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數學計算可交換性益智玩具的最少步數

Calculating the Upper Bounds of the Commutative Puzzles in Mathematics

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Content

Abstract

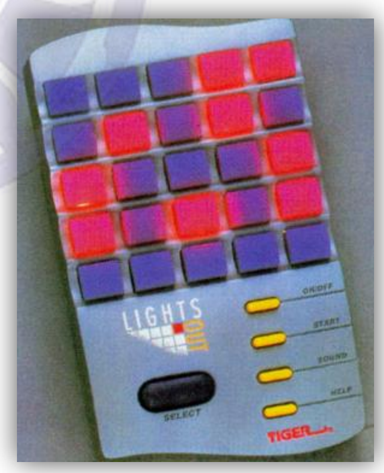
The main purpose of this paper is to describe the optimal solution of Lights Out games and other similar commutative puzzles. In 1998, Anderson and Feil used Linear Algebra to find a solution method for Lights Out games. In 2009, Goldwasser et al. proved the lit-only restriction is not different for the sigma game. In 2014, Schicho and Top discussed many variation of Lights Out. Those results heavily rely on computer. In this paper, we use mathematical methods to find an upper bound of minimal solutions, and furthermore, give an estimation algorithm to the upper bound.

1. Introduction

Lights Out

A standard Lights Out puzzle game consists of a 5×5 grid of buttons which also have light in them. Whenever a button is pressed, its light and the adjacent lights would change, turn on if it was off, and vise versa. Given a pattern of lights, the goal is to turn all the lights off by pressing the buttons.

There are various names of this puzzle, such as Button Madness, Fiver, FlipIt, Lights Out, Magic Square, XL-25, Token Flip, and Orbix. The first electronic version of this game was called XL-25, which was produced by Vulcan Electronics Ltd. in 1983. It was invented by László Mérõ, who is a Hungarian research psychologist and popular science writer.



Linear Solution

A Lights Out puzzle is commutative, in other words, the order of buttons pressed does not affect its outcome. (See fig.1.) Therefore, any lights out puzzle pattern can be solved by finding its linear solution, by simply solving linear equations. However, this may not be the best solution that solves the pattern. This leads to a question: How do we find a solution which requires the minimal number of button pressed?

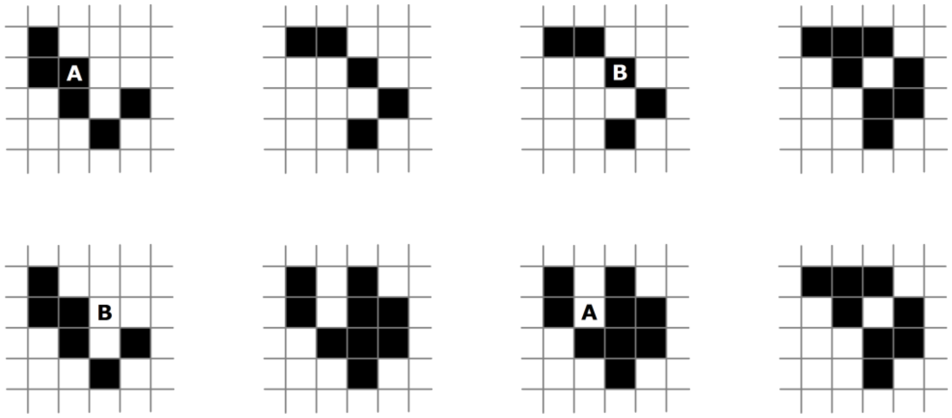


Fig.1. The result doesn't change whenever it goes from A to B or from B to A.

Lie Algebra, Dynkin Diagram and Lit-Only Games

The original idea of the study to Lit-Only Games comes from Lie Algebra and representation theory. The root system of simple Lie Algebra can be represented as on the Finite Dynkin diagram (see fig.2.), with each point colored black or white. Two root systems are categorized as the same type if one can be transformed to another, with the rule very similar to Lights Out games, except that only the black points are allowed to press. This leads to the study of Lit-Only σ games, which is the variation of standard Lights Out game with the restriction that only the lit buttons can be pressed. Josef Schicho and Jaap Top have proved that a Lit-Only game is solvable if and only if the same situation can be solved without the Lit-Only restriction [3]. Hsin-Jung Wu has extended the results to some general graphs [2].

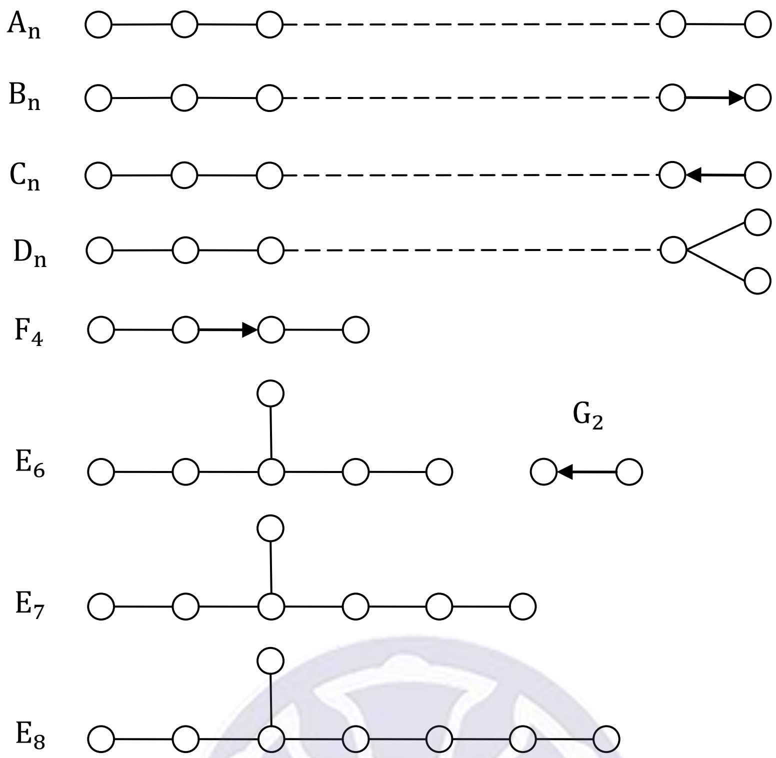


Fig.2. Finite Dynkin diagrams.

In Chapter 2, we will use some mathematical approaches to get the bounds of the optimal solution on standard Lights Out games. The method can be extended to some variation of Lights Out games, such as n-ary models, or even other commutative puzzles. We will see this in Chapter 3 and 4.

2. The standard Lights Out

Standard Lights Out of $n \times n$ grids

Definition 2.1

We begin with some definition for $n \times n$ grids Lights Out Games, and note that the following arithmetic only works in module 2. First, we give some notation of position vectors and movement vectors.

Let $[n] = \{1,2,\dots,n\}$. Let $\mathbb{Z}_2$ be the field consist only $\{0,1\}$.

