\infty \int_\varepsilon^M |z_s(y,s)|^2 dyds < \infty.$$

For each $S > 0$, we thus deduce

$$\int_{-S}^S \int_\varepsilon^M |\partial_s z_\infty(y,s)|^2 dyds \leq \liminf_{j\to\infty} \int_{s_j-S}^\infty \int_\varepsilon^M |z_s(y,s)|^2 dyds = 0,$$

hence $\partial_s z_\infty(y,s) = 0$ and $z_\infty = z_\infty(y)$ satisfies

$$z'' - \alpha\gamma y z^{\gamma-1}z' + \beta z^\gamma - z^q = 0, \quad y > 0. \tag{2.5.13}$$

Using (2.5.13) and (2.5.10), it follows that z_∞ can be extended to a C^2 solution of (2.5.13) on $[0,\infty)$ with $\partial_y z_\infty(0) = 0$, hence to a symmetric C^2 solution on $\mathbb{R}$, in view of the symmetry of z in y. Note that z_∞ is monotone in $y > 0$. Therefore, the conclusion follows from (2.3.4)-(2.3.5) and Proposition 2.4.1. This completes the proof of the theorem. □

Chapter 3

Dynamics for a complex-valued heat equation

3.1 Introduction

In this chapter, we study the following equation

$$z_t = z_{xx} - \frac{1}{z}, \tag{3.1.1}$$

where $z = z(x,t)$ is a complex-valued function of the spatial variable $x \in \mathbb{R}$ and the time variable $t \geq 0$. If we set $z(x,t) = u(x,t) + iv(x,t)$, where $i = \sqrt{-1}$ and $u(x,t)$, $v(x,t) \in \mathbb{R}$, then (3.1.1) can be written as a system of parabolic equations

$$\begin{cases} u_t = u_{xx} - u/(u^2+v^2), \\ v_t = v_{xx} + v/(u^2+v^2). \end{cases} \tag{3.1.2}$$

If $z(x,t)$ is real-valued (i.e., $v \equiv 0$), then the system is reduced to the equation

$$u_t = u_{xx} - \frac{1}{u}.$$

An initial boundary value problem for the above equation was first studied by Kawarada [37] in 1975. For more general negative power nonlinearity, we refer the reader to, e.g., [22, 31, 38] and the references cited therein. The goal of this chapter is to study the dynamics of solutions of the system (3.1.2) with $v \not\equiv 0$.

First of all, we consider a spatially homogeneous solution of (3.1.2), namely, $(u,v) = (U(t), V(t))$. We obtain that $(U(t), V(t))$ satisfies the following ODE system:

$$\begin{cases} U_t = -U/(U^2+V^2), \\ V_t = V/(U^2+V^2). \end{cases} \tag{3.1.3}$$

Given $(U(0), V(0)) \in \mathbb{R}^2 \setminus \{(0,0)\}$. By a simple computation, we obtain that

$$U(t)V(t) = U(0)V(0) := C, \quad \forall t \geq 0. \tag{3.1.4}$$

for some constant $C \in \mathbb{R}$.

If $U(0) = 0$, then the trajectory stays on the the V-axis, exists globally and tends to $\pm\infty$ as $t \to \infty$. On the other hand, if $V(0) = 0$, then $V(t) \equiv 0$ and U tends to zero in finite time. When $C \neq 0$, by (3.1.3) and (3.1.4) we have $(U(t), V(t)) \to (0, \pm\infty)$ as $t \to \infty$.

In this chapter, we consider the initial value problem (P) for (3.1.2) with the initial condition

$$(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0). \tag{3.1.5}$$

In the sequel, we shall always assume that

$$u_0 > 0, \quad v_0 \geq 0, \quad u_0, v_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \quad \inf_{\mathbb{R}} u_0 + \inf_{\mathbb{R}} v_0 > 0.$$

Then the problem (P) has a unique solution $(u, v) \in (C([0,T); L^\infty(\mathbb{R})))^2$, where $T = T(u_0, v_0) \in (0, \infty]$ is the maximal existence time of the solution. Furthermore, we have either $T = \infty$, or

$$T < \infty \;\text{ and }\; \liminf_{t \to T} \{\inf_{x \in \mathbb{R}} u(x,t) + \inf_{x \in \mathbb{R}} v(x,t)\} = 0.$$

In the first case, we have the *global* existence. For the second case, we say that the solution of (P) *quenches* in a finite time T in which T is called the *quenching time*. Moreover, we say that $x_Q \in \mathbb{R}$ is a (finite) *quenching point* for (u, v) if there exists a sequence $\{(x_j, t_j)\}$ such that $x_j \to x_Q$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$. We shall investigate the global and non-global existence of solutions of (P).

