3 Gröbner basis for minors of a fixed size of a generic matrix

Theorem 3.1 ([2], [6]) *Let $X = (x_{ij})$ be a generic $m \times n$ matrix over a field K and let $R = K[X]$. Let G_p be the set of all p-minors of X. Let I be the ideal of R generated by G_p; then G_p is a Gröbner basis for I with respect to the lexicographic term order τ_X defined in Section 2.*

Theorem 3.2 ([4]) *Let $Y = (y_{ij})$ be an $n \times n$ symmetric matrix of indeterminates and let $R = K[Y]$ be a polynomial ring over a field K. Let $\bar{G}_p$ be the set of all p-minors of Y. Let I be the ideal of R generated by $\bar{G}_p$; then $\bar{G}_p$ is a Gröbner basis for I with respect to the lexicographic term order τ_Y defined in Section 2.*

Now we give another proofs to Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1.

Let $A, B \in G_p$ and $A = [i_1 \cdots i_p | j_1 \cdots j_p]$, $B = [k_1 \cdots k_p | l_1 \cdots l_p]$.

Suppose that $|c(A^*) \cap c(B^*)| = u$.

If $u = 0$, then $S(A, B)$ can be reduced by Lemma 2.7.

If $u = p$, then $S(A, B) = 0$, so $S(A, B)$ can be reduced by Lemma 2.4.

From above we may assume that $1 \leq u \leq p - 1$ and assume that

$$c(A^*) \cap c(B^*) = \{x_{g_1,h_1}, \cdots, x_{g_u,h_u}\}.$$

After rearranging the p rows and p columns, respectively, we may get

$$A = [i_1 \cdots i_{p-u} g_1 \cdots g_u | j_1 \cdots j_{p-u} h_1 \cdots h_u]$$

and

$$B = [g_1 \cdots g_u k_1 \cdots k_{p-u} | h_1 \cdots h_u l_1 \cdots l_{p-u}],$$

where $i_1 < \cdots < i_{p-u}$, $j_1 < \cdots < j_{p-u}$, $g_1 < \cdots < g_u$ and $h_1 < \cdots < h_u$.

Let

$$S = \{i_1, \cdots, i_{p-u}, g_1, \cdots, g_u, k_1, \cdots, k_{p-u}\}$$

and

$$T = \{j_1, \cdots, j_{p-u}, h_1, \cdots, h_u, l_1, \cdots, l_{p-u}\}.$$

Note that either $i_1 \neq k_1$ or $j_1 \neq l_1$.

(1) Suppose that $i_1 < k_1$ or ($i_1 = k_1$ and $l_1 < j_1$).

Let E be the determinant of the following $(2p-u) \times (2p-u)$ matrix

$$M = \begin{pmatrix}
x_{i_1,j_1} & \cdots & x_{i_1,j_{p-u}} & x_{i_1,h_1} & \cdots & x_{i_1,h_u} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
x_{i_{p-u},j_1} & \cdots & x_{i_{p-u},j_{p-u}} & x_{i_{p-u},h_1} & \cdots & x_{i_{p-u},h_u} & 0 & \cdots & 0 \\
x_{g_1,j_1} & \cdots & x_{g_1,j_{p-u}} & x_{g_1,h_1} & \cdots & x_{g_1,h_u} & x_{g_1,l_1} & \cdots & x_{g_1,l_{p-u}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
x_{g_u,j_1} & \cdots & x_{g_u,j_{p-u}} & x_{g_u,h_1} & \cdots & x_{g_u,h_u} & x_{g_u,l_1} & \cdots & x_{g_u,l_{p-u}} \\
x_{k_1,j_1} & \cdots & x_{k_1,j_{p-u}} & x_{k_1,h_1} & \cdots & x_{k_1,h_u} & x_{k_1,l_1} & \cdots & x_{k_1,l_{p-u}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
x_{k_{p-u},j_1} & \cdots & x_{k_{p-u},j_{p-u}} & x_{k_{p-u},h_1} & \cdots & x_{k_{p-u},h_u} & x_{k_{p-u},l_1} & \cdots & x_{k_{p-u},l_{p-u}}
\end{pmatrix}.$$

Then

$$E = \sum_{e_i \in T-\{l_1,\cdots,l_{p-u}\}} \pm[i_1 \cdots i_{p-u}|\, e_1 \cdots e_{p-u}][S-\{i_1,\cdots,i_{p-u}\}|\, T-\{e_1,\cdots,e_{p-u}\}]$$

$$= \sum_{e'_i \in S-\{i_1,\cdots,i_{p-u}\}} \pm[e'_1 \cdots e'_{p-u}|\, l_1 \cdots l_{p-u}][S-\{e'_1,\cdots,e'_{p-u}\}|\, T-\{l_1,\cdots,l_{p-u}\}].$$

