## 2 The First Form : $X^aY^b + Y^cZ^d$

Let $K$ be a field of characteristic $p > 0$ and

$$S = K\left[X_1, \ldots, X_r, Y_1, \ldots, Y_s, Z_1, \ldots, Z_t\right].$$

In this section we shall determine the Hilbert-Kunz function of the hypersurface of the following form :

$$f := X^aY^b + Y^cZ^d$$

where $X^a = X_1^{a_1} \ldots X_r^{a_r}$, $Y^b = Y_1^{b_1} \ldots Y_s^{b_s}$, $Y^c = Y_1^{c_1} \ldots Y_s^{c_s}$, $Z^d = Z_1^{d_1} \ldots Z_t^{d_t}$, and $r \geq 1$. Let $q = p^n$, $J = \left\{j \mid b_j > c_j\right\}$, and set $R = S/ < f >$. Then $0 \leq |J| := m \leq s$, and *w.l.o.g.*, we assume that $b_1 > c_1, \ldots, b_m > c_m, b_{m+1} \leq c_{m+1}, \ldots, b_s \leq c_s$. We shall determine the assignment

$$HK_R(q) := dim_K\Big( S/ < X_1^q, \ldots, X_r^q, Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, f > \Big).$$

Let $f + X^{[q]}$ be the ideal of $S$ generated by all $X_i^q$'s, $Y_j^q$'s, $Z_k^q$'s, and $f$. Fix a term order on $S$, and denote by $in\big(f + X^{[q]}\big)$ the initial ideal of $f + X^{[q]}$. Then by Lemma 1.1, we get that $HK_R(q)$ is equal to $dim_K\Big(S/in\big(f + X^{[q]}\big)\Big)$. By making use of Gröbner basis, we understand which monomials one has to add to fill the gap between $in\big(f+X^{[q]}\big)$ and the ideal $in(f)+X^{[q]}$.

Throughout this section, it is not restrictive to assume that $a_1 \geq a_2 \geq \cdots \geq a_r > 0$ , $b_1 - c_1 \geq b_2 - c_2 \geq \cdots \geq b_m - c_m > 0$ , $c_{m+1} - b_{m+1} \geq c_{m+2} - b_{m+2} \geq \cdots \geq c_s - b_s \geq 0$, and $d_1 \geq d_2 \geq \cdots \geq d_t > 0$. Let $u$ be the maximum of the integers $a_1$ , $b_1 - c_1$ , $c_{m+1} - b_{m+1}$, and $d_1$; that is, $u$ is the greatest integer among all $a_i$'s, $(b_j - c_j)$'s, $(c_h - b_h)$'s, and $d_k$'s. We also denote by $[y]$ the greatest integer less than or equal to $y$, and $S_{in}(x)$ the elementary symmetric polynomial of degree $i$ in $n$ indeterminates $x = (x_1, \ldots, x_n)$. Let $I_q$ be the ideal $f + X^{[q]}$ and define $(v)_+ = max\left\{0, v\right\}$.

In order to make it more easy to determine the Hilbert-Kunz function of $R$, we shall prove the following lemma.

**Lemma 2.1.** *Let $S = K[X_1, \ldots, X_r, Y_1, \ldots, Y_s]$, $r \geq 1$, $s \geq 1$, and the $a_i$, $b_j$, and $c_j$ are all positive integers with $b_1 - c_1 \geq b_2 - c_2 \geq \cdots \geq b_m - c_m > 0$, $c_{m+1} - b_{m+1} \geq c_{m+2} - b_{m+2} \geq \cdots \geq c_s - b_s \geq 0$. We denote by $G$ the ideal generated by*

$$X_1^q, \ldots, X_r^q, Y_1^q, \ldots, Y_s^q, X_1^{[q-\alpha a_1]_+} Y^{e+\alpha(c-b)_+}, \ldots, X_r^{[q-\alpha a_r]_+} Y^{e+\alpha(c-b)_+},$$

$$Y_1^{[q-\alpha(b_1-c_1)-c_1]_+} Y^{e+\alpha(c-b)_+}, \ldots, Y_m^{[q-\alpha(b_m-c_m)-c_m]_+} Y^{e+\alpha(c-b)_+}, \quad \text{and} \quad X^a Y^b,$$

*where $\alpha$ is a positive integer, $e = (e_1, \ldots, e_s)$, $e_1 = c_1, \ldots, e_m = c_m, e_{m+1} = b_{m+1}, \ldots, e_s = b_s$, $(c-b)_+ = (0, \ldots, 0, c_{m+1} - b_{m+1}, \ldots, c_s - b_s)$. Then the dimension of $S/G$ is equal to*

$$\begin{aligned} q^{r+s} - \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) - q^r \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h]_+ \\ + \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h]_+ \\ + \prod_{i=1}^{r}(q-\alpha a_i)_+ \prod_{j=1}^{m} [q - \alpha(b_j - c_j) - c_j\,]_+ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h]_+ \\ - \prod_{i=1}^{r} [q - (\alpha+1)a_i\,]_+ \prod_{j=1}^{m} [q - (\alpha+1)(b_j - c_j) - c_j\,]_+ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h]_+ . \end{aligned}$$

**Proof :** If $[q - \alpha a_i]_+ = 0$ for some $i$ or $[q - \alpha(b_j - c_j) - c_j\,]_+ = 0$ for some $j$, then $G$ is generated by

$$X_1^q, \ldots,\ X_r^q,\ Y_1^q, \ldots,\ Y_s^q,\ Y^{e+\alpha(c-b)_+}, \quad \text{and} \quad X^a Y^b.$$

Hence,

$$\begin{aligned} & dim_K\big(S/G\big) \\ & = q^{r+s} - \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) - q^r \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h]_+ \\ & \quad + \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h]_+ \ . \end{aligned}$$

