4 Some useful propositions

Wang [7] gave another proofs to Theorem 1.1 and Theorem 1.2. From his proofs, we get some remarks as follows.

Remark 4.1 *Let $X = (x_{ij})$ be a generic $m \times n$ matrix over a field K and let $R = K[X]$. Let $a_1 \leq \cdots \leq a_r \leq m$, $b_1 \leq \cdots \leq b_r \leq n$ be nonnegative integers and $\eta_1, \cdots, \eta_{r+1}$ be positive integers. Let $D_t(X)$ be the part of the matrix X consisting of the first a_t rows and the first b_t columns. Let $G_t(X)$ be the set of all η_t-minors of $D_t(X)$, $t = 1, \cdots, r$ and set $G_{r+1}(X)$ be the set of all η_{r+1}-minors of X. Let I be the ideal of R generated by the $G(X) = \cup_{t=1}^{r+1} G_t(X)$; then $G(X)$ is a Gröbner basis for I with respect to the lexicographic term order induced from the variable order*

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}.$$

Remark 4.2 *Let $Y = (y_{ij})$ be an $n \times n$ symmetric matrix of indeterminates, and let $R = K[Y]$. Let $a_1 \leq \cdots \leq a_r \leq m$, $b_1 \leq \cdots \leq b_r \leq n$ be nonnegative integers and $\eta_1, \cdots, \eta_{r+1}$ be positive integers. Let $D_t(Y)$ be the part of the matrix Y consisting of the first a_t rows and the first b_t columns. Let $G_t(Y)$ be the set of all η_t-minors of $D_t(Y)$, $t = 1, \cdots, r$ and set $G_{r+1}(Y)$ be the set of all η_{r+1}-minors of Y. Let I be the ideal of R generated by the $G(Y) = \cup_{t=1}^{r+1} G_t(Y)$; then $G(Y)$ is a Gröbner basis for I with respect to the lexicographic term order* *induced from the variable order*

$$y_{11} > y_{12} > \cdots > y_{1n} > y_{22} > \cdots > y_{2n} > \cdots > y_{nn}.$$

From Remark 4.1 and Remark 4.2, we have some propositions as follows.

Proposition 4.3 *Let $X = (x_{ij})$ be a generic $m \times n$ matrix over a field K, and let $R = K[X]$. Let b, p, q are positive integers satisfying $q \leq n$ and $p \leq b \leq n$. Let $D_b(X)$ be the part of the matrix X consisting of the first b columns. Let $G(X)$ be the set of all p-minors of $D_b(X)$ and all q-minors of X. Let I be the ideal of R generated by the $G(X)$; then $G(X)$ is a Gröbner basis for I with respect to the lexicographic term order induced from the variable order*

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}.$$

Proof . In Remark 4.1, taking $r = 1$, $a_1 = 0$, $b_1 = b$, $\eta_1 = p$ and $\eta_2 = q$, then we are done. $\square$

Proposition 4.4 *Let $Y = (y_{ij})$ be an symmetric $n \times n$ matrix of indeterminates, and let $R = K[X]$. Let b, p, q are positive integers satisfying $q \leq n$ and $p \leq b \leq n$. Let $D_b(Y)$ be the part of the matrix Y consisting of the first b columns. Let $G(Y)$ be the set of all p-minors of $D_b(Y)$ and all q-minors of Y. Let I be the ideal of R generated by the $G(Y)$; then $G(Y)$ is a Gröbner basis for I with respect to the lexicographic term order induced from the variable order*

$$y_{11} > y_{12} > \cdots > y_{1n} > y_{22} > \cdots > y_{2n} > \cdots > y_{nn}.$$

Proof . In Remark 4.2, taking $r = 1$, $a_1 = 0$, $b_1 = b$, $\eta_1 = p$ and $\eta_2 = q$, then we are done. □