Let

$$\begin{aligned}
A_{\underline{e}} &= [S-\{i_1,\cdots,i_{p-u}\}|\ T-\{e_1,\cdots,e_{p-u}\}],\\
B_{\underline{e'}} &= [S-\{e'_1,\cdots,e'_{p-u}\}|\ T-\{l_1,\cdots,l_{p-u}\}],\\
C_{\underline{e}} &= [i_1 \cdots i_{p-u}|\ e_1 \cdots e_{p-u}],\\
D_{\underline{e'}} &= [e'_1 \cdots e'_{p-u}|\ l_1 \cdots l_{p-u}],\\
A_0 &= [i_1 \cdots i_{p-u}|\ j_1 \cdots j_{p-u}],\\
B_0 &= [k_1 \cdots k_{p-u}|\ l_1 \cdots l_{p-u}],
\end{aligned}$$

where $\underline{e} = \{e_1,\cdots,e_{p-u}\}$ and $\underline{e'} = \{e'_1,\cdots,e'_{p-u}\}$.

Then

$$B_0A - A_0B = \sum_{\substack{e_i \in T-\{l_1,\cdots,l_{p-u}\}\\ \underline{e} \neq \{j_1,\cdots,j_{p-u}\}}} \pm C_{\underline{e}}A_{\underline{e}} + \sum_{\substack{e'_i \in S-\{i_1,\cdots,i_{p-u}\}\\ \underline{e'} \neq \{k_1,\cdots,k_{p-u}\}}} \pm D_{\underline{e'}}B_{\underline{e'}}\ ,$$

so that

$$\begin{aligned}
S(A,B) &= B_0^*A - A_0^*B\\
&= \sum_{\substack{e_i \in T-\{l_1,\cdots,l_{p-u}\}\\ \underline{e} \neq \{j_1,\cdots,j_{p-u}\}}} \pm C_{\underline{e}}A_{\underline{e}} + \sum_{\substack{e'_i \in S-\{i_1,\cdots,i_{p-u}\}\\ \underline{e'} \neq \{k_1,\cdots,k_{p-u}\}}} \pm D_{\underline{e'}}B_{\underline{e'}}\\
&\quad +(A_0 - A_0^*)B - (B_0 - B_0^*)A.
\end{aligned}$$

Let $L = LCM(A^*, B^*)$.

Therefore, by Lemma 2.4, to show that $S(A,B)$ can be reduced, it is sufficient to show that $C^*_{\underline{e}}A^*_{\underline{e}} < L$, $D^*_{\underline{e'}}B^*_{\underline{e'}} < L$, $(A_0 - A^*_0)^*B^* < L$ and $(B_0 - B^*_0)^*A^* < L$. To show that $C^*_{\underline{e}}A^*_{\underline{e}} < L$ and $D^*_{\underline{e'}}B^*_{\underline{e'}} < L$, we consider the submatrices N_1 and N_2 of M, where

$$N_1 = \begin{pmatrix} x_{i_1,j_1} & \cdots & x_{i_1,j_{p-u}} & x_{i_1,h_1} & \cdots & x_{i_1,h_u} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{i_{p-u},j_1} & \cdots & x_{i_{p-u},j_{p-u}} & x_{i_{p-u},h_1} & \cdots & x_{i_{p-u},h_u} \\ x_{g_1,j_1} & \cdots & x_{g_1,j_{p-u}} & x_{g_1,h_1} & \cdots & x_{g_1,h_u} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{g_u,j_1} & \cdots & x_{g_u,j_{p-u}} & x_{g_u,h_1} & \cdots & x_{g_u,h_u} \end{pmatrix}$$

and

$$N_2 = \begin{pmatrix} x_{g_1,h_1} & \cdots & x_{g_1,h_u} & x_{g_1,l_1} & \cdots & x_{g_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{g_u,h_1} & \cdots & x_{g_u,h_u} & x_{g_u,l_1} & \cdots & x_{g_u,l_{p-u}} \\ x_{k_1,h_1} & \cdots & x_{k_1,h_u} & x_{k_1,l_1} & \cdots & x_{k_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k_{p-u},h_1} & \cdots & x_{k_{p-u},h_u} & x_{k_{p-u},l_1} & \cdots & x_{k_{p-u},l_{p-u}} \end{pmatrix}.$$