From now on, we assume $q - \alpha a_i > 0$ for each $i$ and $q - \alpha(b_j - c_j) - c_j > 0$ for each $j$. Let $l_\alpha$ be the minimum of

$$\left\{ \left[ \frac{q - \alpha(b_j - c_j) - 1}{c_j} \right], \left[ \frac{q-1}{b_h + \alpha(c_h - b_h)} \right] \ \middle|\ j = 1, \ldots, m,\ h = m+1, \ldots, s \right\}.$$

Then we have $q-\alpha(b_{j_0}-c_{j_0}) \leq (l_\alpha+1)c_{j_0}$ for some $j_0$ with $1 \leq j_0 \leq m$ or $q \leq (l_\alpha+1)(b_{h_0}+\alpha(c_{h_0}-b_{h_0}))$ for some $h_0$ with $m+1 \leq h_0 \leq s$, and $q-\alpha(b_j-c_j)-l_\alpha c_j \geq 1$ for each $j$ and $q-l_\alpha(b_h+\alpha(c_h-b_h)) \geq 1$ for each $h$.
We consider the ideals $G_\beta = G : Y^{\beta\left[e+\alpha(c-b)_+\right]}$, for $\beta = 0,1,2,\ldots,l_\alpha+1$. Since $G_0 = G$, $G_{l_\alpha+1} = S$, and $G_{\beta+1} = G_\beta : Y^{e+\alpha(c-b)_+}$, we have the exact sequence of K-modules :

$$0 \longrightarrow S/G_{\beta+1} \overset{Y^{e+\alpha(c-b)_+}}{\longrightarrow} S/G_\beta \longrightarrow S\ /<G_\beta, Y^{e+\alpha(c-b)_+}> \longrightarrow 0.$$

It follows that

$$dim_K\big(S/G\big) = dim_K\big(S/G_0\big) = \sum_{\beta=0}^{l_\alpha} dim_K\Big(S\ /<G_\beta, Y^{e+\alpha(c-b)_+}>\Big).$$

We determine $dim_K\Big(S\ /<G_\beta, Y^{e+\alpha(c-b)_+}>\Big)$ as follows :
For $\beta=0$, the ideal $<G_0, Y^{e+\alpha(c-b)_+}>$ is generated by

$$X_1^q,\ldots,\ X_r^q,\ Y_1^q,\ldots,\ Y_s^q,\ Y^{e+\alpha(c-b)_+},\quad \text{and}\quad X^aY^b.$$

Hence,

$$\begin{aligned}
&dim_K\Big(S\ /<G_0, Y^{e+\alpha(c-b)_+}>\Big)\\
&= q^{r+s} - \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\prod_{h=m+1}^{s}\left[q-\alpha(c_h-b_h)-b_h\right]_+\\
&\qquad + \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\prod_{h=m+1}^{s}\left[q-\alpha(c_h-b_h)-b_h\right]_+ .
\end{aligned}$$

For $1 \leq \beta \leq l_\alpha$, the ideal $G_\beta = G : Y^{\beta\left[e+\alpha(c-b)_+\right]}$ is generated by

$$X_1^{q-\alpha a_1},\ldots,X_r^{q-\alpha a_r},Y_1^{q-\alpha(b_1-c_1)-\beta c_1},\ldots,Y_m^{q-\alpha(b_m-c_m)-\beta c_m},Y_{m+1}^{q-\beta[b_{m+1}+\alpha(c_{m+1}-b_{m+1})]},\ldots,$$
$$Y_s^{q-\beta[b_s+\alpha(c_s-b_s)]},\ \text{and}\ \ X^aY_1^{(b_1-\beta c_1)_+}\cdots Y_m^{(b_m-\beta c_m)_+}Y_{m+1}^{[b_{m+1}-\beta(b_{m+1}+\alpha(c_{m+1}-b_{m+1}))]_+}\cdots$$
$$Y_s^{[b_s-\beta(b_s+\alpha(c_s-b_s))]_+}.$$

Hence, the ideal $<G_\beta, Y^{e+\alpha(c-b)_+}>$ is generated by

$$X_1^{q-\alpha a_1},\ldots,X_r^{q-\alpha a_r},Y_1^{q-\alpha(b_1-c_1)-\beta c_1},\ldots,Y_m^{q-\alpha(b_m-c_m)-\beta c_m},Y_{m+1}^{q-\beta[b_{m+1}+\alpha(c_{m+1}-b_{m+1})]},\ldots,$$
$$Y_s^{q-\beta[b_s+\alpha(c_s-b_s)]},\ X^aY_1^{(b_1-\beta c_1)_+}\cdots Y_m^{(b_m-\beta c_m)_+}Y_{m+1}^{[b_{m+1}-\beta(b_{m+1}+\alpha(c_{m+1}-b_{m+1}))]_+}\cdots$$
$$Y_s^{[b_s-\beta(b_s+\alpha(c_s-b_s))]_+},\ \text{and}\ \ Y^{e+\alpha(c-b)_+}.$$

Thus,

$$
\begin{aligned}
&dim_K\Big(S \;/ < G_\beta, Y^{e+\alpha(c-b)_+} >\Big)\\
&=\prod_{i=1}^{r}(q-\alpha a_i)\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j\;]_+\prod_{h=m+1}^{s}[q-\beta(b_h+\alpha(c_h-b_h))]_+ -\\
&\quad\prod_{i=1}^{r}[q-(\alpha+1)a_i]_+\prod_{j=1}^{m}\big[q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+\big]_+\prod_{h=m+1}^{s}[q-\beta(b_h+\alpha(c_h-b_h))]_+\\
&\quad-\prod_{i=1}^{r}(q-\alpha a_i)\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-(\beta+1)c_j\;]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\\
&\quad+\prod_{i=1}^{r}[q-(\alpha+1)a_i]_+\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\\
&=\prod_{i=1}^{r}(q-\alpha a_i)\times\Bigg\{\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j\;]_+\prod_{h=m+1}^{s}[q-\beta\,(b_h+\alpha\,(c_h-b_h))]_+\\
&\qquad\qquad-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-(\beta+1)c_j\;]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\Bigg\}\\
&\quad-\prod_{i=1}^{r}[q-(\alpha+1)a_i]_+\times\Bigg\{\prod_{j=1}^{m}\big[q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+\big]_+\prod_{h=m+1}^{s}[q-\beta(b_h+\alpha(c_h-\\
&\qquad b_h))]_+-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\Bigg\},
\end{aligned}
$$

where $u_{j\beta}=max\{\;c_j\;,\;[b_j-\beta c_j\;]_+\;\}$.

