A THEOREM ON PRECISE R-ORDER OF CONVERGENCE FOR ITERATIVE PROCEDURES

GWO-DONG LIN

(在重複程序中一個關於精確R一收斂次數的定理)

(林國棟)

Abstract. This paper gives a theorem on precise R-order of convergence of iterative procedures for finding a zero of a nonlinear function defined on R^n. Some special cases were given by J. Ortega and W. Rheinboldt [1] and Robert G. Voigt [2].

1. Definition and Lemma

Let F be a function defined from R^n into R^n. Assume that $\bar{x}$ is a zero of $F(x)$ and $C(I, \bar{x})$ is the set of all sequences generated by the iterative process I with limit $\bar{x}$

$$I : x^{k+1} = G(x^k, x^{k-1}, \cdots, x^{k-\ell+1}), k = \ell-1, \ell, \cdots,$$

where $\ell \geqq 1$ and G is a function defined from $(R^n)^\ell$ into R^n
From now on, we let $\|\cdot\|$ be an arbitrary norm on R^n.

DEFINITION 1.1. We define the R-factors of a sequence $\{x^k\} \in C(I, \bar{x})$ by

$$R_t\{x^k\} = \begin{cases} \limsup\limits_{k\to\infty} \|x^k - \bar{x}\|^{\frac{1}{k}}, \text{ if } t = 1, \\ \limsup\limits_{k\to\infty} \|x^k - \bar{x}\|^{\frac{1}{t^k}}, \text{ if } t > 1; \end{cases}$$

the R-factors of I at $\bar{x}$ by

$$R_t(I, \bar{x}) = \sup\{R_t\{x^k\} \mid \{x^k\} \in C(I, \bar{x})\}, \ 1 \leqq t < \infty;$$

and the R-order of I at $\bar{x}$ by

$$O_R(I, \bar{x}) = \begin{cases} \infty, \text{ if } R_t(I, \bar{x}) = 0, \text{ for every } t \in [1, \infty), \\ \inf\{t \in [1, \infty) \mid R_t(I, \bar{x}) = 1\}, \text{ ortherwise.} \end{cases}$$

LEMMA 1.2. Let I be an iterative process with limit $\bar{x}$ Then

(i) If $R_t(I, \bar{x}) < 1$ for some $t \in [1, \infty)$, then $O_R(I, \bar{x}) \geqq t$;

(ii) If $R_s(I, \bar{x}) > 0$ for some $t \in [1, \infty)$, then $O_R(I, \bar{x}) \leqq s$.

PROOF: The proof was given by Ortega and Rheinboldt[1] in page 290.

2. The main result

We shall prove the following theorem,

THEOREM. Let $C(I, \bar{x})$ be defined as above, $p \in (1, \infty)$ and $m \in \{\ell, \ell+1, \cdots\cdots\}$. Then

(a) If, for any sequence $\{x^k\} \in C(I, \bar{x})$, there exists $m+1$ nonnegative constants $r_0, r_1, \cdots, r_m$, and $k_0 \geqq m$ such that

$$\|x^{k+1} - \bar{x}\| \leqq \|x^k - \bar{x}\| \sum_{j=0}^{m} r_j \|x^{k-j} - \bar{x}\|^p \text{ for every } k \geqq k_0, \quad (1)$$

then $O_R(I, \bar{x}) \geq \tau$, where τ is the unique positive root of $t^{m+1} - t^m - p = 0$.

(b) If, in addition, there exists a $\beta > 0$ and some sequence $\{x^k\} \in C(I, \bar{x})$ such that

$$\|x^{k+1} - \bar{x}\| \geqq \beta \|x^k - \bar{x}\| \; \|x^{k-m} - \bar{x}\|^p > 0, \text{ for every } k \geqq k_0, \quad (2)$$

then $O_R(I, \bar{x}) = \tau$.

PROOF. (A) Let us fix a $\eta \in (0, 1)$. Without loss of generality, we may assume that $k_0 = m$, the m+1 nonnegative constants are not all zero for any sequence $\{x^k\} \in C(I, \bar{x})$ and

$$\|x^i - \bar{x}\| \leqq \eta^p r^{-1/p}, \text{ for every } i = 0, 1, ---, m,$$

where $r = \sum_{j=0}^{m} r_j > 0$

Set

$$\epsilon_k = \|x^k - \bar{x}\|, \quad \text{for } k = 0, 1, 2, ---.$$

Define

$$\sigma_{k+1} = \sigma_k + p\,\sigma_{k-m}, \qquad k \geqq m\ ;$$
$$\sigma_i = p\,, \qquad i = 0, 1, 2, \cdots\cdots, m.$$

Then we have the following inequalities by induction.

$$\epsilon_k \leq \eta^{\sigma_k} r^{-1/p}, \qquad k = 0, 1, 2, \cdots\cdots \tag{3}$$

In fact, by definition, (3) holds for $k=0, 1, \cdots, m$, and if it is valid for $k=0, 1, 2, \cdots\cdots, q$, $(q \geqq m)$, then

$$\epsilon_{q+1} \leqq \epsilon_q \sum_{i=0}^{m} r_j \epsilon_{p-j} \qquad (\text{by } (1))$$
$$= \epsilon_q \sum_{j=0}^{m} (r_j/r)(r^{1/p} \epsilon_{q-j})^p$$
$$\leqq \eta^{\sigma_q} r^{-1/p} \sum_{j=0}^{m} (r_j/r) \eta^{p\sigma_{q-j}}$$
$$\leqq \eta^{\sigma_q} r^{-1/p} \sum_{j=0}^{m} (r_j/r) \eta^{p\sigma_{q-m}}$$
$$= r^{-1/p} \eta^{\sigma_q + p\sigma_{q-m}}$$
$$= r^{-1/p} \eta^{\sigma_{q+1}}$$

So we have shown that (3) holds.