Since for every entry x in N_1 (resp. N_2) which is not in diagonal, there is a diagonal entry x' in N_1(resp. N_2) such that

$$x <_{(1,3)} x',$$

and for every $s = 1, \cdots, p-u$, $t = 1, \cdots, p-u$, we have

$$x_{k_s,j_t} \leq_{(1,3,4)} x_{k_1,j_1} <_{(1,3)} \max\{x_{i_1,j_1}, x_{k_1,l_1}\}.$$

Hence, for every entry x in the matrix M which is not in diagonal, there is a diagonal entry x' in M such that

$$x <_{(1,3,4)} x'.$$

For all $\underline{e} \neq \{j_1, \cdots, j_{p-u}\}$ and $\underline{e}' \neq \{k_1, \cdots, k_{p-u}\}$, it is obvious that

$$c(C^*_{\underline{e}} A^*_{\underline{e}}) \neq c(L) \quad \text{and} \quad c(D^*_{\underline{e'}} B^*_{\underline{e'}}) \neq c(L),$$

then we can conclude that

$$C^*_{\underline{e}} A^*_{\underline{e}} < L \quad \text{and} \quad D^*_{\underline{e'}} B^*_{\underline{e'}} < L.$$

Furthermore, if $m = n$, we have

$$C^*_{\underline{e}} A^*_{\underline{e}} <_\pi L \quad \text{and} \quad D^*_{\underline{e'}} B^*_{\underline{e'}} <_\pi L.$$

Finally, it is clear that

$$(A_0 - A_0^*)^* B^* < A_0^* B^* = L$$

and

$$(B_0 - B_0^*)^* A^* < B_0^* A^* = L.$$

Also, if $m = n$, we have

$$(A_0 - A_0^*)^* B^* <_\pi A_0^* B^* = L$$

and

$$(B_0 - B_0^*)^* A^* <_\pi B_0^* A^* = L.$$

Therefore $S(A, B)$ can be reduced with respect to G_p.

(2) Suppose that $i_1 < k_1$ or $(i_1 = k_1$ and $l_1 < j_1)$.

Let E be the determinant of the following $(2p-u)\times(2p-u)$ matrix

$$M' = \begin{pmatrix} x_{i_1,j_1} & \cdots & x_{i_1,j_{p-u}} & x_{i_1,h_1} & \cdots & x_{i_1,h_u} & x_{i_1,l_1} & \cdots & x_{i_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{i_{p-u},j_1} & \cdots & x_{i_{p-u},j_{p-u}} & x_{i_{p-u},h_1} & \cdots & x_{i_{p-u},h_u} & x_{i_{p-u},l_1} & \cdots & x_{i_{p-u},l_{p-u}} \\ x_{g_1,j_1} & \cdots & x_{g_1,j_{p-u}} & x_{g_1,h_1} & \cdots & x_{g_1,h_u} & x_{g_1,l_1} & \cdots & x_{g_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{g_u,j_1} & \cdots & x_{g_u,j_{p-u}} & x_{g_u,h_1} & \cdots & x_{g_u,h_u} & x_{g_u,l_1} & \cdots & x_{g_u,l_{p-u}} \\ 0 & \cdots & 0 & x_{k_1,h_1} & \cdots & x_{k_1,h_u} & x_{k_1,l_1} & \cdots & x_{k_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{k_{p-u},h_1} & \cdots & x_{k_{p-u},h_u} & x_{k_{p-u},l_1} & \cdots & x_{k_{p-u},l_{p-u}} \end{pmatrix}.$$

Then

$$E = \sum_{e_i\in T-\{j_1,\cdots,j_{p-u}\}} \pm[k_1\cdots k_{p-u}|\, e_1\cdots e_{p-u}][S-\{k_1,\cdots,k_{p-u}\}|\, T-\{e_1,\cdots,e_{p-u}\}]$$

$$= \sum_{e'_i\in S-\{k_1,\cdots k_{p-u}\}} \pm[e'_1\cdots e'_{p-u}|\, j_1\cdots j_{p-u}][S-\{e'_1,\cdots,e'_{p-u}\}|\, T-\{j_1,\cdots,j_{p-u}\}].$$

Let

$$\begin{aligned} A_{\underline{e}} &= [S-\{k_1,\cdots,k_{p-u}\}|\, T-\{e_1,\cdots,e_{p-u}\}], \\ B_{\underline{e'}} &= [S-\{e'_1,\cdots,e'_{p-u}\}|\, T-\{j_1,\cdots,j_{p-u}\}], \\ C_{\underline{e}} &= [k_1\cdots k_{p-u}|\, e_1\cdots e_{p-u}], \\ D_{\underline{e'}} &= [e'_1\cdots e'_{p-u}|\, j_1\cdots j_{p-u}], \\ A_0 &= [i_1\cdots i_{p-u}|\, j_1\cdots j_{p-u}], \\ B_0 &= [k_1\cdots k_{p-u}|\, l_1\cdots l_{p-u}], \end{aligned}$$

where $\underline{e} = \{e_1, \cdots, e_{p-u}\}$ and $\underline{e'} = \{e'_1, \cdots, e'_{p-u}\}$.