Now, we have

$$
\begin{aligned}
&dim_K\big(S/G\big)\\
&=q^{r+s}-\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)-q^r\prod_{j=1}^{m}(q-c_j)\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
&\quad+\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
&\quad+\prod_{i=1}^{r}(q-\alpha a_i)\;\times\Bigg\{\sum_{\beta=1}^{l_\alpha}\Bigg[\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j\;]_+\prod_{h=m+1}^{s}[q-\beta(b_h+\alpha(c_h-b_h))]_+\\
&\qquad-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-(\beta+1)c_j\;]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\Bigg]\Bigg\}
\end{aligned}
$$

$$-\prod_{i=1}^{r}[q-(\alpha+1)a_i]_+ \times \Bigg\{ \sum_{\beta=1}^{l_\alpha} \Bigg[ \prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+]_+ \prod_{h=m+1}^{s}[q-\beta(b_h+$$

$$\alpha(c_h-b_h))]_+ - \prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}]_+ \prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\Bigg]\Bigg\}.$$

Let $(*)$ be the term

$$\sum_{\beta=1}^{l_\alpha}\Bigg[\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j\ ]_+ \prod_{h=m+1}^{s}[q-\beta(b_h+\alpha(c_h-b_h))]_+$$

$$-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-(\beta+1)c_j\ ]_+ \prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\Bigg].$$

Since $q-\alpha(b_{j_0}-c_{j_0}) \le (l_\alpha+1)c_{j_0}$ for some $j_0$ with $1 \le j_0 \le m$ or
$q \le (l_\alpha+1)\big(b_{h_0}+\alpha(c_{h_0}-b_{h_0})\big)$ for some $h_0$ with $m+1 \le h_0 \le s$, $(*)$ is equal to

$$\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ]_+ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+ .$$

Let $(**)$ be the term

$$\sum_{\beta=1}^{l_\alpha}\Bigg[\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+]_+ \prod_{h=m+1}^{s}[q-\beta(b_h+\alpha(c_h-b_h))]_+$$

$$-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}\ ]_+ \prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\Bigg].$$

Since

$$\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}\ ]_+ \prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+$$

$$=\prod_{j=1}^{m}\big[q-\alpha(b_j-c_j)-(\beta+1)c_j-(b_j-(\beta+1)c_j)_+\big]_+ \prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ ,$$

where $\beta=1,\ 2,\ \ldots,\ l_\alpha-1$, the term $(**)$ is equal to

$$\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+$$

$$-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-l_\alpha c_j-u_{jl_\alpha}\ ]_+ \prod_{h=m+1}^{s}[q-(l_\alpha+1)(b_h+\alpha(c_h-b_h))]_+ .$$

Since $q-\alpha(b_{j_0}-c_{j_0}) \leq (l_\alpha+1)c_{j_0}$ or $q \leq (l_\alpha+1)\big(b_{h_0}+\alpha(c_{h_0}-b_{h_0})\big)$, we have

$$\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-l_\alpha c_j-u_{jl_\alpha}\ ]_+=0 \quad \text{or} \quad \prod_{h=m+1}^{s}[q-(l_\alpha+1)(b_h+\alpha(c_h-b_h))]_+=0\ .$$

Thus, $(**)$ is equal to

$$\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\ .$$

So,

$$\begin{aligned}
&dim_K\big(S/G\big)\\
&=q^{r+s}-\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)-q^r\prod_{j=1}^{m}(q-c_j)\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
&\quad+\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
&\quad+\prod_{i=1}^{r}(q-\alpha a_i)_+\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
&\quad-\prod_{i=1}^{r}[q-(\alpha+1)\,a_i\ ]_+\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\ .
\end{aligned}$$

□

Since $u$ is the maximum of the integers among all $a_i$'s, $(b_j-c_j)$'s, $(c_h-b_h)$'s, and $d_k$'s, we have

$$\left[\tfrac{q-v}{u}\right]:=min\left\{\left[\tfrac{q-1}{a_i}\right],\left[\tfrac{q-c_j-1}{b_j-c_j}\right],\left[\tfrac{q-b_h-1}{c_h-b_h}\right],\left[\tfrac{q-1}{d_k}\right]\ \middle|\ \begin{matrix}1\leq i\leq r, & b_j-c_j>0, & c_h-b_h>0\\ 1\leq j\leq m, & m+1\leq h\leq s, & 1\leq k\leq t\end{matrix}\right\}$$

for $q\gg 0$, where $v=1$ or $1+c_j$ for some $j$ or $1+b_h$ for some $h$.
Let $l_u$ be the integer $\left[\frac{q-v}{u}\right]$, and $\epsilon$ be the remainder of $q-v$ divided by $u$. Then $l_u=\frac{q-v-\epsilon}{u}$ and one has $q-l_ua_i>0$, $q-l_u(b_j-c_j)-c_j>0$, $q-l_u(c_h-b_h)-b_h>0$, and $q-l_ud_k>0$ for all $i$, $j$, $h$, and $k$. On the other hand, by the definition of $l_u$, at least one of $[q-(l_u+1)a_i]_+$'s, $[q-(l_u+1)(b_j-c_j)-c_j\ ]_+$'s, $[q-(l_u+1)(c_h-b_h)-b_h]_+$'s, and $[q-(l_u+1)d_k]_+$'s must be zero.

