5 Gröbner basis for ladder determinantal ideals

From Theorem 3.1 and Theorem 3.2, we have the following corollaries :

Corollary 5.1 *Let $X = (x_{ij})$ be a generic $m \times n$ matrix over a field K, and let $R = K[X]$. Let $a \leq m$, $b \leq n$ be nonnegative integers and $p \leq min\{a, b\}$ a positive integer. Let $D(X)$ be the part of the matrix X consisting of the last a rows and the first b columns. Let $G(X)$ be the set of all p-minors of $D(X)$. Let I_X be the ideal of R generated by the $G(X)$; then $G(X)$ is a Gröbner basis for I_X with respect to the lexicographic term order induced from the variable order*

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}.$$

Proof . Let $A, B \in G(X)$ and

$$c(A^*) = \{y_1, \cdots, y_p\} \quad with \quad y_1 > \cdots > y_p,$$

$$c(B^*) = \{z_1, \cdots, z_p\} \quad with \quad z_1 > \cdots > z_p.$$

Assume that $y_1 = x_{i_1,j_1}$, $y_p = x_{i_p,j_p}$, $z_1 = x_{k_1,l_1}$ and $y_p = x_{k_p,l_p}$.

If ($i_1 > a$ and $k_1 > a$) or ($j_p < b$ and $l_p < b$), then we can conclude that $S(A, B)$ can be reduced with respect to $G(X)$ by Theorem 3.1. Therefore, we just have to consider two other cases: $(i_1 > a, l_p < b)$ and $(k_1 > a, j_p < b)$.

We may assume without loss of generality that $k_1 > a \geq i_1$ and $j_p < b \leq l_p$.

Let

$$A = [i_1 \cdots i_p | j_1 \cdots j_p]$$

and

$$B = [k_1 \cdots k_p | l_1 \cdots l_p].$$

Suppose that $|c(A^*) \cap c(B^*)| = u$.

If $u = 0$, then $S(A, B)$ can be reduced by Lemma 2.7.

If $u = p$, then $S(A, B) = 0$, so that $S(A, B)$ can be reduced by Lemma 2.4.

From above we may assume that $0 < u \leq p - 1$ and assume that

$$c(A^*) \cap c(B^*) = \{x_{g_1,h_1}, \cdots x_{g_u,h_u}\}.$$

After rearranging the p rows and p columns, respectively, we may further assume that

$$A = [i_1 \cdots i_{p-u} g_1 \cdots g_u | j_1 \cdots j_{p-u} h_1 \cdots h_u]$$

and

$$B = [g_1 \cdots g_u k_1 \cdots k_{p-u} | h_1 \cdots h_u l_1 \cdots l_{p-u}],$$

where $i_1 < \cdots < i_{p-u}$, $j_1 < \cdots < j_{p-u}$, $g_1 < \cdots < g_u$ and $h_1 < \cdots < h_u$.

Let E be the determinant of the following $(2p-u) \times (2p-u)$ matrix

$$M = \begin{pmatrix} x_{i_1,j_1} & \cdots & x_{i_1,j_{p-u}} & x_{i_1,h_1} & \cdots & x_{i_1,h_u} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{i_{p-u},j_1} & \cdots & x_{i_{p-u},j_{p-u}} & x_{i_{p-u},h_1} & \cdots & x_{i_{p-u},h_u} & 0 & \cdots & 0 \\ x_{g_1,j_1} & \cdots & x_{g_1,j_{p-u}} & x_{g_1,h_1} & \cdots & x_{g_1,h_u} & x_{g_1,l_1} & \cdots & x_{g_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{g_u,j_1} & \cdots & x_{g_u,j_{p-u}} & x_{g_u,h_1} & \cdots & x_{g_u,h_u} & x_{g_u,l_1} & \cdots & x_{g_u,l_{p-u}} \\ x_{k_1,j_1} & \cdots & x_{k_1,j_{p-u}} & x_{k_1,h_1} & \cdots & x_{k_1,h_u} & x_{k_1,l_1} & \cdots & x_{k_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k_{p-u},j_1} & \cdots & x_{k_{p-u},j_{p-u}} & x_{k_{p-u},h_1} & \cdots & x_{k_{p-u},h_u} & x_{k_{p-u},l_1} & \cdots & x_{k_{p-u},l_{p-u}} \end{pmatrix}.$$

