2 Preliminary

Definition 2.1 *Let $R = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K. If $f \in R$, then f^* is the leading term of f with respect to some lexicographical order τ on R. If I is an ideal, then I^* is the ideal of R generated by the set $\{f^* | f \in I\}$.*

Definition 2.2 *Let I be an ideal of a ring $R = K[x_1, \cdots, x_n]$. A subset $G = \{g_1, \cdots, g_n\}$ of I is called a Gröbner basis of I with respect to some lexicographical order τ if $I^* = (g_1^*, \cdots, g_n^*)$.*

In order to check whether a set of polynomials is a Gröbner basis, we introduce the notion of S-polynomials.

Definition 2.3 *Let $R = K[x_1, \cdots, x_n]$ be a polynomial ring over a field. If A and B are two polynomials of R, then the S-polynomial of A and B, denoted by $S(A, B)$, is*

$$\frac{q}{A^*} \cdot A - \frac{q}{B^*} \cdot B,$$

where q is the least common multiple of A^* and B^*.

Lemma 2.4 ([7]) *Let G be a subset of homogeneous polynomials of R. Then the following are equivalent for G:*

(1) G is a Gröbner basis.

(2) For every $A, B \in G$, either $S(A, B) = 0$ or there are forms $f_1, \cdots$, and $C_1, \cdots$ in G such that

(*a*) $S(A,B) = \sum_{i\geq 1} f_i C_i$, *and*

(*b*) $f_i^* C_i^* < LCM(A^*, B^*)$ *for every* i.

(3) If there are monomials $f_1, \cdots, f_n$ *and elements* $A_1, \cdots, A_n$ *in* G *such that the leading monomials of* $\{f_1A_1, \cdots, f_nA_n\}$ *are the same but the leading monomial of* $\sum f_iA_i$ *is smaller than the one of* f_1A_1, *then there are forms* $g_1, \cdots$, *and* $C_1, \cdots \in G$ *such that*

(*a*) $\sum f_iA_i = \sum g_iC_i$, *and*

(*b*) $g_i^*C_i^* < f_1A_1^*$.

Proof . (1)$\Rightarrow$ (2) is obvious.

(2)$\Rightarrow$ (3) Let q be the LCM of the leading monomials of A_1 and A_2. Since the leading monomials of f_1A_1 and f_2A_2 are the same, f_1A_1 and f_2A_2 are both divisible by q, therefore there are units $\alpha, \beta \in K$ and a monomial g such that $\alpha f_1A_1 - \beta f_2A_2 = gS(A_1, A_2)$. By (2), there are forms $g_1, \cdots$, and $C_1, \cdots \in G$ such that $S(A_1, A_2) = \sum g_iC_i$ and $g_i^*C_i^* < q$. So,

$$\sum f_iA_i = (1 + \alpha^{-1}\beta)f_2A_2 + \sum_{i\geq 3} f_iA_i + \alpha^{-1}\sum gg_iC_i$$

and $gg_i^*C_i^* < f_1A_1^*$. Now, the assertion follows by induction.

(3)$\Rightarrow$ (1) Let I be the ideal of R generated by G and let $P \in I$. Then there are forms f_i and elements $A_i \in G$ such that P can be expressed as $\sum_{i=1}^t f_iA_i$ with $f_1^*A_1^* \geq f_2^*A_2^* \geq \cdots$. We may ask that such an expression is the smallest in the sense that if $P = \sum_{i=1}^l b_iB_i$ for some forms g_i and elements $B_i \in G$ such that $g_1^*B_1^* \geq g_2^*B_2^* \geq \cdots$, then $f_1^*A_1^* \leq g_1^*B_1^*$. Since we can always find the smallest expression, we assume that $P = \sum f_i A_i$ is the smallest one. However, this implies that $P^* = f_i^* A_i^*$. For, if not, there is an integer n such that the leading monomials of $\{f_1^* A_1, \cdots, f_n^* A_n\}$ are the same but the leading monomial of $\sum_{i=1}^n f_i^* A_i$ is smaller than the one of $f_1^* A_1$, so that by (3), P has an expression that is smaller than the expression $\sum f_i A_i$, which is impossible. Thus, P^* is indeed divisible by A_1^*. This shows (1). □

Let G be a subset of homogeneous polynomials of R and A, B be two polynomials in G. We shall say that $S(A, B)$ can be "reduced" with respect to G if A, B satisfy (2) of Lemma 2.4.