**Proposition 2.2.** *Let* $f := X^aY^b + Y^cZ^d$. *Then*

$$
\begin{aligned}
HK_R(q) = q^{r+s+t} &- q^t \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) \\
&- q^r \prod_{j=1}^{m}(q-c_j)\times \\
&\left\{ \sum_{\alpha=1}^{l_u} \left[ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right] \times \left[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \right] \right\} \\
&+ \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\times \\
&\left\{ \sum_{\alpha=1}^{l_u} \left[ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right] \times \left[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \right] \right\} \\
&+ \sum_{\alpha=1}^{l_u} \left\{ \prod_{i=1}^{r}(q-\alpha a_i)\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ] \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h] \right. \\
&\quad \left. - \prod_{i=1}^{r}[q-(\alpha+1)a_i]_+ \prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j]_+ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h] \right\} \\
&\quad \times \left\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \right\},
\end{aligned}
$$

*where* $l_u$ *is the integer* $\left[\frac{q-v}{u}\right]$, *and* $0 \le m \le s$ .

**Proof :** Let $u$ be the maximum of the integers $a_1$, $b_1 - c_1$, $c_{m+1} - b_{m+1}$, and $d_1$. Let $<$ be the lexicographic order on $S$ and define

$$e_j = c_j \quad \text{for } j = 1, \dots, m \quad \text{and} \quad e_h = b_h \quad \text{for } h = m+1, \dots, s.$$

Then $X^aY^b$ is bigger than $Y^cZ^d$ and $Y^e = Y_1^{e_1} \dots Y_s^{e_s} = Y_1^{c_1} \dots Y_m^{c_m} Y_{m+1}^{b_{m+1}} \dots Y_s^{b_s}$.
We determine a Gröbner basis of the ideal

$$I_q = < X_1^q, \dots, X_r^q, Y_1^q, \dots, Y_s^q, Z_1^q, \dots, Z_t^q, f >,$$

by means of Buchberger's algorithm (Algorithm 1.9). By this algorithm, the elements

$$X_1^q, \dots, X_r^q, Y_1^q, \dots, Y_s^q, Z_1^q, \dots, Z_t^q, X_i^{(q-\delta a_i)_+} Y^{e+\delta(c-b)_+} Z^{\delta d},\ i = 1, \dots, r,\ \delta = 1, \dots, l,$$

$$Y_j^{[q-\delta(b_j-c_j)-c_j]_+} Y^{e+\delta(c-b)_+} Z^{\delta d},\ j = 1, \dots, m,\ \delta = 1, \dots, l, \text{ and } X^aY^b + Y^cZ^d,$$

form a Gröbner basis of the ideal $I_q$, where $l = \left[\frac{q-1}{d_1}\right]$. Thus, the ideal $in(I_q)$ is generated by

$$X_1^q, \dots, X_r^q, Y_1^q, \dots, Y_s^q, Z_1^q, \dots, Z_t^q, X_i^{(q-\delta a_i)_+} Y^{e+\delta(c-b)_+} Z^{\delta d},\ i = 1, \dots, r,\ \delta = 1, \dots, l,$$

$$Y_j^{[q-\delta(b_j-c_j)-c_j]_+} Y^{e+\delta(c-b)_+} Z^{\delta d},\ j = 1, \dots, m,\ \delta = 1, \dots, l, \text{ and } X^aY^b.$$

Now we have to compute the dimension of $S/in(I_q)$. In order to do this, we consider the ideals $K_\alpha = in(I_q) : Z^{\alpha d}$ for $\alpha = 0, 1, \ldots, l+1$, where $Z^{\alpha d} = Z_1^{\alpha d_1} \ldots Z_t^{\alpha d_t}$. Since $K_0 = in(I_q)$, $K_{l+1} = S$, and $K_{\alpha+1} = K_\alpha : Z^d$, we have the exact sequence of K-modules :

$$0 \longrightarrow S/K_{\alpha+1} \xrightarrow{Z^d} S/K_\alpha \longrightarrow S \,/ < K_\alpha, Z^d > \longrightarrow 0 \,.$$

It follows that

$$dim_K\big(S/in(I_q)\big) = dim_K\big(S/K_0\big) = \sum_{\alpha=0}^{l} dim_K\Big(S \,/ < K_\alpha, Z^d >\Big).$$

We compute $dim_K\Big(S \,/ < K_\alpha, Z^d >\Big)$ as follows :

For $\alpha = 0$, the ideal $< K_0, Z^d >$ is generated by

$$X_1^q, \ldots,\ X_r^q,\ Y_1^q, \ldots,\ Y_s^q,\ Z_1^q, \ldots,\ Z_t^q,\ X^a Y^b, \text{ and } Z^d.$$

Let $S_1 = K\,[X_1, \ldots, X_r, Y_1, \ldots, Y_s]$, and $S_2 = K\,[Z_1, \ldots, Z_t]$. Then

$$\begin{aligned}
&dim_K\Big(\ S \,/ < K_0, Z^d >\ \Big) \\
&= dim_K\Big(\ S_1 \,/ < X_1^q, \ldots, X_r^q, Y_1^q, \ldots, Y_s^q, X^a Y^b >\ \Big) \ \times\ dim_K\Big(\ S_2 \,/ < Z_1^q, \ldots, Z_t^q, Z^d >\ \Big) \\
&= \left[\ q^{r+s} - \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)\ \right] \times \left[\ q^t - \prod_{k=1}^{t}(q-d_k)\ \right] \\
&= q^{r+s+t} - q^t\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) - q^{r+s}\prod_{k=1}^{t}(q-d_k) + \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)\prod_{k=1}^{t}(q-d_k).
\end{aligned}$$