The position vector $\mathbf{b} = \left(b_{11}, b_{12}, \dots, b_{ij}, \dots, b_{nn}\right)^T \in \mathbb{Z}_2^{n^2}, b_{ij} \in \mathbb{Z}_2, i,j \in [n]$.

The movement vector $\mathbf{m}_{ij} = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{kl}, \dots, \alpha_{nn})^T \in \mathbb{Z}_2^{n^2}$, with

$$\alpha_{kl} = \begin{cases} 1, |k-i| + |l-j| \leq 1 \\ 0, otherwise \end{cases}, i,j,k,l \in [n].$$

Then the movement matrix $M = \begin{bmatrix} | & | & & | & & | \\ m_{11} & m_{12} & \dots & m_{ij} & \dots & m_{nn} \\ | & | & & | & & | \end{bmatrix}$.

The movement matrix records all the movement we do to the Lights Out buttons.

Clearly, $\mathrm{col}(M) = \Lambda$ is a subspace of $\mathbb{Z}_2^{n^2}$.

If $\mathbf{b} \in \Lambda$ then $\mathbf{b}$ is called solvable, otherwise unsolvable.

Whenever $\mathbf{b} \in \Lambda$, there exists a linear combination of $\mathbf{m}_{ij}$ such that

$$\mathbf{b} = \sum_{i,j \in [n]} c_{ij} \mathbf{m}_{ij} = \mathrm{M}\mathbf{c}.$$

Then we can define the solution set of $\mathbf{b}$:

$$\Gamma_b \coloneqq \left\{ \boldsymbol{c} = \left(c_{11}, c_{12}, \dots, c_{ij}, \dots c_{nn}\right)^T \middle| \mathbf{b} = M\mathbf{c} \right\} \subseteq \mathbb{Z}_2^{n^2}.$$

For each solution, we can calculate the moves, which is the number of nonzero entries in module 2. Thus we define the function $\tau\colon \mathbb{Z}_2^{n^2} \to n^2 \cup \{0\}$ by

$$\tau(\mathbf{c}) = \sum_{c_{ij} \neq 0} 1.$$

and the best solution for each position vector $\mathbf{b}$:

$$\gamma_b = \min\{\tau(\mathbf{c}) | \mathbf{c} \in \Gamma_b\}.$$

Among all the solvable position vectors, we can get the best solution, i.e., the minimal solution of Lights Out and denote by

$$\gamma = \max\{\gamma_b | \mathbf{b} \in \Lambda\}.$$

Example 2.2

Consider a 2×2 grids Lights Out game. There are four movement vectors, $\mathbf{m}_{11}, \mathbf{m}_{12}, \mathbf{m}_{21}, \mathbf{m}_{22}$ as follows (see fig.3.):

$$\mathbf{m}_{11} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_{12} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{m}_{21} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{m}_{22} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

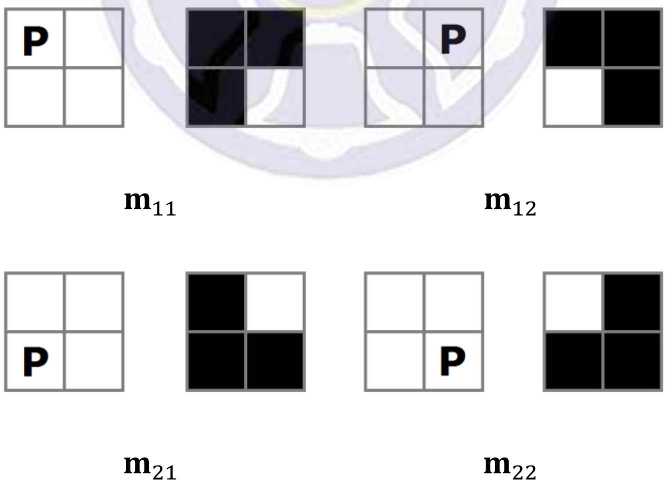


Fig.3. On the left side, the "P" marks the button pressed, and the right side shows the movement affect to the grids.

Thus the movement matrix is

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

and the column space

$$\Lambda = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \mathbb{Z}_2^4.$$

Then every position vector is considered solvable.

For example,

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{m}_{21} + \mathbf{m}_{22}$$

is a linear combination of the movement vectors.

Definition 2.3

We define the null space of M

$$\operatorname{Null}(M) = \{\mathbf{v} | M\mathbf{v} = \mathbf{0}\} \subseteq \mathbb{Z}_2^{n^2}$$

which is a subspace of $\mathbb{Z}_2^{n^2}$ as well.

Any vector in $\operatorname{Null}(M)$ is called a null vector, and the basis of $\operatorname{Null}(M)$ is the null basis.

Proposition 2.4

For any solution vector $\mathbf{c} \in \Gamma_b$, null vector $\mathbf{v} \in \operatorname{Null}(M)$, $\mathbf{c} + \mathbf{v} \in \Gamma_b$.

For any two solution vectors $\mathbf{c}, \mathbf{c'} \in \Gamma_b$, $\mathbf{c} - \mathbf{c'} \in \operatorname{Null}(M)$.

Proof. (a) Let $\mathbf{b}$ be the corresponding position vector to $\mathbf{c}$, then $\mathbf{b} = M\boldsymbol{c}$. Since $\mathbf{v} \in \operatorname{Null}(M)$, $M\mathbf{v} = 0$. Then $\mathbf{b} = M\mathbf{c} + M\mathbf{v} = M(\boldsymbol{c} + \mathbf{v})$, thus $\mathbf{c} + \mathbf{v}$ is also a solution vector of $\mathbf{b}$. For (b), As $\mathbf{b} = M\mathbf{c}$ and $\mathbf{b} = M\mathbf{c}', M(\mathbf{c} - \mathbf{c}') = M\mathbf{c} - M\mathbf{c}' = 0$, so $\mathbf{c} - \mathbf{c}' \in \operatorname{Null}(M)$.

Theorem 2.5

If $\mathrm{Null}(M) = \{\mathbf{0}\}$, then every position vector is solvable, and the corresponding solution set is a singleton.

Proof. From the Rank-nullity Theorem we know that

$$\dim(\Lambda) + \dim(\mathrm{Null}(M)) = n^2.$$

Now as $\mathrm{Null}(M) = \{\mathbf{0}\}$, $\dim(\Lambda) = n^2$, thus $\Lambda = \mathbb{Z}_2^{n^2}$. So every position vector, which is in $\mathbb{Z}_2^{n^2}$, belongs to Λ and therefore solvable. Assume there are two solution vectors, $\mathbf{c}_1, \mathbf{c}_2$. From Proposition 2.4(b) we know that

$$\mathbf{c}_1 - \mathbf{c}_2 \in \mathrm{Null}(M) = \{\mathbf{0}\},$$

so $\mathbf{c}_1 = \mathbf{c}_2$. Therefore the solution is unique.

Example 2.6

From Example 2.2 we know that on 2×2 grids Lights Out game, $\Lambda = \mathbb{Z}_2^4$. As the null space only contains the zero vector, every position on 2×2 Lights Out game has a unique solution.