We show next that the σ_k satisfy

$$\sigma_k \geqq \alpha \tau^k, \text{ for } k = 0, 1, 2, \cdots\cdots, \tag{4}$$

where $\alpha = p\tau^{-m} > 0$. In fact, since $\tau > 1$, (4) holds for $k = 0$, $1, 2, \cdots\cdots, m$, and if it is valid for $k = 0, 1, 2, \cdots\cdots, q$, ($(q \geqq m)$,

then

$$\sigma_{q+1} = \sigma_q + p\sigma_{q-m}$$
$$\geqq \alpha\tau^q + p\alpha\tau^{q-m}$$
$$= \alpha\tau^{q-m}(\tau^m + p)$$
$$= \alpha\tau^{q-m}\tau^{m+1}$$
$$= \alpha\tau^{q+1}$$

So we have shown that (4) holds by induction.

From (3) and (4), we have

$$\lim_{k\to\infty} \sup \epsilon_k^{1/\tau^k} \leqq \lim_{k\to\infty} \sup (\eta^{\sigma_k} r^{-1/p})^{1/\tau^k}$$

$$= \limsup_{k \to \infty} \eta^{\sigma_k/\tau^k} r^{-1/(p\tau^k)}$$

$$\leqq \eta^{\alpha} < 1 \qquad\qquad \geqq \tau$$

Therefore $R_{\tau}(I, \bar{x}) < 1$, and $O_R(I, \bar{x})$ by the lemma in section 1.

This proves first part of the theorem.

(B) With the condition in (b), we may assume that

$$0 < v = \min \{\beta^{1/p}\epsilon_0, \beta^{1/p}\epsilon_1, \cdots, \beta^{1/p}\epsilon_m\} < 1,$$

where $\epsilon_0, \epsilon_1, \cdots, \epsilon_m$ are defined as in (A).

Define

$$u_{k+1} = u_k + p u_{k-m}, \text{ if } k \geqq m;$$
$$u_i = 1, \qquad \text{if } i = 0, 1, \cdots, m.$$

Then we show that the u_k satisfy

$$\beta^{1/p}\epsilon_k \geqq v^{u_k}, \text{ for } k = 0, 1, 2, \cdots\cdots \tag{5}$$

In fact, by definition, (5) holds for $k = 0, 1, \cdots, m$, and if it is valid for $k = 0, 1, \cdots, q$, $(q \geqq m)$, then, from (2), we have

$$\begin{aligned}\beta^{1/p}\epsilon_{q+1} &\geqq \beta^{1/p}(\beta\epsilon_q\ \epsilon^p_{q-m})\\ &= (\beta^{1/p}\epsilon_q)(\beta^{1/p}\epsilon_{q-m})^p\\ &\geqq v^{u_q} v^{p u_{q-m}}\\ &= v^{u_{q+1}}\end{aligned}$$

Now we claim that

$$u_k \leqq \tau^k, \text{ for } k = 0, 1, 2, \cdots\cdots. \tag{6}$$

Since $\tau > 1$, (6) holds for $k=0,1,\cdots,m$, and if it is valid for $k=0,1,2,\cdots,q$, $(q \geqq m)$, then

$$
\begin{aligned}
u_{q+1} = u_q + p u_{q-m} &\leqq \tau^{q} + p\tau^{q-m} \\
&= \tau^{q+1}(\tau^{-1} + p\tau^{-m-1}) \\
&= \tau^{q+1}
\end{aligned}
$$

So we have shown that (6) holds by induction.

From (5) and (6), we have

$$
\begin{aligned}
\limsup_{k\to\infty} \| x^k - \bar{x} \|^{1/\tau^k} &= \limsup_{k\to\infty} \epsilon_k^{1/\tau^k} \\
&= \limsup_{k\to\infty} (\beta^{-1/p})^{1/\tau^k} (\beta^{1/p}\epsilon_k)^{1/\tau^k} \\
&\geqq \limsup_{k\to\infty} (\beta^{-1/p})^{1/\tau^k} v^{u_k/\tau^k} \\
&\geqq v > 0
\end{aligned}
$$

Therefore $R_\tau(I,\bar{x}) > 0$, and $O_R(I,\bar{x}) \leqq \tau$ by the lemma in section 1. Together with (A), we have shown that $O_R(I,\bar{x}) = \tau$

We have finished the proof of the theorem.

REMARK. In above theorem, we have two special cases. If p=1, it is the result of Ortega and Rheinboldt [1] on pages 291-293. If p=2, it is the result of Robert G. Voigt [2] on page 227-228.

REFERENCES

[1] J. Ortega and W. Rheinboldt. Iterative solution of nonlinear equations in several variables. Acadmic press. New York 1970.

[2] Robert G. Voigt. Orders of convergence for iterative procedures. SIAM J. Numerical Analysis, Vol. 8, no. 2, June 1971. pp. 222-243.