For $1 \le \alpha \le l$, the ideal $K_\alpha = in(I_q) : Z^{\alpha d}$ is generated by

$$X_1^q, \ldots, X_r^q, Y_1^q, \ldots, Y_s^q, Z_1^{q-\alpha d_1}, \ldots, Z_t^{q-\alpha d_t},\ X_i^{(q-\delta a_i)_+} Y^{e+\delta(c-b)_+} Z^{(\delta-\alpha)d},\ i = 1, \ldots, r,$$

$$\delta = 1, \ldots, l,\ Y_j^{[q-\delta(b_j-c_j)-c_j]_+} Y^{e+\delta(c-b)_+} Z^{(\delta-\alpha)d},\ j = 1, \ldots, m,\ \delta = 1, \ldots, l, \text{ and } X^a Y^b.$$

Hence, the ideal $< K_\alpha, Z^d >$ is generated by

$$X_1^q, \ldots, X_r^q, Y_1^q, \ldots, Y_s^q, Z_1^{q-\alpha d_1}, \ldots, Z_t^{q-\alpha d_t},\ X_1^{(q-\alpha a_1)_+} Y^{e+\alpha(c-b)_+}, \ldots,\ X_r^{(q-\alpha a_r)_+} Y^{e+\alpha(c-b)_+},$$

$$Y_1^{[q-\alpha(b_1-c_1)-c_1]_+} Y^{e+\alpha(c-b)_+}, \ldots,\ Y_m^{[q-\alpha(b_m-c_m)-c_m]_+} Y^{e+\alpha(c-b)_+},\ X^a Y^b, \text{ and } Z^d.$$

Let $S_1 = K[X_1, \ldots, X_r, Y_1, \ldots, Y_s]$, and $S_2 = K[Z_1, \ldots, Z_t]$. Then by Lemma 2.1, we have

$$\begin{aligned}
& dim_K\Big(S \;/ < K_\alpha, Z^d >\Big) \\
&= dim_K\Big(S_1 \;/ < X_1^q, \ldots, X_r^q, Y_1^q, \ldots, Y_s^q,\; X_1^{(q-\alpha a_1)_+} Y^{e+\alpha(c-b)_+}, \ldots,\; X_r^{(q-\alpha a_r)_+} Y^{e+\alpha(c-b)_+}, \\
&\qquad\qquad Y_1^{[q-\alpha(b_1-c_1)-c_1]_+} Y^{e+\alpha(c-b)_+}, \ldots,\; Y_m^{[q-\alpha(b_m-c_m)-c_m]_+} Y^{e+\alpha(c-b)_+},\; X^a Y^b >\Big) \times \\
&\quad dim_K\Big(S_2 \;/ < Z_1^{q-\alpha d_1}, \ldots, Z_t^{q-\alpha d_t}, Z^d >\Big) \\
&= \Bigg\{ q^{r+s} - \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) - q^r \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
&\quad + \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
&\quad + \prod_{i=1}^{r}(q-\alpha a_i)_+ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j\,]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
&\quad - \prod_{i=1}^{r}[q-(\alpha+1)a_i]_+ \prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \\
&\quad \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \Bigg\}.
\end{aligned}$$

Since $dim_K\big(S/in(I_q)\big)$ can be written as

$$dim_K\Big(S \;/ < K_0, Z^d >\Big) \;+\; \sum_{\alpha=1}^{l} dim_K\Big(S \;/ < K_\alpha, Z^d >\Big),$$

$$\begin{aligned}
& dim_K\big(S/in(I_q)\big) \\
&= q^{r+s+t} - q^t \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) - q^{r+s}\prod_{k=1}^{t}(q-d_k) + \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)\prod_{k=1}^{t}(q-d_k) \\
&\quad + \Bigg[ q^{r+s} - \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) \Bigg] \times \Bigg\{ \sum_{\alpha=1}^{l} \Bigg[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \Bigg] \Bigg\} \\
&\quad - \sum_{\alpha=1}^{l} \Bigg\{ q^r \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \Bigg\} \\
&\quad + \sum_{\alpha=1}^{l} \Bigg\{ \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \\
&\qquad \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \Bigg\}
\end{aligned}$$

$$+\sum_{\alpha=1}^{l}\Bigg\{\prod_{i=1}^{r}(q-\alpha a_i)_+\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+$$

$$-\prod_{i=1}^{r}[q-(\alpha+1)a_i]_+\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\Bigg\}$$

$$\times\Bigg\{\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\Bigg\}.$$

Let $(***)$ be the term

$$\Bigg[q^{r+s}-\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)\Bigg]\times\Bigg\{\sum_{\alpha=1}^{l}\Bigg[\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\Bigg]\Bigg\}.$$

Since $l=\left[\frac{q-1}{d_1}\right]$, we have $q\le(l+1)d_1$, and so $[q-(l+1)d_1]_+=0$. Hence, the term $(***)$ is equal to

$$q^{r+s}\prod_{k=1}^{t}(q-d_k)-\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)\prod_{k=1}^{t}(q-d_k).$$