Let

$$S = \{i_1, \cdots, i_{p-u}, g_1, \cdots, g_u, k_1, \cdots, k_{p-u}\}$$

and

$$T = \{j_1, \cdots, j_{p-u}, h_1, \cdots, h_u, l_1, \cdots, l_{p-u}\}.$$

Then

$$\begin{aligned} E &= \sum_{e_i \in T-\{l_1,\cdots,l_{p-u}\}} \pm [i_1 \cdots i_{p-u} |\, e_1 \cdots e_{p-u}][S-\{i_1, \cdots, i_{p-u}\} |\, T-\{e_1, \cdots, e_{p-u}\}] \\ &= \sum_{e'_i \in S-\{i_1,\cdots,i_{p-u}\}} \pm [e'_1 \cdots e'_{p-u} |\, l_1 \cdots l_{p-u}][S-\{e'_1, \cdots, e'_{p-u}\} |\, T-\{l_1, \cdots, l_{p-u}\}]. \end{aligned}$$

Let

$$\begin{aligned}
A_{\underline{e}} &= [S - \{i_1, \cdots, i_{p-u}\}|\ T - \{e_1, \cdots, e_{p-u}\}], \\
B_{\underline{e'}} &= [S - \{e'_1, \cdots, e'_{p-u}\}|\ T - \{l_1, \cdots, l_{p-u}\}], \\
C_{\underline{e}} &= [i_1 \cdots i_{p-u}|\ e_1 \cdots e_{p-u}], \\
D_{\underline{e'}} &= [e'_1 \cdots e'_{p-u}|\ l_1 \cdots l_{p-u}], \\
A_0 &= [i_1 \cdots i_{p-u}|\ j_1 \cdots j_{p-u}], \\
B_0 &= [k_1 \cdots k_{p-u}|\ l_1 \cdots l_{p-u}],
\end{aligned}$$

where $\underline{e} = \{e_1, \cdots, e_{p-u}\}$ and $\underline{e'} = \{e'_1, \cdots, e'_{p-u}\}$.

Then

$$B_0 A - A_0 B = \sum_{\substack{e_i \in T - \{l_1, \cdots, l_{p-u}\} \\ \underline{e} \neq \{j_1, \cdots, j_{p-u}\}}} \pm C_{\underline{e}} A_{\underline{e}} + \sum_{\substack{e'_i \in S - \{i_1, \cdots, i_{p-u}\} \\ \underline{e'} \neq \{k_1, \cdots, k_{p-u}\}}} \pm D_{\underline{e'}} B_{\underline{e'}} \ ,$$

so that

$$\begin{aligned}
S(A,B) &= B_0^* A - A_0^* B \\
&= \sum_{\substack{e_i \in T - \{l_1, \cdots, l_{p-u}\} \\ \underline{e} \neq \{j_1, \cdots, j_{p-u}\}}} \pm C_{\underline{e}} A_{\underline{e}} + \sum_{\substack{e'_i \in S - \{i_1, \cdots, i_{p-u}\} \\ \underline{e'} \neq \{k_1, \cdots, k_{p-u}\}}} \pm D_{\underline{e'}} B_{\underline{e'}} \\
&\quad + (A_0 - A_0^*) B - (B_0 - B_0^*) A.
\end{aligned}$$

As before, we first show that $A_{\underline{e}}, B_{\underline{e'}} \in G(X)$. $A_{\underline{e}}, B_{\underline{e'}} \in G(X)$ as it is clear that every entry x_{ij} in the matrix M can not satisfy both $i \leq a$ and $j \geq b$. Let $L = LCM(A^*, B^*)$. Therefore, by Lemma 2.4, to show that $S(A,B)$ can be reduced, it is sufficient to show that $C_{\underline{e}}^* A_{\underline{e}}^* < L$, $D_{\underline{e'}}^* B_{\underline{e'}}^* < L$, $(A_0 - A_0^*)^* B^* < L$ and $(B_0 - B_0^*)^* A^* < L$.