The first result is about the global existence and (time) asymptotic behavior of solution of the problem (P).

Theorem 3.1.1 *Suppose that the initial data satisfy*

$$\begin{cases} u_0(x) > 0, \; v_0(x) > 0, \; \forall x \in \mathbb{R}, \; u_0 \text{ and } v_0 \text{ are bounded in } \mathbb{R}, \\ u_0(x)v_0(x) \geq K, \; \forall x \in \mathbb{R}, \; \text{for some constant } K > 0. \end{cases} \tag{3.1.6}$$

Then the solution of (3.1.2) *with* (3.1.5) *exists globally in time and* (u, v) *converges to* $(0, \infty)$ *as* $t \to \infty$ *uniformly in* $\mathbb{R}$.

For $t \geq 0$, we set

$$\mathcal{R}(t) := \big\{(u(x,t), v(x,t)) \in \mathbb{R}^2; \; x \in \mathbb{R}\big\}$$

to be the image of the solution on (u, v)-plane. We remark that, under the hypothesis of Theorem 3.1.1, the closure of the convex hull of $\mathcal{R}(0)$ lies in the first quadrant of (u, v)-plane. Indeed, under the condition (3.1.6), we shall see that $\mathcal{R}(t)$ stays in the first quadrant for all $t > 0$. This implies the global existence of solutions.

On the other hand, if the initial data do not satisfy (3.1.6), in view of the dynamics of (3.1.3), it is interesting to see what happen. One question is to see under what conditions the quenching occurs. From (3.1.2) it is easy to see that both u and v quench simultaneously whenever quenching occurs. On the contrary, there might be *non-simultaneous quenching* in which just one component quenches and the other remains bounded away from zero. For this, we refer the reader to, e.g., [9, 43, 61, 45, 60].

To find solutions quenching in finite time, we consider the case when the initial data are *asymptotically constants*. Namely, we impose the following conditions on initial data:

$$u_0, v_0 \in C^1(\mathbb{R}), \quad u_0 \geq M, \quad u_0 \not\equiv M, \quad v_0 \geq 0, \quad v_0 \not\equiv 0, \tag{3.1.7}$$

$$\lim_{|x|\to\infty} u_0(x) = M, \quad \lim_{|x|\to\infty} v_0(x) = N \tag{3.1.8}$$

for some constants $M > 0$ and $N \geq 0$.

The following theorem shows that the solution of (3.1.2) with initial data satisfying (3.1.7) and (3.1.8) with $N > 0$ behaves like the solution the ODE system (3.1.3) with $(U(0), V(0)) = (M, N)$.

Theorem 3.1.2 *Let* (u, v) *be a solution of* (3.1.2) *with initial data* (u_0, v_0) *satisfying* (3.1.7) *and* (3.1.8). *If* $N > 0$, *then the solution of* (3.1.2) *with* (3.1.5) *exists globally for all* $t \geq 0$ *and* (u, v) *converges to* $(0, \infty)$ *as* $t \to \infty$ *uniformly in* $\mathbb{R}$.

On the other hand, if the initial data of (3.1.2) satisfy (3.1.7) and (3.1.8) with $N = 0$, then the solution of (3.1.2) and (3.1.5) quenches only at space infinity. Namely, there are no (finite) quenching points, while there exists a sequence $\{(x_j, t_j)\}$ such that $|x_j| \to \infty$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$.

