Distance Between Two Convex Sets

by

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Let X be a Banach space satisfying the condition:

(E) for any convex set C in X, every sequence $\{ x_n \}$ in C with $\lim_{n \to \infty} \| x_n \| = \inf \{ \| x \|, x \in C \}$ is a Cauchy sequence.

For example, every uniformly convex Banach space satisfies the condition E, but not the converse (See K. Fan and I. Glicksbery [2], [3]).

It was shown by K. Fan and I. Glicksberg in [3] that for each nonempty closed convex subset K of X the proximity map P of X onto K which to each point x of X associates the point Px in K closest to x is continuous. In the present article we consider two such subsets K_1, K_2 and first show (Theorem 1) that the boundedness of either of them is sufficient for the existence of a minimum distance between them. Then, assuming X* to be strictly convex and letting $Q = P_1 P_2$ denote the composition of the proximity mappings P_1 and P_2 of X onto K_1 and K_2 respectively, we prove (Theorems 2 and 3) that the following three conditions are equivalent to one another:

(a) there exists a minimum distance between K_1 and K_2,

(b) Q has a fixed point,

(c) for some $x \in X$, the sequence $\{ Q^n x \}$ has a cluster point.

Theorems 2 and 3 also show that any cluster point of a sequence $\{ Q^n x \}$, $x \in X$, – and thus, in particular, any fixed point of Q – is a point of K_1 closest to K_2. For the particular case that X is a Hilbert space, similar results were obtained by W. Cheney and A. A. Goldstein [1].

We first need the following lemma.

Lemma 1. If K is a nonempty closed convex set in X, then for every $x \in X$, there is a unique element Px in K which is closest to x. Furthermore, the map P is continuous on X.

This lemma was proved by K. Fan and I. Glieksberg (Theorem 8, [3]).

Theorem 1. Let K_1 and K_2 be two closed convex subsets of X, and let K_1 be bounded. Then there is a point y in K_1 such that $\| y - P_2 y \| = D(K_1, K_2)$;

where P_2 is the proximity map of X onto K_2 and $D(K_1, K_2) = \inf \{ \|x-y\|; x \in K_1, y \in K_2 \}$.

Proof. For simplicity, we set $r = D(K_1, K_2)$. Then for each n, there is an $x_n \in K_1$ such that $\|x_n - P_2 x_n\| \leqslant r + \frac{1}{n}$. Since K_1 is weakly compact, there exists a weakly convergent subsequence we then have

$\{ x_{n_i} \}$ of $\{ x_n \}$. For each $j \geq 0$, we let $z_j = \sum_{i \geq j} a_i x_{n_i}$ with $\sum_{i \geq j} a_i = 1$ and

$a_i \geq 0$,

$$\| z_j - P_2 z_j \| \leqslant \| z_j - \sum_{i\ j} a_i P_2 x_{n_i} \| \leqslant \sum_{i \geq j} a_i \| x_{n_i} - P x_{n_i} \| \leqslant r + \frac{1}{n_j}$$

If the weak limit of $\{ x_{n_i} \}$ is denoted by y, then for any $j > 0$, Mazur's Theorem states that y is the strong limit of some sequence of linear combinations of $\{ x_{n_i}; i \geq j \}$. Thus it follows from the above inequality, and from the continuity of P_2, that $\| y - P_2 y \| \leqslant r + \frac{1}{n_j}$ for all $j > 0$. Thus $\| y - P_2 y \| = r = D(K_1, K_2)$.

Let X^* be the dual space of X and for $x \in X$, $x^* \in X^*$ let $x^*(x)$ be denoted by $[x, x^*]$. If X^* is strictly convex and if F denotes the duality mapping of X into X^* namely, for each $x \in X$ we have $Fx \in X^*$ with $\| Fx \| = \| x \|$ and $[x, Fx] = \| x \|^2$, it was shown by T. Kato [4] that for every $x, y \in X$ if $\| x + \lambda y \| \geqq \| x \|$ for all $\lambda \geqslant 0$, then $[y, Fx] \geqslant 0$. Using Kato's result, we have the following lemma.

Lemma 2. Let K be a closed convex subset of X, and let X^* be strictly convex. Then $[Px-y, F(x-Px)] \geqslant 0$ for all $x \in X$, $y \in K$.

Proof. By the definition of P, we have

$$\| x - Px \| \leqslant \| x - ((1-t)Px + ty) \| = \| (x-Px) + t(Px-y) \|$$

for all $x \in X$, $y \in K$ and $0 \leqslant t \leqslant 1$, and then $\| x-Px \| \leqslant \| (x-Px) + t(Px-y) \|$ for all $t \geqslant 0$, so that by Kato's result we have $[Px-y, F(x-Px)] \geqslant 0$.