Let K be a field. Let $X_{m,n} = (x_{ij})$ be a generic $m \times n$ matrix over K and $Y = (y_{ij})$ be a generic $n \times n$ symmetric matrix over K. Let τ_X (resp. τ_Y) be the lexicographical term order on $K[X_{m,n}]$ (resp. $K[Y]$) induced by the variable order

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}.$$

$$(resp. \;\; y_{11} > y_{12} > \cdots > y_{1n} > y_{22} > \cdots > y_{2n} > \cdots > y_{nn}.)$$

Let f^* be the leading term of f with respect to τ_X (resp. τ_Y) if f is an element of $X_{m,n}$ (resp. Y). When $m = n$, we let $\pi : K[X_{n,n}] \longrightarrow K[Y]$ be the canonical ring epimorphism by sending x_{ij} to y_{ij}. Notice that if f and g are two monomials of $X_{m,n}$ with $f > g$, it is not necessary that $\pi(f) > \pi(g)$.

For convenience, we use the abbreviation notations for the rest of the paper : Denote R to be either $K[X_{n,n}]$ or $K[Y]$.

(1). For a monomial $f \in R$, $c(f)$ is the set of all variables in f.

(2). For a monomial $f \in R$, $l(f)$ is the largest variable in $c(f)$, and $s(f)$ is the smallest variable in $c(f)$.

(3). Let x_{ij}, x_{kl} be two variables in $X_{m,n}$ with $x_{ij} > x_{kl}$. We write $x_{ij} >_1 x_{kl}$ if $i = k$ and $j < l$, $x_{ij} >_2 x_{kl}$ if $i < k$ and $j > l$, $x_{ij} >_3 x_{kl}$ if $i < k$ and $j = l$, $x_{ij} >_4 x_{kl}$ if $i < k$ and $j < l$. Also, we write $x_{ij} >_{(r,s)} x_{kl}$ if $x_{ij} >_r x_{kl}$ or $x_{ij} >_s x_{kl}$. Similarly, we write $x_{ij} >_{(r,s,t)} x_{kl}$ if $x_{ij} >_r x_{kl}$ or $x_{ij} >_s x_{kl}$ or $x_{ij} >_t x_{kl}$.

(4). Let x_{ij}, x_{kl} be two variables in $X_{m,n}$. We write $x_{ij} \geq_r x_{kl}$ if $x_{ij} = x_{kl}$ or $x_{ij} >_r x_{kl}$. Also, we write $x_{ij} \geq_{(r,s)} x_{kl}$ if $x_{ij} = x_{kl}$ or $x_{ij} >_{(r,s)} x_{kl}$. Similarly, we write $x_{ij} \geq_{(r,s,t)} x_{kl}$ if $x_{ij} = x_{kl}$ or $x_{ij} >_{(r,s,t)} x_{kl}$.

(5). Let f and g be two monomials of $X_{m,n}$. We write $f >_\pi g$ if $\pi(f) > \pi(g)$.

Definition 2.5 *The determinant of a submatrix of X is called a minor of X. Let p be a positive integer, the minor corresponding to the submatrix of X with rows $i_1, \cdots, i_p$ and columns $j_1, \cdots, j_p$ is denoted by $[i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_p]_X$, or $[i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_p]$ if there is no ambiguity, which called a p-minor of X.*

Remark 2.6 (1). *A minor $[i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_p]$ of Y is called a doset minor in the sense of* [4] *if $i_1 \leq j_1, \cdots, i_p \leq j_p$. From* [4], *$\bar{G}_p$ is the set of the doset p-minors of Y. Therefore, there is a one to one correspondence between the set $\{A \in G_p | c(A^*) \subseteq X'\}$ and $\bar{G}_p$, where $X' = \{x_{ij} | i \leq j\}$ is a subset of $X_{n,n}$.*
(2). *If x_1 and x_2 are two variables in $X_{n,n}$ with $x_1 >_i x_2$, where $i = 1,\ 3,\ 4,$*

then $\pi(x_1) > \pi(x_2)$.

(3). *Let* f *and* g *be two monomials of* $K[X_{n,n}]$. *If for every* $x \in c(f)$ *there is a variable* $x' \in c(g)$ *such that* $x >_i x'$ *where* $i = 1, 3$ *or* 4, *then* $f >_\pi g$.

Lemma 2.7 ([7]) *Let* G *be a set of minors of* $X_{m,n}$ *or a set of minors of* Y, *and let* $A, B \in G$. *If* $|c(A^*) \cap c(B^*)| = \phi$, *then* $S(A, B)$ *can be reduced with respect to* $\{A, B\}$.

Proof . Since A and B are minors, there are monomials $f_1 = A^* > f_2 > \cdots$ and $g_1 = B^* > g_2 > \cdots$ such that $A = f_1 + f_2 + \cdots$ and $B = g_1 + g_2 + \cdots$. Therefore,

$$S(A, B) = g_1 A - f_1 B = (\sum_{i \geq 2} f_i)B - (\sum_{i \geq 2} g_i)A.$$

It follows that $S(A, B)$ can be reduced with respect to {A,B}. $\square$