Theorem 3.1.3 *Let* (u, v) *be a solution of* (3.1.2) *and* (3.1.5) *with the initial data* (u_0, v_0) *satisfying* (3.1.7) *and* (3.1.8) *with* $M > 0$ *and* $N = 0$. *Then the solution of* (3.1.2) *with* (3.1.5) *quenches at the finite time* $t = T = M^2/2$. *Moreover, the solution quenches only at space infinity.*

Note that the problem of quenching at space infinity for scalar equation was studied by Giga-Seki-Umeda [20, 21]. In [20], they characterized that, with suitable initial data, solutions of the following Cauchy problem $u_t = u_{xx}/(1 + u_x^2) - (n-1)/u$ quenching only at space infinity. In [21], they estimated its profile at the quenching time from above and below.

The motivation of this study is from a work of Guo-Ninomiya-Shimojo-Yanagida [28]. In [28], they considered, instead of (3.1.1), the following complex-valued equation:

$$z_t = \Delta z + z^2 \tag{3.1.9}$$

where $z = z(x,t) = u(x,t) + iv(x,t)$ is a complex-valued function of $x \in \mathbb{R}^m$ ($m \in \mathbb{N}$) and $t \geq 0$. To obtain the asymptotical behavior of the solution, our method is close to that in [28] by using an invariant set argument. But, instead of considering the invariant subset in (u,v)-plane, we transform our problem in (u,w)-plane where $w := 1/v$. Also, the solution blows up non-simultaneously at space infinity for the case (3.1.9) with asymptotically constant initial data. But, in our case (3.1.1), quenching can only occurs simultaneously.

This chapter is organized as follows. In section 3.2, we provide a sufficient condition for the existence of global solutions and study the asymptotic behavior of solutions as $t \to \infty$. In section 3.3, we study the solution of (3.1.2) with asymptotically constant initial data.

3.2 Global existence and Convergence

In this section we give a proof of Theorem 3.1.1. Let us first recall some properties about invariant sets (cf. [56]).

Lemma 3.2.1 *Suppose that $\Omega(t) \subset \mathbb{R}^2$ is convex for each $t \geq 0$ and $\{\Omega(t)\}_{t\geq 0}$ is (positively) invariant under the flow* (3.1.3) *in the sense that $(U(t), V(t)) \in \Omega(t)$ for all $t > 0$, if $(U(0), V(0)) \in \Omega(0)$. Then $\{\Omega(t)\}_{t\geq 0}$ is also invariant under the flow* (3.1.2). *That is, if $\mathcal{R}(t_0) \subset \Omega(t_0)$ for some $t_0 \geq 0$, then $\mathcal{R}(t) \subset \Omega(t)$ for all $t > t_0$.*

To construct invariant sets, the following lemma is very useful.

Lemma 3.2.2 *Let $\{F_i\}_{1\leq i\leq m}$ be a set of C^1 functions from $\mathbb{R}^3$ to $\mathbb{R}$. Suppose that $\Omega(t)$ is expressed as*

$$\Omega(t) = \bigcap_{i=1}^{m} \{(u,v) \in \mathbb{R}^2 \ ; \ F_i(u,v,t) < 0\}, \quad t \geq 0.$$

Then $\{\Omega(t)\}_{t\geq 0}$ is invariant under the flow (3.1.3) *if*

$$\frac{d}{dt} F_i(U(t), V(t), t) \leq 0 \quad \text{on} \quad \{(u,v) \in \partial\Omega \ ; F_i(u,v,t) = 0\}$$

for all $i = 1, \ldots, m$.

With these lemmas, we are ready to prove the global existence of the solution of (3.1.2) and (3.1.5).

Proof of Theorem 3.1.1. Set

$$D_1 := \left\{(u,v) \in \mathbb{R}^2;\ u > 0,\ v > 0 \ \text{ and } \ -uv + K \le 0\right\}.$$

By assumption, we have $\mathcal{R}(0) \subset D_1$. For $(U,V) \in \partial D_1$, we compute

$$\frac{d}{dt}(-UV + K) = -(U_t V + U V_t) = 0.$$

Thus D_1 is invariant under the flow (3.1.3) by Lemma 3.2.2.

Since u_0 is bounded, there exists a constant $A > 0$ such that $u_0(x) \le A$, $\forall x \in \mathbb{R}$. Set

$$D_2 := \left\{(u,v) \in \mathbb{R}^2;\ u > 0,\ v > 0 \text{ and } u \le A\right\}.$$

Note that $D_1 \cap D_2$ is convex. For $(U,V) \in D_1 \cap \partial D_2$, we compute

$$\frac{d}{dt}(U - A) = -\frac{U}{U^2 + V^2} < 0.$$

Therefore, $D_1 \cap D_2$ is invariant under the flow (3.1.3) by Lemma 3.2.2.