Then

$$B_0 A - A_0 B = \sum_{\substack{e_i \in T - \{j_1, \cdots, j_{p-u}\} \\ \underline{e} \neq \{l_1, \cdots, l_{p-u}\}}} \pm C_{\underline{e}} A_{\underline{e}} + \sum_{\substack{e'_i \in S - \{k_1, \cdots, k_{p-u}\} \\ \underline{e'} \neq \{i_1, \cdots, i_{p-u}\}}} \pm D_{\underline{e'}} B_{\underline{e'}} \ ,$$

so that

$$\begin{aligned} S(A,B) &= B_0^* A - A_0^* B \\ &= \sum_{\substack{e_i \in T - \{j_1, \cdots, j_{p-u}\} \\ \underline{e} \neq \{l_1, \cdots, l_{p-u}\}}} \pm C_{\underline{e}} A_{\underline{e}} + \sum_{\substack{e'_i \in S - \{k_1, \cdots, k_{p-u}\} \\ \underline{e'} \neq \{i_1, \cdots, i_{p-u}\}}} \pm D_{\underline{e'}} B_{\underline{e'}} \\ &\qquad + (A_0 - A_0^*) B - (B_0 - B_0^*) A. \end{aligned}$$

Let $L = LCM(A^*, B^*)$.

Therefore, by Lemma 2.4, to show that $S(A,B)$ can be reduced, it is sufficient to show that $C_{\underline{e}}^* A_{\underline{e}}^* < L$, $D_{\underline{e'}}^* B_{\underline{e'}}^* < L$, $(A_0 - A_0^*)^* B^* < L$ and $(B_0 - B_0^*)^* A^* < L$. Similar to the proof of (1), for all $\underline{e} \neq \{j_1, \cdots, j_{p-u}\}$ and $\underline{e'} \neq \{k_1, \cdots, k_{p-u}\}$, we can conclude that

$$C_{\underline{e}}^* A_{\underline{e}}^* < L \quad \text{and} \quad D_{\underline{e'}}^* B_{\underline{e'}}^* < L.$$

Furthermore, if $m = n$, we have

$$C_{\underline{e}}^* A_{\underline{e}}^* <_\pi L \quad \text{and} \quad D_{\underline{e'}}^* B_{\underline{e'}}^* <_\pi L.$$

Finally, it is clear that

$$(A_0 - A_0^*)^* B^* < A_0^* B^* = L$$

and

$$(B_0 - B_0^*)^* A^* < B_0^* A^* = L.$$

Also, if $m = n$, we have

$$(A_0 - A_0^*)^* B^* <_\pi A_0^* B^* = L$$

and

$$(B_0 - B_0^*)^* A^* <_\pi B_0^* A^* = L.$$

Therefore $S(A, B)$ can be reduced with respect to G_p.

From the discussion of (1) and (2), G_p is a Gröbner basis by Lemma 2.4. $\square$

From the proof of Theorem 3.1, Theorem 3.2 is immediate.

Proof of Theorem 3.2.

Let $A, B \in \bar{G}_p$. Let $X' = \{x_{ij} | i \leq j\}$ be a subset of $X_{n,n}$. Then by Remark 2.6(1), we can find $\mathcal{A}, \mathcal{B} \in G_p$ such that $\pi(\mathcal{A}) = A$, $\pi(\mathcal{B}) = B$ and $c(\mathcal{A}^*), c(\mathcal{B}^*) \subseteq X'$. Therefore $\pi(S(\mathcal{A}, \mathcal{B})) = S(A, B)$. Now, by Theorem 3.1, we see that there are $(p-u)$-minors f_i and p-minors C_i such that $S(\mathcal{A}, \mathcal{B}) = \sum f_i C_i$ and $LCM(\mathcal{A}^*, \mathcal{B}^*) >_\pi f_i^* C_i^*$ for every i. It follows that $S(A, B) = \sum \pi(f_i)\pi(C_i)$ and $LCM(A^*, B^*) > \pi(f_i)\pi(C_i)$. $\square$