However, it is possible that the null space is nontrivial. In such situation, we can calculate the number of solutions.

Theorem 2.7

If the nullity of M is m, then there are 2^{n^2-m} distinct solvable position vectors, and each corresponds to 2^{m} solution vectors. Furthermore, for each solvable position vector $\mathbf{b}$, if $\mathbf{c}_0$ is a solution vector to it, then

$$\Gamma_b = \{\mathbf{c}_0 + \mathbf{v} | \mathbf{v} \in \mathrm{Null}(M)\}.$$

Proof. From the nullity we know that $|\text{Null}(M)| = 2^m$. By Proposition 2.4 (a), for each solvable position vector $\mathbf{b}$, there are 2^m solution vectors, i.e. $\Gamma_b = \{\mathbf{c}_0 + \mathbf{v} | \mathbf{v} \in \text{Null}(M)\}$, and $|\Gamma_b| = 2^m$. We know that on $\text{n} \times \text{n}$ grid, each button can either be pressed or not, which implies there are 2^{n^2} ways of button pressing. Then there are $\frac{2^{n^2}}{2^m} = 2^{n^2-m}$ distinct solvable position vectors.

From Theorem 2.7 we can conclude that in order to get the minimal solution, it is necessary to find the basis of the null space first. Here we'll first give some properties of the solvable solution set.

Definition 2.8

A matrix M is symmetric if $M = M^T$, or equivalently, $m_{ij} = m_{ji}$ for $i, j \in [n]$. A Lights Out game is symmetric if the movement matrix is symmetric.

A matrix M is reflexive if for all $i \in [n], m_{ii} = 1$. A Lights Out game is reflexive if the movement matrix is reflexive.

Remark. The standard Lights Out games are all symmetric and reflexive.

There are few solvability test theorems to those games.

Theorem 2.9

In a symmetric game, a position vector $\mathbf{b}$ is solvable if and only if $\mathbf{b} \cdot \mathbf{v} = \mathbf{0}$ for each $\mathbf{v} \in \text{Null}(M)$, where $\mathbf{b} \cdot \mathbf{v}$ denotes the dot product.

Proof. Assume $\mathbf{v} \in \text{Null}(M)$. Since M is symmetric, $\Lambda^{\perp} = \text{Null}(M)$, i.e. $\mathbf{b} \cdot \mathbf{v} = \mathbf{0}$. Conversely, suppose $\mathbf{b} \cdot \mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \text{Null}(M)$.Then $\mathbf{b} \in \text{Null}(M)^{\perp} =$

$\mathrm{Null}(M^T)^{\perp} = \mathrm{Col}(M) = \Lambda$, therefore $\mathbf{b}$ is solvable.

Corollary 2.10

In a reflexive, symmetric game, the position vector with all entries 1, is solvable.

Proof. Let $\mathbf{b} = (1,1,\dots,1)$ be the position vector. Suppose $\mathbf{v} = (v_{11},\dots,v_{ij},\dots,v_{nn})$, then

$$\mathbf{b}\cdot\mathbf{v} = \sum_{i,j\in[n]} v_{ij} = \sum_{i,j\in[n]} v_{ij}^2 = \mathbf{v}\cdot\mathbf{v}$$

since we are working on module 2. Since M is reflexive, the diagonal terms are all 1. Then $\mathbf{v}^T M\mathbf{v} = \mathbf{v}^T\mathbf{v} = \mathbf{v}\cdot\mathbf{v}$ as the off-diagonal terms cancel each other by symmetric on module 2, leaves all the diagonal terms (See the following example for details.) Then by Theorem 2.9, $\mathbf{b}$ is solvable.

$$\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$$
$$= [a_{11}+a_{31}+a_{41}+a_{13}+a_{33}+a_{43}+a_{14}+a_{34}+a_{44}]$$
$$= [a_{11}+a_{33}+a_{44}] = [1+1+1] = \begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}.$$

Now we give some inspection about the solution set. By Theorem 2.7, for each solvable vector $\mathbf{b}$, the corresponding solution set is $\Gamma_b = \{\mathbf{c}_0 + \mathbf{v} | \mathbf{v} \in \mathrm{Null}(M)\}$, where $\mathbf{c}_0$ is a solution vector to $\mathbf{b}$. Thus we can list out all the vectors in null space, and adding those vector to $\mathbf{c}_0$ gives out all the possible solution to $\mathbf{b}$.

Let $S = \{\mathbf{v}_1,\dots,\mathbf{v}_m\}$ be a basis of the null space. Since all the null vectors are generated by the basis S, assume in the i^{th} entry, a vector $\mathbf{v'}$ has a unit 1 in module. Now if there is a null vector with the i^{th} entry in $\mathbf{v}$ is p, then $\mathbf{v} = a_1\mathbf{v}_1 +$

$\cdots + a_i\mathbf{v}_i + \cdots + a_m\mathbf{v}_m$, where $a_i \in \mathbb{Z}_2$. Similarly, $\mathbf{v}' = b_1\mathbf{v}_1 + \cdots + b_i\mathbf{v}_i + \cdots + b_m\mathbf{v}_m$. Then

$$\mathbf{v} + \mathbf{v}' = (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_i + b_i)\mathbf{v}_i + \cdots + (a_m + b_m)\mathbf{v}_m$$

is another solution vector to the same position vector, with the i^{th} entry changed to $p + 1$. Therefore if we assume $T_i = \{\mathbf{v}|\mathbf{v} \in \mathrm{Null(M)}$ with the i^{th} entry$= p\}$, then the set $T_i^{v'} = \{\mathbf{v} + \mathbf{v}'|\mathbf{v} \in T_i\}$ is another vector set with equal amount, with all the i^{th} entry changed to zero and $T_i \cap T_i^{v'} = \{\phi\}$. From this we have the following lemma:

Lemma 2.11

Consider the matrix Q which columns are all the null vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_{2^m}\}$:

$$\mathrm{Q} = \begin{bmatrix} -\mathbf{v}_1 - \\ \cdots \\ -\mathbf{v}_{2^m} - \end{bmatrix}$$

Then if some column is nonempty, then the column consists of $\{0,1\}$ with equal amounts. For such column the number of entries 0 and 1 are both 2^{m-1}, where m is the nullity.

Now it comes to the main result of standard $n \times n$ Lights out games.