It follows from Lemma 3.2.1 that $u(x,t) > 0$, $v(x,t) > 0$, $u(x,t)v(x,t) \ge K$ and $u(x,t) \le A$ for all $x \in \mathbb{R}$ and $t \ge 0$, as long as v stays finite. Using $u^2 + v^2 \ge 2uv \ge 2K$, we have

$$v_t \le v_{xx} + v/(2K).$$

From this, it follows that the solution of (3.1.2) and (3.1.5) with (3.1.6) exists globally in time.

Next, we shall prove the asymptotic behavior of the solution (u,v) as $t \to \infty$. We set $w := 1/v$. Then (3.1.2) is equivalent to

$$\begin{cases} u_t = u_{xx} - uw^2/(u^2w^2+1), \\ w_t = w_{xx} - 2w_x^2/w - w^3/(u^2w^2+1). \end{cases}$$

Moreover, it follows from (3.1.6) that

$$(3.2.1)\quad \begin{cases} u_0(x) > 0,\ w_0(x) := 1/v_0(x) > 0,\ u_0(x), w_0(x) \text{ are bounded, } \forall x \in \mathbb{R}, \\ u_0(x) \ge K w_0(x),\ \forall x \in \mathbb{R}, \quad \text{for some constant } K > 0. \end{cases}$$

Therefore, it is enough to prove that (u,w) converges to $(0,0)$ as $t \to \infty$.

For this, we first consider the spatially homogeneous solution $(u,w) = (U(t), W(t))$. Then (U,W) satisfies the following ODE system:

$$(3.2.2)\qquad \begin{cases} U_t = -UW^2/(U^2W^2+1), \\ W_t = -W^3/(U^2W^2+1). \end{cases}$$

We set

$$D_3 := \left\{(u,w) \in \mathbb{R}^2;\ u > 0,\quad w > 0 \ \text{ and } Kw - u \leq 0\right\}.$$

Then, by (3.2.1), we obtain that $\mathcal{S}(0) \subset D_3$. Hereafter $\mathcal{S}(t) := \{(u(x,t), w(x,t)) \in \mathbb{R}^2;\ x \in \mathbb{R}\}$.

For $(U,W) \in \partial D_3$, we have

$$\begin{aligned}
\frac{d}{dt}(KW - U) &= KW_t - U_t \\
&= -\frac{KW^3}{U^2W^2+1} + \frac{UW^2}{U^2W^2+1} \\
&= \frac{-W^2(KW-U)}{U^2W^2+1} = 0.
\end{aligned}$$

Hence D_3 is invariant under the flow (3.2.2) by Lemma 3.2.2.

Next, we set

$$D_4 := \left\{(u,w) \in \mathbb{R}^2;\ u > 0,\quad w > 0 \ \text{ and } \ -w + au^2 \leq 0\right\}$$

for some positive constant a such that $\mathcal{S}(0) \subset D_4$. This can be done due to (3.2.1). Note that $D_3 \cap D_4$ is convex and

$$\partial D_3 \cap \partial D_4 = \left\{(0,0),\ (1/(aK), 1/(aK^2))\right\}.$$

For $(U,W) \in D_3 \cap \partial D_4$, we have

$$\begin{aligned}
\frac{d}{dt}(-W + aU^2) &= -W_t + 2aUU_t \\
&= \frac{W^3}{U^2W^2+1} + 2aU\left(\frac{-UW^2}{U^2W^2+1}\right) \\
&= \frac{W^2}{U^2W^2+1}[W - 2aU^2] \\
&= \frac{W^2}{U^2W^2+1}[aU^2 - 2aU^2] = \frac{-aU^2W^2}{U^2W^2+1} \leq 0.
\end{aligned}$$

Hence $D_3 \cap D_4$ is invariant under the flow (3.2.2).