It follows that

$$dim_K\big(S/in(I_q)\big)$$

$$=q^{r+s+t}-q^t\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)$$

$$-q^r\prod_{j=1}^{m}(q-c_j)\times$$

$$\Bigg\{\sum_{\alpha=1}^{l}\Bigg[\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\Bigg]\times\Bigg[\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\Bigg]\Bigg\}$$

$$+\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\times$$

$$\Bigg\{\sum_{\alpha=1}^{l}\Bigg[\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\Bigg]\times\Bigg[\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\Bigg]\Bigg\}$$

$$+\sum_{\alpha=1}^{l}\Bigg\{\prod_{i=1}^{r}(q-\alpha a_i)\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\,]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h\,]_+$$

$$-\prod_{i=1}^{r}[q-(\alpha+1)a_i]_+\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\Bigg\}$$

$$\times\Bigg\{\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\Bigg\}.$$

By the definition of $l_u$, we have

$$HK_R(q) = dim_K\big(S/in(I_q)\big) = q^{r+s+t} - q^t \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)$$

$$- q^r \prod_{j=1}^{m}(q-c_j)\times$$

$$\left\{ \sum_{\alpha=1}^{l_u} \left[ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right] \times \left[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \right] \right\}$$

$$+ \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)\times$$

$$\left\{ \sum_{\alpha=1}^{l_u} \left[ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right] \times \left[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \right] \right\}$$

$$+ \sum_{\alpha=1}^{l_u} \left\{ \prod_{i=1}^{r}(q-\alpha a_i) \prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ] \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h] \right.$$

$$\left. - \prod_{i=1}^{r}[q-(\alpha+1)a_i\ ]_+ \prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h] \right\}$$

$$\times \left\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ \right\}.$$

□

In order to make it more easy to observe the behavior of the Hilbert-Kunz function of $R$, we shall prove some lemmas.

**Lemma 2.3.** *Let* $a_i$ *and* $b_j$ *be all positive integers. Then*

$$q^{r+s+t} - q^t \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j)$$

$$= \Big(S_{1r}(a) + S_{1s}(b)\Big)q^{r+s+t-1} \ + \ (\ terms\ \ of\ \ degree\ \le\ r+s+t-2\ \ in\ \ q\ \ over\ Z\ ),$$

*where* $a = (a_1, \ldots, a_r)$, *and* $b = (b_1, \ldots, b_s)$ .

**Proof :** Let $a = (a_1, \ldots, a_r)$, and $b = (b_1, \ldots, b_s)$. Then

$$
\begin{aligned}
& q^{r+s+t} - q^t \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{s}(q-b_j) \\
&= q^{r+s+t} - q^t \left[\, q^r - S_{1r}(a)q^{r-1} + \sum_{i=2}^{r}(-1)^i S_{ir}(a)q^{r-i} \,\right] \times \\
&\qquad\qquad \left[\, q^s - S_{1s}(b)q^{s-1} + \sum_{j=2}^{s}(-1)^j S_{js}(b)q^{s-j} \,\right] \\
&= q^{r+s+t} - \Big[\, q^{r+s+t} - \left(S_{1r}(a) + S_{1s}(b)\right) q^{r+s+t-1} \\
&\qquad\qquad + (\text{ terms of degree } \le r+s+t-2 \text{ in } q \text{ over } Z \,)\Big] \\
&= \Big(S_{1r}(a) + S_{1s}(b)\Big) q^{r+s+t-1} + (\text{ terms of degree } \le r+s+t-2 \text{ in } q \text{ over } Z \,).
\end{aligned}
$$

□

**Lemma 2.4.** *Let* $a_i$, $b_j$, *and* $c_j$ *be all positive integers. Then*

$$
\begin{aligned}
& \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r \prod_{j=1}^{m}(q-c_j) \\
&= -\Big(S_{1r}(a) + S_{1m}(b' - c')\Big)\, q^{r+m-1} + (\textit{ terms of degree } \le r+m-2 \textit{ in } q \textit{ over } Z \,),
\end{aligned}
$$

*where* $a = (a_1, \ldots, a_r)$, *and* $b' - c' = (b_1 - c_1, \ldots, b_m - c_m)$ .

**Proof :** Let $a = (a_1, \ldots, a_r)$, $b' = (b_1, \ldots, b_m)$, and $c' = (c_1, \ldots, c_m)$. Then

$$
\begin{aligned}
& \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r \prod_{j=1}^{m}(q-c_j) \\
&= \left[\, q^r - S_{1r}(a)q^{r-1} + \sum_{i=2}^{r}(-1)^i S_{ir}(a)q^{r-i} \,\right] \times \left[\, q^m - S_{1m}(b')q^{m-1} + \sum_{j=2}^{m}(-1)^j S_{jm}(b')q^{m-j} \,\right] \\
&\quad - q^r \left[\, q^m - S_{1m}(c')q^{m-1} + \sum_{j=2}^{m}(-1)^j S_{jm}(c')q^{m-j} \,\right] \\
&= \Big[\, q^{r+m} - \Big(S_{1r}(a) + S_{1m}(b')\Big)\, q^{r+m-1} + (\text{ terms of degree } \le r+m-2 \text{ in } q \text{ over } Z) \,\Big] \\
&\quad - \Big[\, q^{r+m} - S_{1m}(c')q^{r+m-1} + (\text{ terms of degree } \le r+m-2 \text{ in } q \text{ over } Z \,) \Big] \\
&= -\Big(S_{1r}(a) + S_{1m}(b') - S_{1m}(c')\Big) q^{r+m-1} + (\text{ terms of degree } \le r+m-2 \text{ in } q \text{ over } Z \,) \\
&= -\Big(S_{1r}(a) + S_{1m}(b' - c')\Big) q^{r+m-1} + (\text{ terms of degree } \le r+m-2 \text{ in } q \text{ over } Z \,).
\end{aligned}
$$

□

**Lemma 2.5.**

$$\sum_{\alpha=1}^{l_u-1}\left\{\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)]\right\}\times\left\{\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]\right\}$$

$$=\left[\sum_{j=0,\ k=1}^{s-m,\ t}(-1)^{1+j+k}S_{j(s-m)}(c''-b'')S_{kt}(d)\ \frac{k}{(j+k)u^{j+k}}\right]q^{s-m+t}$$

$$+\ (\ terms\ \ of\ \ degree\ \le\ s-m+t-1\ in\ \ q\ \ over\ Q\,[\epsilon]\ ),$$

*where* $c''-b''=(c_{m+1}-b_{m+1},\dots,c_s-b_s)$, $d=(d_1,\dots,d_t)$, $S_{0(s-m)}(c''-b''):=1$, *and* $Q\,[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$ .