Theorem 2.12

Let Q be the matrix as in Lemma 2.11. Denote the set of all the columns by N, the set of columns with some nonzero entries by N_1 and the set of columns with all the entries equals to zero by N_0. Then:

$$\gamma \leq \frac{1}{2}\#(N_1) + \#(N_0).$$

Proof. For every solution $\mathbf{c} = (c_{11}, \ldots, c_{ij}, \ldots, c_{nn})$, we can treat it as some pattern in the following table:

$d_{1,1,0}$	$d_{1,2,0}$	$\ldots$	$d_{n,n,0}$
$d_{1,1,1}$	$d_{1,2,1}$		$d_{n,n,1}$

with $d_{i,j,l} = \begin{cases} 1, c_{ij} = l \\ 0, \text{otherwise} \end{cases}$, and

$$\sum_l d_{i,j,l} = 1 \ \forall i,j \in [n].$$

To calculate the minimal solution, we assume that for a solvable vector $\mathbf{b}$, there is a solution $\mathbf{c}_0 = (c_{11}, \ldots, c_{ij}, \ldots, c_{nn})$ such that

$$\gamma_b = \tau(\mathbf{c}_0).$$

In table form, this means that

$$\gamma_b = \tau(\mathbf{c}_0) = \sum_{i,j} d_{i,j,1}.$$

For any null vector $\mathbf{v}$, we denote the set of 0 positions by $s_0(\mathbf{v})$ and the set of 1 positions by $s_1(\mathbf{v})$. For any other solution vector $\mathbf{c}$, $\mathbf{c} = \mathbf{c}_0 + \mathbf{v}$ for some null vector $\mathbf{v}$. Viewing this in table form, from $\tau(\mathbf{c}_0) = \gamma_b \leq \tau(\mathbf{c})$ we can see that

$$\sum_{i,j} d_{i,j,1} \leq \sum_{i,j \in s_0(\mathbf{v})} d_{i,j,1} + \sum_{i,j \in s_1(\mathbf{v})} d_{i,j,0}$$

since the entries changes only in $s_1(\mathbf{v})$. For each null vector $\mathbf{v}$, we can get one inequality, and sum over all the inequalities gives the result:

$$\sum_{\mathbf{v} \in \mathrm{Null}(M)} \sum_{i,j} d_{i,j,1} \leq \sum_{\mathbf{v} \in \mathrm{Null}(M)} \sum_{i,j \in s_0(\mathbf{v})} d_{i,j,1} + \sum_{\mathbf{v} \in \mathrm{Null}(M)} \sum_{i,j \in s_1(\mathbf{v})} d_{i,j,0}.$$

Since

$$\sum_l d_{i,j,l} = 1 \ \forall i,j \in [n]$$

we can change the inequality to

$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j} d_{i,j,1} \le \sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_0(\mathbf{v})} d_{i,j,1} + \sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_1(\mathbf{v})} (1 - d_{i,j,1})$$
$$= \sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_0(\mathbf{v})} d_{i,j,1} + \sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_1(\mathbf{v})} 1 - \sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_1(\mathbf{v})} d_{i,j,1}.$$

From Lemma 2.11, for each nonempty column, the column consists equal amount of 0 and 1. Changing the summation from rows to columns leads to:

$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j} d_{i,j,1} = 2^m \sum_{N} d_{i,j,1}.$$
$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_1(\mathbf{v})} 1 = \sum_{N_1} 2^{m-1} = 2^{m-1}(\#(N_1)).$$
$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_0(\mathbf{v})} d_{i,j,1} = \sum_{N_0} 2^m d_{i,j,1} + \sum_{N_1} 2^{m-1} d_{i,j,1}.$$
$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_1(\mathbf{v})} d_{i,j,1} = \sum_{N_1} 2^{m-1} d_{i,j,1}.$$

Thus

$$2^m \sum_{N} d_{i,j,1} \le 2^{m-1}(\#(N_1)) + \sum_{N_0} 2^m d_{i,j,1}.$$
$$\gamma_b = \tau(\mathbf{c}_0) = \sum_{N} d_{i,j,1} \le \frac{1}{2}(\#(N_1)) + \sum_{N_0} d_{i,j,1} \le \frac{1}{2}(\#(N_1)) + \#(N_0).$$

gives an upper bound for the best solution of each solvable position vector **b.** As all the solvable vectors follow the inequality,

$$\gamma \le \frac{1}{2}\#(N_1) + \#(N_0).$$

Example 2.13

Here we calculate the minimal solution for 5×5 Lights Out game. The movement matrix is

$$M=\begin{bmatrix}
1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1
\end{bmatrix}$$

The null space matrix is

$$Q=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&1\\
0&1&1&1&0&1&0&1&0&1&1&1&0&1&1&1&0&1&0&1&0&1&1&1&0\\
1&1&0&1&1&0&0&0&0&0&1&1&0&1&1&0&0&0&0&0&1&1&0&1&1
\end{bmatrix}$$

There are 5 empty columns and 20 nonempty columns. From Theorem 2.12,

$$\gamma \le \frac{1}{2}\#(N_1)+\#(N_0)=\frac{1}{2}\times 20+5=15.$$

Example 2.14

Here we do the calculation for 4×4 Lights Out as well. The movement matrix is

$$M=\begin{bmatrix}
1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&1&1&1&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&1&1&1&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&1&1&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&1&1&1&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&1&1&1&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1
\end{bmatrix}$$

And the null space matrix:

$$Q=\begin{bmatrix}
0&0&0&1&0&0&1&1&0&1&0&1&1&1&1&0\\
0&0&1&0&0&1&1&1&1&0&0&0&1&0&1&1\\
0&0&1&1&0&1&0&0&1&1&0&1&0&1&0&1\\
0&1&0&0&1&1&1&0&0&0&0&1&1&1&0&1\\
0&1&0&1&1&1&0&1&0&1&0&0&0&0&1&1\\
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0\\
0&1&1&1&1&0&1&0&1&1&0&0&1&0&0&0\\
1&0&0&0&1&1&0&0&1&0&1&0&0&1&1&1\\
1&0&0&1&1&1&1&1&1&1&1&1&1&0&0&1\\
1&0&1&0&1&0&1&1&0&0&1&0&1&1&0&0\\
1&0&1&1&1&0&0&0&0&1&1&1&0&0&1&0\\
1&1&0&0&0&0&1&0&1&0&1&1&1&0&1&0\\
1&1&0&1&0&0&0&1&1&1&1&0&0&1&0&0\\
1&1&1&0&0&1&0&1&0&0&1&1&0&0&0&1\\
1&1&1&1&0&1&1&0&0&1&1&0&1&1&1&1
\end{bmatrix}$$

There are no empty columns and 16 nonempty columns. From Theorem 2.12,

$$\gamma \le \frac{1}{2}\#(N_1)+\#(N_0)=\frac{1}{2}\times 16+0=8.$$

Remark. The other Lights Out puzzle of different size can be done similarly. However, on the size of $2,3,6,7,8$, and many others, the null space is empty. Thus from Theorem 2.5, the minimal moves γ is the number of grids: $4, 9, 36, 49, 64$

respectively.

Lights Out cube

The version is similar to a standard Lights Out game, but played on a $3 \times 3 \times 3$ cube. Whereas the standard Lights Out has edges, the cube does not. Each button always has 4 neighbors, thus each button press changes 5 lights.

We can define the order of number as in fig.4:

			3	6	9						
			2	5	8						
			1	4	7						
21	24	27	12	15	18	39	42	45	30	33	36
20	23	26	11	14	17	38	41	44	29	32	35
19	22	25	10	13	16	37	40	43	28	31	34
			48	51	54						
			47	50	53						
			46	49	52						

Fig.4.