Finally, we set

$$D_5(t) := \left\{(u,w) \in \mathbb{R}^2;\ u > 0,\quad w > 0 \text{ and } w - h(t) \leq 0\right\},\ t \geq 0,$$

where $h(t)$ is a positive smooth decreasing function to be specified later. Note that $D_3 \cap D_4 \cap D_5(t)$ is convex. We choose $h(0) = 1/(aK^2)$ such that $\mathcal{S}(0) \subset D_3 \cap D_4 \cap D_5(0)$. For $(U,W) \in D_3 \cap D_4 \cap \partial D_5(t)$, we compute

$$\frac{d}{dt}(W - h) = W_t - h_t = \frac{-W^3}{U^2W^2+1} - h_t.$$

Hence $\{D_3 \cap D_4 \cap D_5(t)\}_{t\geq 0}$ is invariant under the flow (3.2.2), if

$$(3.2.3)\quad h_t = \sup_{(U,W)\in D_3\cap D_4\cap \partial D_5(t)} \frac{-W^3}{U^2W^2+1} = \frac{-h^3}{c^2h^2+1}, \text{ where } c := 1/(aK) > 0.$$

Therefore, let $h(t)$ be the solution of

$$(3.2.4)\qquad c^2 \ln h(t) - \frac{1}{2h^2(t)} = c^2 \ln h(0) - \frac{1}{2h^2(0)} - t,$$

we have that $h(t)$ satisfies (3.2.3) and $\{D_3 \cap D_4 \cap D_5(t)\}_{t\geq 0}$ is invariant under the flow (3.2.2). Moreover, by (3.2.3) and (3.2.4) we obtain that $h(t)$ decreases to 0 as $t \to \infty$. Therefore, (u, w) converges to $(0, 0)$ as $t \to \infty$. Since $v = 1/w$, we have (u, v) converges to $(0, \infty)$ as $t \to \infty$. This completes the proof of the theorem. □

3.3 Asymptotically constant initial data

This section is devoted to the study the solution of (3.1.2) with asymptotically constant initial data. We first consider the following ODE system:

$$(3.3.1)\qquad \begin{cases} U_t = -U/(U^2+V^2), \\ V_t = V/(U^2+V^2), \end{cases}$$

for $t \geq 0$ with the initial condition $(U(0), V(0)) = (M, 0)$ for some constant $M > 0$. Then it is easy to see that the solution is given by $(U(t), V(t)) := (\sqrt{M^2-2t}, 0)$. Note that the quenching time of this ODE system is $T = T(M) := M^2/2$.

Next, in order to estimate $u(x,t)$ from below, we consider the following Cauchy problem:

$$(3.3.2)\qquad \begin{cases} \overline{u}_t = \overline{u}_{xx} - 1/\overline{u}, & x \in \mathbb{R}, \quad t \in [0, \overline{T}), \\ \overline{u}(x,0) = \overline{u}_0(x), & x \in \mathbb{R}. \end{cases}$$

where $[0, \overline{T})$ is the maximal existence interval of $\overline{u}$. Also, we consider the following ODE problem corresponding to the problem (3.3.2):

$$(3.3.3)\qquad \overline{U}_t = -1/\overline{U}, \quad t \in [0, T), \quad \overline{U}(0) = M.$$

Note that the solution of (3.3.3) is given by $\overline{U}(t) = \sqrt{M^2-2t}$ with $T = T(M) := M^2/2$.

Motivated by an idea from [28], we have the following lemma. We also refer the reader to [41] for the Fujita equation, [49] for a quasilinear parabolic equation, and [51] for a cooperative parabolic system.

Lemma 3.3.1 *Let $\overline{U}$ be the solution of* (3.3.3) *and let $\overline{u}$ be the solution of* (3.3.2) *defined on $\mathbb{R} \times [0, \overline{T})$. Suppose that there exist $t_0 \in [0, \widehat{T})$, $r_0 \in (0, \infty)$ and a constant $\theta > 1$ such that*

$$\overline{u}(x,t) \geq \theta \overline{U}(t), \quad for \quad |x| \leq r_0, \quad t_0 \leq t < \widehat{T}.$$

where $\widehat{T} := \min\{T, \overline{T}\}$. Then $\overline{u}$ has a positive lower bound in $\{|x| \leq r_0/2\} \times [t_0, \widehat{T})$.