Theorem 2. Let K_1 and K_2 be two closed convex subsets in X and let X^* be strictyly convex. Then for $x_1 \in K_1$ and $x_2 \in K_2$, we have $\| x_1 - x_2 \| = D(K_1, K_2)$ if and only if $x_1 = P_1 x_2$ and $x_2 = P_2 x_1$.

Proof. Assume $x_1 = P_1 x_2$ and $x_2 = P_2 x_1$. Then for $u_1 \in K_1$ and $u_2 \in K_2$, we obtain from Lemma 2 that

$$[u_1 - x_1, F(x_1 - x_2)] \geqslant 0,$$

and
$$[u_2 - x_2, F(x_2 - x_1)] \geqslant 0.$$

Hence $[u_1 - u_2, F(x_1 - x_2)] \geqslant \| x_1 - x_2 \|^2$ and thus $\| u_1 - u_2 \| \geqslant \| x_1 - x_2 \|$.

Conversely, assume $\| x_1 - x_2 \| = D(K_1, K_2)$, $x_1 \in K_1$, and $x_2 \in K_2$. Then from $\| x_1 - x_2 \| \geq \| x_1 - P_2 x_1 \| \geq D(K_1, K_2)$ we have $\| x_1 - x_2 \| = \| x_1 - P_2 x_1 \|$, and thus $x_2 = P_2 x_1$. Similarly, we obtain $x_1 = P_1 x_2$.

Remark. It follows from Theorem 2 that x_1 is a point in K_1 closest to K_2 if and only if x_1 is a fixed point of $P_1 P_2$.

Theorem 3. Let K_1 and K_2 be two closed convex subsets of x and let $x \in K_1$. If x_1 is a cluster point of $\{ Q^n x \}$, then $\| x_1 - P_2 x_1 \| = D(K_1, K_2)$.

Proof. By the definition of P_1 and P_2, we have

$$\| x - P_2 x \| \geqslant \| P_2 x - Qx \| \geqslant \| Qx - P_2 Qx \| \geqslant \ldots$$
$$\geqslant \| Q^n x - P_2 Q^n x \| \geq \| P_2 Q^n x - Q^{n+1} x \| \geq \| Q^{n+1} x - P_2 Q^{n+1} x \| \geq \ldots.$$

Thus $\lim_{n \to \infty} \| Q^n x - P_2 Q^n x \| = \lim_{n \to \infty} \| Q^{n+1} x - P_2 Q^n x \| = r$, say. Then,

since x_1 is a cluster point of $Q^n x$ and P_1, P_2 are continuous, it follows that

$$\| x_1 - P_2 x_1 \| = \| Qx_1 - P_2 x_1 \| = \| P_1 P_2 x_1 - P_2 x_1 \|$$

and hence $x_1 = P_1 P_2 x_1 = Qx_1$. Thus by the Remark of Theorem 2,
we have

$$\| x_1 - P_2 x_1 \| = D(K_1, K_2).$$

Remark. 1. It follows from the Remark of Theorems 2 and 3 that the conditions a), b) and c) are mutually equivalent.

2. Let K_1 and K_2 be two existence sets in a Banach space X and define

$$P_i x = \{ y \in k_i ; \| x - y \| = \text{Min} \{ \| x - z \|, z \in K_i \} \}$$

for every $x \in X$. Then we have the following result which is a generalization of Theorem 2.

Theoren 2′. Let X be Banach space with a strictly convex dual space X^*. If K_1 and K_2 are two existence subsets of X, then $x_1 \in K_1$, $x_2 \in K_2$, and $\| x_1 - x_2 \| = D(K_1, K_2)$ if and only if $x_1 \in P_1 x_2$ and $x_2 \in P_2 x_1$.

References

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二凸集間之距離

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中文摘要

設X爲具性質（E）之巴納赫空間，其對偶空間爲嚴格凸空間，若X之閉凸集 K_1, K_2，則可定義函爲

$$P_i : X \longrightarrow K_i$$

四 $i=1,2$，如下：即 $P_i x \in K_i$ 滿足 $\| x - P_i x \| = \inf\{ \| y - x \| ; y \in K_i \}$，令 $Q = P_1 P_2$，則下列三敍述等價：

（i）存在 $x_i \in k_i$，$i=1,2$，使得 $\| x_1 - x_2 \|$ 爲 K_1 與 K_2 間之距離。

（ii）Q有一定點

（iii）存在某 $x \in X$，序列 $\{ Q^n x \}$ 有一聚點。（定理2及3）

本文亦證明（定理1），當 K_1 或 K_2 爲囿集時，則（i）成立。