The movement matrix of Lights Out Cube is shown in block matrix form

$$M = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$$

where

$$A = \begin{bmatrix}
1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
1&0&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&1 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1 \\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1
\end{bmatrix}$$

$$B=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

$$C=\begin{bmatrix}
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1
\end{bmatrix}$$

Now for the null space, there is a basis as shown in the rows of R:

$$R=\begin{bmatrix}
1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&1&1&1&0&1&0&1&0&1&1&1&1&1&1&1&1&0&1&0&1&0&1&1&1&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&0&1&1&0&0&0&0&1&1&1&0&0&1&0&1&1&0&0&0&1&1&0&0&0&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&1&1&0&1&0&1&0&1&1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&1&1&1&0&1&0&0&0&0&0&1&0&0\\
0&0&0&1&0&1&0&0&0&0&0&1&1&0&1&0&0&1&1&1&0&0&0&0&1&1&0&0&0&1&1&0&1&0&0&1&1&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0&0&1&1&1&0&1&0&1&0&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

Therefore the cube has $2^6 = 64$ vectors in the null space.

From the basis we can see that in the null space matrix Q, there will be 6 empty columns and 48 nonempty columns. Thus by Theorem 2.12:

$$\gamma \le \frac{1}{2}\#(N_1) + \#(N_0) = \frac{1}{2} \times 48 + 6 = 30.$$

3. N-ary models and Variations

The method in chapter 2 can be applied similarly to other commutative puzzles. Here are some examples:

N-ary Lights Out

Lights Out 2000 is a version of Lights Out that seems very similar to the standard one, i.e. a 5×5 grids of buttons with light on them. However the rules is a little bit different: every light have three states: off, red, and green. Each press changes the light from off to red, red to green, or green to off. Thus the puzzle is of module 3.

On such puzzles, we can generalized our definition.

Definition 3.1

All the vectors belong to $\mathbb{Z}_r^{n^2}$ now, where r is the modular. $\mathbb{Z}_r$ denote the set which consist only $\{0,1,\ldots,r-1\}$. All the other symbols follow in Definition 2.1.

Most theorems can be applied directly, except some in the following might need an extended version on different modules.

Theorem 3.2

If the nullity of M is m, then there are r^{n^2-m} distinct solvable position vectors, and each corresponds to r^m solution vectors. Furthermore, for each solvable position vector $\mathbf{b}$, if $\mathbf{c}_0$ is a solution vector to it, then

$$\Gamma_b = \{\mathbf{c}_0 + \mathbf{v} | \mathbf{v} \in \mathrm{Null}(M)\}$$

Proof. From the nullity, $|\mathrm{Null}(M)| = r^m$. Thus for each solvable position vector $\mathbf{b}$, there are r^m solution vectors, i.e. $\Gamma_b = \{\mathbf{c}_0 + \mathbf{v} | \mathbf{v} \in \mathrm{Null}(M)\}$, and $|\Gamma_b| = r^m$. On $n \times n$ grid, each button can either be pressed or not, which implies there are r^{n^2} ways of button pressing. Then there are $\frac{r^{n^2}}{r^m} = r^{n^2-m}$ distinct solvable position vectors.

Lemma 3.3

Consider the matrix Q which columns are all the null vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_{r^m}\}$:

$$\mathrm{Q} = \begin{bmatrix} -\mathbf{v}_1 - \\ \cdots \\ -\mathbf{v}_{r^m} - \end{bmatrix}$$

Then if in some column, an entry is a unit in module *r*, then the column consists of $\{0, 1, \ldots, r-1\}$ with equal amounts. For such column, the amount of each number is r^{m-1}, where *m* is the nullity.

Proof. By Theorem 3.2, for each solvable vector $\mathbf{b}$, the corresponding solution set is $\Gamma_b = \{\mathbf{c}_0 + \mathbf{v} | \mathbf{v} \in \mathrm{Null}(M)\}$, where $\mathbf{c}_0$ is a solution vector to $\mathbf{b}$. Thus we can list out all the vectors in null space, and adding those vector to $\mathbf{c}_0$ gives out all the possible solutions to $\mathbf{b}$.

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a basis of the null space. Since all the null vectors are generated by the basis S, assume in the i^{th} entry, a vector $\mathbf{v'}$ has a unit in module r. Now if there is a null vector with the i^{th} entry in $\mathbf{v}$ is p, then $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_i\mathbf{v}_i + \cdots + a_m\mathbf{v}_m$, where $a_i \in \mathbb{Z}_r$. Similarly, $\mathbf{v}' = b_1\mathbf{v}_1 + \cdots + b_i\mathbf{v}_i + \cdots + b_m\mathbf{v}_m$. Then

$$\mathbf{v} + k\mathbf{v}' = (a_1 + kb_1)\mathbf{v}_1 + \cdots + (a_i + kb_i)\mathbf{v}_i + \cdots + (a_m + kb_m)\mathbf{v}_m$$

is another solution vector to the same position vector, with the i^{th} entry changed to $p + ku$. Since u is a unit, when we change the value of k from 1 to r, $p + ku$ cycles over all numbers in $\{1, \dots, r\}$. (If not, there would be distinct k_1, k_2 that $p + k_1u = p + k_2u$, then $(k_1 - k_2)u = 0$, but since $k_1 \neq k_2$, this leads to a contradiction that u is a unit.) Therefore if we assume $T_i = \{\mathbf{v}|\mathbf{v} \in \mathrm{Null(M)}$ with the i^{th} entry$= p\}$, then the set $T_i^{kv'} = \{\mathbf{v} + k\mathbf{v}'|\mathbf{v} \in T_i\}$ is another distinct vector set with equal amount, with all the i^{th} entry changed to $k + 1$. Thus the amount of each number in those column is r^{m-1}.

On n-ary models, we can measure the number of movements in different ways. One is to simply count the nonzero entries, while the other gives a weight for each entry. For instance, consider the Lights Out puzzle. One can count the red-green-off moves as one or two moves. This leads to the following definition:

Definition 3.4

The whole move metric counts the number of n consecutive moves on one movement as one, while the separate move metric count it as n. i.e.

Whole move metric	$\tau(c) = \sum_{c_{ij}\neq 0} 1$
Separate move metric	$\tau(c) = \sum_{c_{ij}\neq 0} c_{ij}$

With this we can complete the theorems for minimal moves.