Proof. We shall construct a suitable subsolution of (3.3.2) as follows

$$w(x,t) := \widehat{\theta}\sqrt{M^2 - 2t + h(x)},$$

where $\widehat{\theta} \in (1, \theta)$ and

$$h(x) := \varepsilon cos^2(\frac{\pi x}{2r_0})$$

with small $\varepsilon > 0$ to be specified later.

By a simple computation, we obtain that

$$\begin{aligned} w_t - w_{xx} + \frac{1}{w} &= \frac{\widehat{\theta}^2}{w}\Big\{-1 - \frac{1}{2}h'' + \frac{h'^2}{4(M^2 - 2t + h)} + \Big(\frac{1}{\widehat{\theta}}\Big)^2\Big\} \\ &\leq \frac{\widehat{\theta}^2}{w}\Big\{-1 - \frac{1}{2}h'' + \frac{h'^2}{4h} + \Big(\frac{1}{\widehat{\theta}}\Big)^2\Big\}. \end{aligned}$$

By the choice of h, we obtain that both $|h''|$ and h'^2/h are of order ε for $|x| \leq r_0$. Hence, if we choose $\varepsilon > 0$ sufficiently small such that

$$\varepsilon \leq (M^2 - 2t_0)\Big[\Big(\frac{\theta}{\widehat{\theta}}\Big)^2 - 1\Big], \quad -1 - \frac{1}{2}h'' + \frac{h'^2}{4h} + \Big(\frac{1}{\widehat{\theta}}\Big)^2 \leq 0,$$

then we have

$$\begin{cases} w_t \leq w_{xx} - 1/w, & |x| \leq r_0, \quad t_0 \leq t < \widehat{T}, \\ w(x,t_0) \leq \overline{u}(x,t_0), & |x| \leq r_0, \\ w(x,t) \leq \overline{u}(x,t), & |x| = r_0, \quad t_0 \leq t < \widehat{T}, \end{cases} \tag{3.3.4}$$

where we have used the fact $\widehat{\theta} \in (1, \theta)$.

Then it follows from (3.3.4) and the comparison principle that $w(x,t) \leq \overline{u}(x,t)$ for $|x| \leq r_0$ and $t_0 \leq t < \widehat{T}$. Therefore, we have

$$\overline{u}(x,t) \geq \widehat{\theta}\sqrt{M^2 - 2t + h(r_0/2)} = \widehat{\theta}\sqrt{M^2 - 2t + \varepsilon/2} \geq \widehat{\theta}\sqrt{\varepsilon/2} > 0$$

for any $|x| \leq r_0/2$ and $t_0 \leq t < \widehat{T}$. The lemma follows. $\square$

Hereafter, we assume

$$\overline{u}_0 \in C^1(\mathbb{R}), \quad \overline{u}_0 \geq M, \quad \overline{u}_0 \not\equiv M, \tag{3.3.5}$$

$$\lim_{|x|\to\infty} \overline{u}_0(x) = M. \tag{3.3.6}$$

Note that by (3.3.2), (3.3.3), and (3.3.5) we have $\overline{U} \leq \overline{u}$. Therefore, we obtain $\overline{T} \geq T$ and so $\widehat{T} = T$.

The following lemma shows that quenching can occur only at space infinity.

Lemma 3.3.2 *Let $\overline{u}$ be a solution of* (3.3.2) *satisfying* (3.3.5) *and* (3.3.6) *for some constant $M > 0$. Then $\overline{u}$ has a positive lower bound in $\Omega \times [0, T)$ for any compact set $\Omega \subset \mathbb{R}$.*

Proof. In view of Lemma 3.3.1, since $\widehat{T} = T$, it suffices to show that, for any given $R > 0$ there exist $t_0 \in [0, T)$ and $\theta > 1$ such that

$$\overline{u}(x,t) \geq \theta\sqrt{M^2 - 2t}, \quad |x| \leq 2R, \quad t_0 \leq t < T. \tag{3.3.7}$$

For this purpose, we let $\gamma(x,t) := \overline{u}(x,t)/\overline{U}(t)$. Then the function $\gamma = \gamma(x,t)$ satisfies

$$\gamma_t = \gamma_{xx} + \frac{1}{\overline{U}^2}\Big(-\frac{1}{\gamma} + \gamma \Big) \geq \gamma_{xx},$$

since $\gamma \geq 1$. Moreover, by (3.3.5) and (3.3.6) we obtain

$$\gamma(\cdot, 0) = \frac{\overline{u}_0}{M} \geq 1, \quad \gamma(\cdot, 0) \not\equiv 1.$$