**Proof :** Suppose $d=(d_1,\dots,d_t)$, $c''-b''=(c_{m+1}-b_{m+1},\dots,c_s-b_s)$, and let

$$A=\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)]$$

$$=q^{s-m}-\alpha S_{1(s-m)}(c''-b'')q^{s-m-1}\ +\ \sum_{j=2}^{s-m}(-1)^j\alpha^jS_{j(s-m)}(c''-b'')q^{s-m-j},$$

$$B=\prod_{k=1}^{t}(q-\alpha d_k)=q^t-\alpha S_{1t}(d)q^{t-1}\ +\ \sum_{k=2}^{t}(-1)^k\alpha^kS_{kt}(d)q^{t-k},$$

$$C=\prod_{k=1}^{t}[q(\alpha+1)d_k]=q^t-(\alpha+1)S_{1t}(d)q^{t-1}\ +\ \sum_{k=2}^{t}(-1)^k(\alpha+1)^kS_{kt}(d)q^{t-k}\ .$$

Then

$$B-C\ =\ -\Big[\ \alpha-(\alpha+1)\ \Big]S_{1t}(d)q^{t-1}+\Big[\ \alpha^2-(\alpha+1)^2\ \Big]S_{2t}(d)q^{t-2}-\cdots$$

$$+(-1)^t\Big[\ \alpha^t-(\alpha+1)^t\ \Big]S_{tt}(d).$$

Thus,

$$\sum_{\alpha=1}^{l_u-1}A(B-C)=\sum_{\alpha=1}^{l_u-1}\Big\{\ \big[(\alpha+1)-\alpha\big]S_{1t}(d)q^{s-m+t-1}$$

$$-\Big[\big((\alpha+1)^2-\alpha^2\big)\,S_{2t}(d)+\alpha\,((\alpha+1)-\alpha)\,S_{1(s-m)}(c''-b'')S_{1t}(d)\Big]\,q^{s-m+t-2}$$

$$+\Big[\big((\alpha+1)^3-\alpha^3\big)\,S_{3t}(d)+\alpha\,\big((\alpha+1)^2-\alpha^2\big)\,S_{1(s-m)}(c''-b'')S_{2t}(d)$$

$$+\ \alpha^2\big((\alpha+1)-\alpha\big)S_{2(s-m)}(c''-b'')S_{1t}(d)\Big]\,q^{s-m+t-3}$$

$$-\cdots+(-1)^{s-m+t}\Big[\alpha^{s-m}\,\big((\alpha+1)^t-\alpha^t\big)\,S_{(s-m)(s-m)}(c''-b'')S_{tt}(d)\Big]\ \Big\}$$

$$=\sum_{j=0,\ k=1}^{s-m,\ t}\left[(-1)^{1+j+k}S_{j(s-m)}(c''-b'')S_{kt}(d)q^{s-m+t-j-k}\sum_{\alpha=1}^{l_u-1}W_{jk}(\alpha)\right],$$

where $S_{0(s-m)}(c^{''}-b^{''}):=1$, and $W_{jk}(\alpha)=\alpha^j\Big[(\alpha+1)^k-\alpha^k\Big]$, $0\le j\le s-m$, $1\le k\le t$.
Since

$$W_{jk}(\alpha)=\alpha^j\Big[(\alpha+1)^k-\alpha^k\Big]=k\alpha^{j+k-1}+C_2^k\alpha^{j+k-2}+\cdots,$$

it follows that

$$\begin{aligned}\sum_{\alpha=1}^{l_u-1}W_{jk}(\alpha)&=k\sum_{\alpha=1}^{l_u-1}\alpha^{j+k-1}+C_2^k\sum_{\alpha=1}^{l_u-1}\alpha^{j+k-2}+\cdots\\&=\frac{k}{j+k}\,l_u^{\,j+k}\ +\ (\text{ terms of degree }\le\ j+k-1\ \text{ in }\ l_u\ \text{ over } Q\ ).\end{aligned}$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{u}$ , we obtain the following expression

$$\begin{aligned}\sum_{\alpha=1}^{l_u-1}W_{jk}(\alpha)&=\frac{k}{j+k}\left(\frac{q-v-\epsilon}{u}\right)^{j+k}+\left(\text{terms of degree }\le j+k-1\ \text{ in }\ \frac{q-v-\epsilon}{u}\ \text{ over } Q\right)\\&=\frac{k}{(j+k)\ u^{j+k}}\ q^{j+k}\ +\ (\text{ terms of degree }\le\ j+k-1\ \text{ in }\ q\ \text{ over } Q\,[\epsilon]\ ),\end{aligned}$$

where $Q\,[\epsilon]$ is the polynomial ring in $\epsilon$ over $Q$.

Now, we have

$$\begin{aligned}\sum_{\alpha=1}^{l_u-1}A(B-C)&=\left[\sum_{j=0,\ k=1}^{s-m,\ t}(-1)^{1+j+k}S_{j(s-m)}(c^{''}-b^{''})S_{kt}(d)\frac{k}{(j+k)\ u^{j+k}}\right]q^{s-m+t}\\&\qquad+\ (\text{ terms of degree }\le\ s-m+t-1\ \text{ in }\ q\ \text{ over } Q\,[\epsilon]\ ).\qquad\Box\end{aligned}$$

**Lemma 2.6.**

$$\begin{aligned}&\sum_{\alpha=1}^{l_u-1}\left\{\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]\right\}\\&=\left[\sum_{j=0,\ k=1}^{s-m,\ t}(-1)^{1+j+k}S_{j(s-m)}(c^{''}-b^{''})S_{kt}(d)\frac{k}{(j+k)\ u^{j+k}}\right]q^{s-m+t}\\&\qquad+\ (\ \textit{terms of degree}\ \le\ s-m+t-1\ \textit{ in }\ q\ \textit{ over } Q\,[\epsilon]\ ),\end{aligned}$$

*where* $c^{''}-b^{''}=(c_{m+1}-b_{m+1},\dots,c_s-b_s)$, $d=(d_1,\dots,d_t)$, $S_{0(s-m)}(c^{''}-b^{''}):=1$, *and* $Q\,[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$ .