Theorem 3.5 (for whole move metric)

Let Q be the matrix as in Lemma 3.3. Denote the set of all the columns by N, the set of columns with some nonzero entries by N_1 and the set of columns with all the entries equals to zero by N_0. Suppose that each column in N_1 contains a unit, then:

$$\gamma \leq \left(1-\frac{1}{r}\right)\#(N_1) + \#(N_0).$$

Proof. For every solution $\mathbf{c} = (c_{11}, \dots, c_{ij}, \dots, c_{nn})$, we can treat it as some pattern in the following table:

$d_{1,1,0}$	$d_{1,2,0}$	…	$d_{n,n,0}$
$d_{1,1,1}$	$d_{1,2,1}$		$d_{n,n,1}$
…	…		…
$d_{1,1,r-1}$	$d_{1,2,r-1}$	…	$d_{n,n,r-1}$

with $d_{i,j,l} = \begin{cases} 1, c_{ij} = l \\ 0, \text{otherwise} \end{cases}$, and

$$\sum_l d_{i,j,l} = 1 \ \forall i,j \in [n].$$

To calculate the minimal solution, we assume that for a solvable vector $\mathbf{b}$, there is a solution $\mathbf{c}_0 = (c_{11}, \dots, c_{ij}, \dots, c_{nn})$ such that

$$\gamma_b = \tau(\mathbf{c}_0).$$

In table form, this means that

$$\gamma_b = \tau(\mathbf{c}_0) = \sum_{\substack{i,j,l \\ l\neq 0}} d_{i,j,l}.$$

For any null vector $\mathbf{v}$, we denote the position set by $\mathrm{s_k}(\mathbf{v})$ if the number is k. For any other solution $\mathbf{c}$, $\mathbf{c} = \mathbf{c}_0 + \mathbf{v}$ for some null vector $\mathbf{v}$. Viewing this in table form, from $\tau(\mathbf{c}_0) = \gamma_b \leq \tau(\mathbf{c})$ we can see that

$$\sum_{\substack{i,j,l \\ l\neq 0}} d_{i,j,l} \leq \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} d_{i,j,l} + \sum_{\substack{i,j\in s_1(\mathbf{v}) \\ l\neq r-1}} d_{i,j,l} + \cdots + \sum_{\substack{i,j\in s_{r-1}(\mathbf{v}) \\ l\neq 1}} d_{i,j,l}.$$

Since the entries changes only in $s_r(\mathbf{v})$ while $r \neq 0$. For each null vector $\mathbf{v}$, we can get one inequality, and sum over all the inequalities gives the result:

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} d_{i,j,l} \leq \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} d_{i,j,l} + \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_1(\mathbf{v}) \\ l\neq r-1}} d_{i,j,l} + \cdots + \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_{r-1}(\mathbf{v}) \\ l\neq 1}} d_{i,j,l}.$$

Since

$$\sum_{l} d_{i,j,l} = 1 \; \forall i,j \in [n]$$

we can change the inequality to

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} d_{i,j,l} \leq \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} d_{i,j,l} + \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{i,j\in s_1(\mathbf{v})} \left(1 - d_{i,j,r-1}\right)$$
$$+ \cdots + \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{i,j\in s_{r-1}(\mathbf{v})} \left(1 - d_{i,j,1}\right).$$

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} d_{i,j,l} \leq \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} d_{i,j,l} + \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{i,j\notin s_0(\mathbf{v})} 1$$
$$- \sum_{\mathbf{v}\in\mathrm{Null}(M)} \left(\sum_{i,j\in s_1(\mathbf{v})} d_{i,j,r-1} + \cdots + \sum_{i,j\in s_{r-1}(\mathbf{v})} d_{i,j,1} \right).$$

From Lemma 3.3, by assumption, for each nonempty column, the column consists equal amount of numbers in $\{0, \dots, r-1\}$.Changing the summation from rows to columns leads to:

$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} d_{i,j,l} = r^m \sum_{N} \sum_{l\neq 0} d_{i,j,l}\,.$$

$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\notin s_0(\mathbf{v})} 1 = \sum_{N_1} (r-1)\times r^{m-1} = (r-1)\times r^{m-1}\big(\#(N_1)\big)$$

$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} d_{i,j,l} = r^m \sum_{N_0} \sum_{l\neq 0} d_{i,j,l} + r^{m-1} \sum_{N_1} \sum_{l\neq 0} d_{i,j,l}$$

And for each t, $1 \leq t \leq r-1$,

$$\sum_{\mathbf{v}\in \mathrm{Null}(M)} \sum_{i,j\in s_t(\mathbf{v})} d_{i,j,r-t} = \sum_{N_1} r^{m-1}\, d_{i,j,r-t}.$$

Thus

$$r^m \sum_{N} \sum_{l\neq 0} d_{i,j,l} \leq r^m \sum_{N_0} \sum_{l\neq 0} d_{i,j,l} + r^{m-1} \sum_{N_1} \sum_{l\neq 0} d_{i,j,l} + (r-1)\times r^{m-1}\big(\#(N_1)\big)$$
$$- \sum_{N_1} r^{m-1} \sum_{l\neq 0} d_{i,j,l}\,.$$

$$\tau(\mathbf{c}_0) = \sum_{N} \sum_{l\neq 0} d_{i,j,l} \leq \sum_{N_0} \sum_{l\neq 0} d_{i,j,l} + \frac{r-1}{r}\#(N_1) \leq \left(1-\frac{1}{r}\right)\#(N_1) + \#(N_0).$$

gives an upper bound for the best solution of each solvable position vector **b.** As all the solvable vectors follow the inequality,

$$\gamma \leq \left(1-\frac{1}{r}\right)\#(N_1) + \#(N_0).$$

Example 3.6.

Consider the Lights Out 2000 puzzle measured in whole move metric. The movement matrix is exactly the same as in standard 5×5 Lights Out game:

$$M = \begin{bmatrix} 1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&1&1&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&1&1&1&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&1&1&1&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&1&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&1&1&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&1&1&1&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&1&1&1&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1 \end{bmatrix}$$

However the null space matrix Q is different in module 3:

$$
Q=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&2&2&1&2&1&2&0&1&2&0&1&1&2&1&1&2&2&1&1\\
0&0&2&0&2&0&1&1&2&1&2&1&0&2&1&0&2&2&1&2&2&1&1&2&2\\
0&1&0&2&0&2&2&0&1&1&2&1&0&2&1&1&1&0&2&2&2&0&0&0&1\\
0&1&1&2&1&2&1&2&2&0&0&0&0&0&0&1&2&1&1&0&0&2&2&1&2\\
0&1&2&2&2&2&0&1&0&2&1&2&0&1&2&1&0&2&0&1&1&1&1&2&0\\
0&2&0&1&0&1&1&0&2&2&1&2&0&1&2&2&2&0&1&1&1&0&0&0&2\\
0&2&1&1&1&1&0&2&0&1&2&1&0&2&1&2&0&1&0&2&2&2&2&1&0\\
0&2&2&1&2&1&2&1&1&0&0&0&0&0&0&2&1&2&2&0&0&1&1&2&1\\
1&0&0&0&2&2&2&0&1&1&1&2&0&1&2&1&1&0&2&2&0&2&0&1&0\\
1&0&1&0&0&2&1&2&2&0&2&1&0&2&1&1&2&1&1&0&1&1&2&2&1\\
1&0&2&0&1&2&0&1&0&2&0&0&0&0&0&1&0&2&0&1&2&0&1&0&2\\
1&1&0&2&2&1&1&0&2&2&0&0&0&0&0&2&2&0&1&1&2&2&0&1&1\\
1&1&1&2&0&1&0&2&0&1&1&2&0&1&2&2&0&1&0&2&0&1&2&2&2\\
1&1&2&2&1&1&2&1&1&0&2&1&0&2&1&2&1&2&2&0&1&0&1&0&0\\
1&2&0&1&2&0&0&0&0&0&2&1&0&2&1&0&0&0&0&0&1&2&0&1&2\\
1&2&1&1&0&0&2&2&1&2&0&0&0&0&0&0&1&1&2&1&2&1&2&2&0\\
1&2&2&1&1&0&1&1&2&1&1&2&0&1&2&0&2&2&1&2&0&0&1&0&1\\
2&0&0&0&1&1&1&0&2&2&2&1&0&2&1&2&2&0&1&1&0&1&0&2&0\\
2&0&1&0&2&1&0&2&0&1&0&0&0&0&0&2&0&1&0&2&1&0&2&0&1\\
2&0&2&0&0&1&2&1&1&0&1&2&0&1&2&2&1&2&2&0&2&2&1&1&2\\
2&1&0&2&1&0&0&0&0&0&1&2&0&1&2&0&0&0&0&0&2&1&0&2&1\\
2&1&1&2&2&0&2&2&1&2&2&1&0&2&1&0&1&1&2&1&0&0&2&0&2\\
2&1&2&2&0&0&1&1&2&1&0&0&0&0&0&0&2&2&1&2&1&2&1&1&0\\
2&2&0&1&1&2&2&0&1&1&0&0&0&0&0&1&1&0&2&2&1&1&0&2&2\\
2&2&1&1&2&2&1&2&2&0&1&2&0&1&2&1&2&1&1&0&2&0&2&0&0\\
2&2&2&1&0&2&0&1&0&2&2&1&0&2&1&1&0&2&0&1&0&2&1&1&1
\end{bmatrix}
$$

There are 1 empty column and 24 nonempty columns. As all the nonempty columns contains a 1, which is a unit, from Theorem 3.5,

$$\gamma \leq \frac{2}{3}\#(N_1) + \#(N_0) = \frac{2}{3} \times 24 + 1 = 17.$$

Now we give another theorem for separate move metric.

Theorem 3.7 (for separate move metric)

Let Q be the matrix as in Lemma 3.3. Denote the set of all the columns by N, the set of columns with some nonzero entries by N_1 and the set of columns with all the entries equals to zero by N_0. Suppose that each column in N_1 contains a unit, then:

$$\gamma \leq (r-1)\left[\#(N_0) + \frac{1}{2}\#(N_1)\right].$$

Proof. For every solution $\mathbf{c} = (c_{11}, \dots, c_{ij}, \dots, c_{nn})$, we can treat it as some pattern in the following table:

$d_{1,1,0}$	$d_{1,2,0}$	…	$d_{n,n,0}$
$d_{1,1,1}$	$d_{1,2,1}$		$d_{n,n,1}$
…	…		…
$d_{1,1,r-1}$	$d_{1,2,r-1}$	…	$d_{n,n,r-1}$

with $d_{i,j,l} = \begin{cases} 1, c_{ij} = l \\ 0, \text{otherwise} \end{cases}$, and

$$\sum_l d_{i,j,l} = 1 \ \forall i,j \in [n].$$

To calculate the minimal solution, we assume that for a solvable vector $\mathbf{b}$, there is a solution $\mathbf{c}_0 = (c_{11}, \dots, c_{ij}, \dots, c_{nn})$ such that

$$\gamma_b = \tau(\mathbf{c}_0).$$

In table form, this means that

$$\gamma_b = \tau(\mathbf{c}_0) = \sum_{\substack{i,j,l \\ l \neq 0}} l \times d_{i,j,l}.$$

with the weight of $d_{i,j,l} = l$.

For any null vector $\mathbf{v}$, we denote the positions by $s_k(\mathbf{v})$ if the number is k. For any other solution $\mathbf{c}$, $\mathbf{c} = \mathbf{c}_0 + \mathbf{v}$ for some null vector $\mathbf{v}$. Viewing this in table form, from $\tau(\mathbf{c}_0) = \gamma_b \leq \tau(\mathbf{c})$ we can see that

$$\sum_{\substack{i,j,l \\ l \neq 0}} l \times d_{i,j,l} \leq \sum_{\substack{i,j \in s_0(\mathbf{v}) \\ l \neq 0}} l \times d_{i,j,l} + \sum_{\substack{i,j \in s_1(\mathbf{v}) \\ l \neq r-1}} \phi_1(l) \times d_{i,j,l} + \dots + \sum_{\substack{i,j \in s_{r-1}(\mathbf{v}) \\ l \neq 1}} \phi_{r-1}(l) \times d_{i,j,l},$$

where $\phi_t(l) = \begin{cases} l+t, & l < r-t \\ l+t-r, & l \geq r-t \end{cases}$.

Since the entries changes only in $s_r(\mathbf{v})$ while $r \neq 0$. For each null vector $\mathbf{v}$, we can get one inequality, and sum over all the inequalities gives the result:

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} l \times d_{i,j,l} \leq \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} l \times d_{i,j,l}$$
$$+ \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_1(\mathbf{v}) \\ l\neq r-1}} \phi_1(l) \times d_{i,j,l} + \cdots + \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_{r-1}(\mathbf{v}) \\ l\neq 1}} \phi_{r-1}(l) \times d_{i,j,l}\,.$$

Since

$$\sum_{l} d_{i,j,l} = 1 \ \forall i,j \in [n]$$

we can change the inequality to

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} l \times d_{i,j,l} \leq \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} l \times d_{i,j,l}$$
$$+ \sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{t\neq 0} \sum_{\substack{i,j\in s_t(\mathbf{v}) \\ l\neq r-t}} \phi_t(l) \times d_{i,j,l}\,.$$

From Lemma 3.3, by assumption, for each nonempty column, the column consists equal amount of numbers in $\{0, \ldots, r-1\}$.Changing the summation from rows to columns leads to:

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j,l \\ l\neq 0}} l \times d_{i,j,l} = r^m \sum_{N} \sum_{l\neq 0} l \times d_{i,j,l}\,.$$
$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_0(\mathbf{v}) \\ l\neq 0}} l \times d_{i,j,l} = r^m \sum_{N_0} \sum_{l\neq 0} l \times d_{i,j,l} + r^{m-1} \sum_{N_1} \sum_{l\neq 0} l \times d_{i,j,l}\,.$$

And for each t, $1 \leq t \leq r-1$,

$$\sum_{\mathbf{v}\in\mathrm{Null}(M)} \sum_{\substack{i,j\in s_t(\mathbf{v}) \\ l\neq r-t}} \phi_t(l) \times d_{i,j,l} = r^{m-1} \sum_{N_1} \sum_{l\neq r-t} \phi_t(l) \times d_{i,j,l}.$$