From the strong maximum principle, we have that $\gamma(x,t) > 1$ for all $x \in \mathbb{R}$ and $t > 0$. Therefore, for any given $R > 0$, there exist $\theta > 1$ and $t_0 \in (0, T)$ such that

$$\gamma(x,t) \geq \theta, \quad |x| \leq 2R, \quad t_0 \leq t < T.$$

This gives (3.3.7). Therefore we complete the proof. □

To investigate the behavior of the solution of (3.1.2) at space infinity, we recall the following useful property (cf. [28]). We also refer the reader to [51] for the blow-up problem for a cooperative parabolic system.

Theorem 3.3.1 *Let* $\mathbf{u}$ *and* $\widehat{\mathbf{u}}$ *be solutions of*

$$\begin{cases} \mathbf{u}_t = D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}), & x \in \mathbb{R}, \quad t > 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.3.8}$$

where $\mathbf{u}(x,t) = (u(x,t), v(x,t)) \in \mathbb{R}^2$, $\mathbf{f} = (f_1, f_2)$ *is a smooth mapping from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, $D = diag(1,1)$, *with initial data* $\mathbf{u}_0$, $\widehat{\mathbf{u}}_0 \in (L^\infty(\mathbb{R}) \cap C(\mathbb{R}))^2$, *respectively. Suppose that there exist sequences* $\{r_n\}_{n=1}^\infty \subset (0,\infty)$ *and* $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ *with* $r_n \to \infty$ *as* $n \to \infty$ *such that*

$$\lim_{n\to\infty} ||\mathbf{u}_0 - \widehat{\mathbf{u}}_0||_{L^\infty(B_{2r_n}(a_n))} = 0.$$

Then

$$\lim_{n\to\infty} ||\mathbf{u}(\cdot,t) - \widehat{\mathbf{u}}(\cdot,t)||_{L^\infty(B_{r_n}(a_n))} = 0.$$

for any $t \in (0, \widetilde{T})$, *where* $\widetilde{T} = \min\{T(\mathbf{u}_0), T(\widehat{\mathbf{u}}_0)\}$.

Notice that the following corollary is applicable to our system (3.1.2). Since its proof is exactly the same as the one given in [28, Corollary 4.2], we omit is here.

Corollary 3.3.3 *If some solutions of*

$$\mathbf{U}_t = \mathbf{f}(\mathbf{U}) \tag{3.3.9}$$

quenches in a finite time, then there exists a spatially inhomogeneous solutions of (3.3.8) *which quenches in a finite time.*

In the following, we shall focus on the Cauchy problem for (3.1.2) with initial data satisfying (3.1.7) and (3.1.8).

Lemma 3.3.4 *Let* $\overline{u}$ *be a solution of* (3.3.2) *satisfying* (3.3.5) *and* (3.3.6) *for some constant* $M > 0$. *Then* $\overline{u}$ *quenches at the finite time* $T = M^2/2$.

Proof. First, we set $\mathbf{u}(x,t) = \overline{u}(x,t)$, $\widehat{\mathbf{u}}(x,t) = \overline{U}(t)$, $|a_n| = 4n$, and $r_n = n$. By (3.3.6), we have

$$\lim_{n\to\infty} ||\mathbf{u}_0 - \widehat{\mathbf{u}}_0||_{L^\infty(B_{2r_n}(a_n))} = 0. \tag{3.3.10}$$

Notice that $\mathbf{u}$ and $\widehat{\mathbf{u}}$ are solutions of (3.3.2) and (3.3.3) with initial data $\mathbf{u}_0$ and $\widehat{\mathbf{u}}_0$, respectively. Let $\mathbf{f}(\mathbf{u}) = -1/\overline{u}$, $\mathbf{f}(\widehat{\mathbf{u}}) = -1/\overline{U}$. Applying Theorem 3.3.1 to (3.3.2) and (3.3.3), we obtain

$$\lim_{|x|\to\infty} \overline{u}(x,t) = \overline{U}(t), \quad \forall t \in [0,T).$$

On the other hand, by (3.3.2), (3.3.3), (3.3.5), and the comparison principle, we have $\overline{u}(x,t) \geq \overline{U}(t)$ for all $x \in \mathbb{R}$ and $t > 0$. Combining the above two facts, we have the quenching time $\overline{T} = T = M^2/2$. ∎

Now we prove the Theorem 3.1.2 by using Theorems 3.1.1 and 3.3.1.