Let

$$N_1 = \begin{pmatrix} x_{i_1,j_1} & \cdots & x_{i_1,j_{p-u}} & x_{i_1,h_1} & \cdots & x_{i_1,h_u} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{i_{p-u},j_1} & \cdots & x_{i_{p-u},j_{p-u}} & x_{i_{p-u},h_1} & \cdots & x_{i_{p-u},h_u} \\ x_{g_1,j_1} & \cdots & x_{g_1,j_{p-u}} & x_{g_1,h_1} & \cdots & x_{g_1,h_u} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{g_u,j_1} & \cdots & x_{g_u,j_{p-u}} & x_{g_u,h_1} & \cdots & x_{g_u,h_u} \end{pmatrix}$$

and

$$N_2 = \begin{pmatrix} x_{g_1,h_1} & \cdots & x_{g_1,h_u} & x_{g_1,l_1} & \cdots & x_{g_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{g_u,h_1} & \cdots & x_{g_u,h_u} & x_{g_u,l_1} & \cdots & x_{g_u,l_{p-u}} \\ x_{k_1,h_1} & \cdots & x_{k_1,h_u} & x_{k_1,l_1} & \cdots & x_{k_1,l_{p-u}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k_{p-u},h_1} & \cdots & x_{k_{p-u},h_u} & x_{k_{p-u},l_1} & \cdots & x_{k_{p-u},l_{p-u}} \end{pmatrix}.$$

Since for every entry x in N_1 (resp. N_2) which is not in diagonal, there is a diagonal entry x' in N_1(resp. N_2) such that

$$x <_{(1,3)} x',$$

and for every $s = 1, \cdots, p-u$, $t = 1, \cdots, p-u$, we have

$$x_{k_s,j_t} \leq_{(1,3,4)} x_{k_1,j_1} <_{(1,3)} \max\{x_{i_1,j_1}, x_{k_1,l_1}\}.$$

Hence, for every entry x in the matrix M which is not in diagonal, there is a diagonal entry x' in M such that

$$x <_{(1,3,4)} x'.$$

For all $\underline{e} \neq \{j_1, \cdots, j_{p-u}\}$ and $\underline{e}' \neq \{k_1, \cdots, k_{p-u}\}$, it is obvious that

$$c(C^*_{\underline{e}} A^*_{\underline{e}}) \neq c(L) \quad \text{and} \quad c(D^*_{\underline{e}'} B^*_{\underline{e}'}) \neq c(L),$$

then we can conclude that

$$C^*_{\underline{e}} A^*_{\underline{e}} < L \quad \text{and} \quad D^*_{\underline{e}'} B^*_{\underline{e}'} < L.$$

Furthermore, if $m = n$, we have

$$C^*_{\underline{e}} A^*_{\underline{e}} <_\pi L \quad \text{and} \quad D^*_{\underline{e}'} B^*_{\underline{e}'} <_\pi L.$$

Finally, it is clear that

$$(A_0 - A_0^*)^* B^* < A_0^* B^* = L$$

and

$$(B_0 - B_0^*)^* A^* < B_0^* A^* = L.$$

Also, if $m = n$, we have

$$(A_0 - A_0^*)^* B^* <_\pi A_0^* B^* = L$$

and

$$(B_0 - B_0^*)^* A^* <_\pi B_0^* A^* = L.$$

Therefore $S(A, B)$ can be reduced with respect to G_p. $\square$

As in the proof of Theorem 3.2, we can deduce the following result from Lemma 5.1.

Corollary 5.2 *Let $Y = (y_{ij})$ be a generic $n \times n$ symmetric matrix over a field K, and let $R = K[Y]$. Let $a \leq n$, $b \leq n$ be nonnegative integers and*

$p \leq min\{a,b\}$, a positive integer. Let $D(Y)$ be the part of the matrix Y consisting of the last a rows and the first b columns. Let $G(Y)$ be the set of all p-minors of $D(Y)$. Let I_Y be the ideal of R generated by the $G(Y)$; then $G(Y)$ is a Gröbner basis for I_Y with respect to the lexicographic term order induced from the variable order

$$y_{11} > y_{12} > \cdots > y_{1n} > y_{22} > \cdots > y_{2n} > \cdots > y_{nn}.$$