Then

$$
\begin{aligned}
\sum_{N_1}\sum_{t\neq 0}\sum_{l\neq r-t}\phi_t(l)\times d_{i,j,l} &= \sum_{N_1}\sum_{l}\sum_{t\neq r-l}\phi_t(l)\times d_{i,j,l}\\
&= \sum_{N_1}\sum_{l} d_{i,j,l}\sum_{t\neq r-l}\phi_t(l)\\
&= \sum_{N_1}\sum_{l} d_{i,j,l}\left(\sum_{t=1}^{r-l-1}(l+t)+\sum_{t=r-l+1}^{r-1}(l+t-r)\right)\\
&= \sum_{N_1}\sum_{l} d_{i,j,l}\times\frac{1}{2}\big((r-l-1)(l+r)+(l-1)l\big)\\
&= \sum_{N_1}\sum_{l} d_{i,j,l}\times\frac{1}{2}(r^2-r-2l)\\
&= \sum_{N_1}\left[\frac{1}{2}(r^2-r)\sum_{l} d_{i,j,l}-\sum_{l} l\times d_{i,j,l}\right]\\
&= \frac{1}{2}(r^2-r)(\#(N_1))-\sum_{N_1}\sum_{l} l\times d_{i,j,l}\,.
\end{aligned}
$$

The inequality becomes

$$
\begin{aligned}
r^m\sum_{Q}\sum_{l\neq 0} l\times d_{i,j,l} \le r^m\sum_{N_0}\sum_{l\neq 0} l\times d_{i,j,l}+r^{m-1}\sum_{N_1}\sum_{l\neq 0} l\times d_{i,j,l}\\
+\frac{r^{m-1}}{2}(r^2-r)(\#(N_1))-r^{m-1}\sum_{N_1}\sum_{l} l\times d_{i,j,l}\,.
\end{aligned}
$$

$$
\begin{aligned}
\tau(\mathbf{c}_0)=\sum_{Q}\sum_{l\neq 0} l\times d_{i,j,l} &\le \sum_{N_0}\sum_{l\neq 0} l\times d_{i,j,l}+\frac{1}{2}(r-1)(\#(N_1))\\
&\le (r-1)\left[\#(N_0)+\frac{1}{2}\#(N_1)\right].
\end{aligned}
$$

gives an upper bound for the best solution of each solvable position vector **b.** As all the solvable vectors follow the inequality,

$$
\gamma \le (r-1)\left[\#(N_0)+\frac{1}{2}\#(N_1)\right].
$$

Example 3.8

Consider the Lights Out 2000 puzzle measured in separate move metric. The movement matrix M and null space matrix Q are the same as in Example 3.6.

Then, from Theorem 3.7,

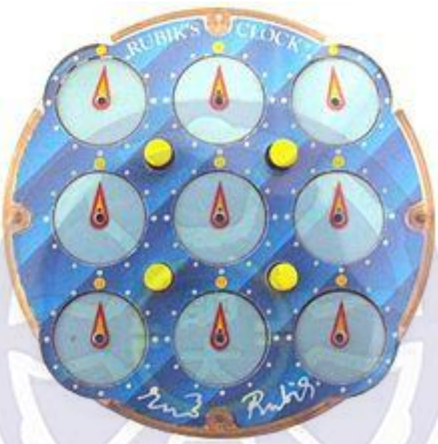
$$\gamma \leq (3-1)\left[\#(N_0)+\frac{1}{2}\#(N_1)\right] = 2\left(1+\frac{1}{2}\times 24\right) = 26.$$

We will show some more variations in Chapter 4.

4. Some other commutative puzzles

Rubik's Clock

Rubik's Clock is a puzzle invented and patented by Christopher C. Wiggs and Christopher J. Taylor. The inventor of Rubik's Cube, Ernő Rubik, who is a Hungarian sculptor and professor of architecture, bought the patent and market the product under his name in 1988.



It has two sides, each contains nine small clocks. Four wheels, one at each corner of the puzzle, is able to twist and affect the small clocks. Four buttons in the middle can be either pressed or not pressed. Whenever some button is up, and if the corresponding wheel on the corner twisted, the adjacent clocks of the button also rotates, and the corners on the back side rotates as well.

The goal of the puzzle is to set all the small clocks to 12 o'clock. This puzzle is also commutative since the order that button pressed and wheel rotated does not affect the state at the end.

With the theorems in chapter 3, we can calculate the minimal solution of it.

The Rubik's Clock is of module 12. Since there are four wheels, four buttons and 2 way of wheel rotating depend on the buttons (rotate the wheels of pressed button or unpressed button), there are $4 \times 4 \times 2 - 2 = 30$ movements (With 2 case of all button pressed or unpressed)

Thus the movement matrix is as follow:

$$M = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

for the null space, there is a basis as shown in the rows of R:

$$R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 1 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 0
\end{bmatrix}$$

From Theorem 3.2, there are 12^{16} null vectors in total. From the null basis, in the null vector matrix Q, each column contains at least a unit 1. Measured in whole turn metric, from Theorem 3.5,

$$\gamma \leq \frac{11}{12} \times 30 = 27.5.$$

Remark. On n-ary models, as n becomes large, Theorem 3.5 may not be appropriate. As on Rubik's Clock, since the nullity is 16, all the solvable position vectors can be solved in $30 - 16 = 14$ moves. However, in 2014, Jakob Kogler found the upper bound of γ is 12 with computer. [4]

Gear Shift

Gear Shift is a cube shaped puzzle invented by Oskar van Deventer, who lives in Leidschendam, the Netherlands. The cube consists eight corners that are gear-shaped. Four bigger corners have gears with 8 teeth and four smaller one have gears with 5 teeth. Whenever a corner is twisted, all the same sized corner twisted in the same direction, and the different sized ones twist in another direction. One can also pulled apart the puzzle to two opposite faces, so they become independent systems. In this way you can twist four corners on one face, independently of the rest corners.

On Gear Shift cube, there are 7 possible movements, including one 8 corners twist and six 4 corners twist on different faces. Thus the movement matrix is

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

And the null space is of nullity 3, with a basis shown in the rows of R:

$$R = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

On Gear Shift cube, the large corners are in module 8 while the smaller ones in module 5. Thus it can be viewed as in module 40, and therefore there are $40^3 = 64000$ null vectors. By the basis, there is no empty columns in the null space matrix Q, and each column contains a unit 1. Measured in separate cube metric, by Theorem 3.7,

$$\gamma \leq 39\left(\frac{1}{2} \times 7\right) = 136.5.$$

which means it can always be solved in 136 teeth.

Remark. When measured in whole cube metric, there is a simple upper bound from the nullity 3, $\gamma \leq 7 - 3 = 4$.

5. Conclusion

In this paper we describe the method to find the upper bounds, but for the lower bounds, there is no general way to find it. It relies on computer search or even case by case. On n-ary models, the upper bound is not "tight" enough when $n \geq 3$ (we have seen it on Rubik's Clock). Also on non-commutative puzzles, there is no systematic proof such as lit-only Lights Out games. All those things are the targets in the future.

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