Proof of Theorem 3.1.2. First, we have the local existence of (u, v) for $t \in [0, \sigma]$ for some $\sigma > 0$. Let $\mathbf{u}(x, t) = (u(x, t), v(x, t))$, $\widehat{\mathbf{u}}(x, t) = (U(t), V(t))$ and

$$\mathbf{f}(\mathbf{u}) = \Big(\frac{-u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\Big),$$

where (u, v) and (U, V) are solutions of (3.1.2) and (3.1.3), respectively. By applying Theorem 3.3.1 to (3.1.2) and (3.1.3) with $|a_n| = 4n$ and $r_n = n$, we have

$$\lim_{|x|\to\infty} u(x, t) = U(t), \quad \text{and} \quad \lim_{|x|\to\infty} v(x, t) = V(t), \quad \forall t \in [0, \sigma]. \tag{3.3.11}$$

Also, it follows from (3.1.4) and (3.1.8) with $N > 0$ that

$$\lim_{|x|\to\infty} u(x, t)v(x, t) = U(t)V(t) = U(0)V(0) = \lim_{|x|\to\infty} u_0(x)v_0(x) = MN > 0.$$

Hence the assumption (3.1.6) is satisfied for all x with $|x| \geq R$ at $t = \sigma$ for some constants R sufficient large and $K > 0$.

Moreover, by the strong maximum principle, we obtain $v > 0$ in $\mathbb{R} \times [0, \sigma]$. It implies that the assumption (3.1.6) holds for all x with $|x| \leq R$ at $t = \sigma$ with the positive constant K (taking a smaller one if necessary). Therefore, by applying Theorem 3.1.1 to the Cauchy problem (3.1.2) starting at $t = \sigma$, we obtain that the solution (u, v) exists globally in time and (u, v) converges to $(0, \infty)$ as $t \to \infty$. This completes the proof of Theorem 3.1.2. □

Finally, we give a proof of Theorem 3.1.3.

Proof of Theorem 3.1.3. We choose $\overline{u}_0 = u_0$. Then, by the comparison principle, we obtain

$$u(x, t) \geq \overline{u}(x, t), \quad x \in \mathbb{R}, \quad \text{for } t > 0 \ \text{ such that } u \text{ and } \overline{u} \text{ exist.} \tag{3.3.12}$$

Suppose that the solution (u, v) quenches at time T^*. By (3.3.12), we have $T^* \geq T$. On the other hand, by Lemmas 3.3.2 and 3.3.4, the solution $\overline{u}$ quenches at finite time $T = M^2/2$ only at space infinity. Thus the inequality (3.3.12) implies that

$$u \geq \overline{u} > 0 \ \text{ in } \ \mathbb{R} \times [0, T). \tag{3.3.13}$$

Moreover, we set $\mathbf{u}(x, t) = (u(x, t), v(x, t))$, $\widehat{\mathbf{u}}(x, t) = (U(t), V(t)) = (U(t), 0)$ and

$$\mathbf{f}(\mathbf{u}) = \Big(\frac{-u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\Big),$$

where (u, v) and (U, V) are solutions of (3.1.2) and (3.3.1), respectively. Applying Theorem 3.3.1 to (3.1.2) and (3.3.1) with $|a_n| = 4n$ and $r_n = n$ again, we have

$$\lim_{|x|\to\infty} u(x, t) = \overline{U}(t), \quad \lim_{|x|\to\infty} v(x, t) = V(t) = 0, \quad \forall t \in [0, T). \tag{3.3.14}$$

Hence we obtain $T^* = T$. From Lemma 3.3.2, $\overline{u}$ quenches only at space infinity. Combining this with (3.3.13), we conclude that the quenching of the solution (u, v) occurs only at space infinity. This proves the theorem. $\square$

Chapter 4

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