**Proof :** Since

$$\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]$$
$$=\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)]-\sum_{m+1\,\le\,h_1\,\le\,s}\left[b_{h_1}\prod_{h\neq h_1}[q-\alpha(c_h-b_h)]\right]$$
$$+\sum_{m+1\,\le\,h_1\,<\,h_2\,\le\,s}\left[b_{h_1}b_{h_2}\prod_{h\,\neq\,h_1,\,h_2}[q-\alpha\,(c_h-b_h)]\right]-\cdots+(-1)^{s-m}\;b_{m+1}\cdots b_s,$$

by applying Lemma 2.5, this lemma can be easily proved . □

**Lemma 2.7.**

$$\left[\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)-q^r\prod_{j=1}^{m}(q-c_j)\right]\times$$
$$\left\{\sum_{\alpha=1}^{l_u-1}\left[\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]\right]\times\left[\prod_{k=1}^{t}(q-\alpha d_k)-\prod_{k=1}^{t}[q-(\alpha+1)d_k]\right]\right\}$$
$$=\left[\sum_{j=0,\;k=1}^{s-m,\;t}(-1)^{j+k}S_{1(r+m)}(e^{'})S_{j(s-m)}(c^{''}-b^{''})S_{kt}(d)\frac{k}{(j+k)\;u^{j+k}}\right]q^{r+s+t-1}$$
$$+\;(\;\textit{terms of degree}\;\le\;r+s+t-2\;\textit{ in }\;q\;\;\textit{over}\;Q\,[\epsilon]\;)\,,$$

*where* $e^{'}=(a_1,\ldots,a_r,b_1-c_1,\ldots,b_m-c_m)$, *and* $S_{0(s-m)}(c^{''}-b^{''}):=1$ .

**Proof :** By applying Lemma 2.4 and Lemma 2.6, this lemma can be proved. □

**Lemma 2.8.**

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j)-q^r\prod_{j=1}^{m}(q-c_j)\right\}\times\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times$$
$$\left\{\prod_{k=1}^{t}(q-l_ud_k)-\prod_{k=1}^{t}[q-(l_u+1)d_k]_+\right\}\qquad(****)$$
$$=\textit{terms of degree}\;\le\;r+s+t-2\;\;\textit{in}\;\;q\;\;\textit{over}\;Q\,[\epsilon]\,,\;\textit{for}\;\;q\gg 0.$$

**Proof :** We prove this lemma by discussing on $u$.

*Case 1 :* Suppose $u = d_1$, that is, $\prod_{k=1}^{t}[q-(l_u+1)d_k]_+ = 0$, and $l_u = \frac{q-v-\epsilon}{d_1}$ . Then the term $(****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_ud_k)\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{d_1}$ , we obtain the following expression for $(****)$ :

$$\frac{v+\epsilon}{{d_1}^{s-m+t-1}} \left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times$$
$$\left\{\prod_{h=m+1}^{s}[(d_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-d_1b_h]\right\} \times \left\{\prod_{k=2}^{t}[(d_1-d_k)q+(v+\epsilon)d_k]\right\},$$

which is a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$.

*Case 2 :* Suppose $u = a_1$, that is, $l_u = \frac{q-v-\epsilon}{a_1}$ .
If $a_1 = d_1$, then $\prod_{k=1}^{t}[q-(l_u+1)d_k]_+ = 0$, and $(****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_ud_k)\right\}.$$

Similarly, the term $(****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$.
If $a_1 > d_1$, then $q-(l_u+1)d_k > 0$ for $q > \frac{(a_1-1)d_k}{a_1-d_k}$ .
It follows that for $q \gg 0$, $(****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times$$
$$\left\{\prod_{k=1}^{t}(q-l_ud_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k]\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{a_1}$ , we obtain the following expression for $(****)$ :

$$\frac{1}{{a_1}^{s-m+t}} \left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times$$
$$\left\{\prod_{h=m+1}^{s}[(a_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-a_1b_h]\right\} \times$$
$$\left\{\prod_{k=1}^{t}[(a_1-d_k)q+(v+\epsilon)d_k] - \prod_{k=1}^{t}[(a_1-d_k)q+(v+\epsilon-a_1)d_k]\right\}.$$

Therefore, the term $(****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$, for $q \gg 0$.

*Case 3* $(m \neq 0)$ *:* Suppose $u = b_1 - c_1$, that is, $l_u = \frac{q-v-\epsilon}{b_1-c_1}$ .
If $b_1 - c_1 = d_1$, then $\prod_{k=1}^{t} [q-(l_u+1)d_k]_+ = 0$, and $(****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_ud_k)\right\}.$$

Similarly, the term $(****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$.
If $b_1 - c_1 > d_1$, then $q-(l_u+1)d_k > 0$ for $q > \frac{(b_1-1-2c_1)d_k}{(b_1-c_1)-d_k}$ .
It follows that for $q \gg 0$, $(****)$ has the form

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times$$
$$\left\{\prod_{k=1}^{t}(q-l_ud_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k]\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{b_1-c_1}$ , we obtain the following expression for $(****)$ :

$$\frac{1}{(b_1-c_1)^{\,s-m+t}} \left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times$$
$$\left\{\prod_{h=m+1}^{s}[(b_1-c_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-(b_1-c_1)b_h]\right\} \times$$
$$\left\{\prod_{k=1}^{t}[(b_1-c_1-d_k)q+(v+\epsilon)d_k] - \prod_{k=1}^{t}[(b_1-c_1-d_k)q+(v+\epsilon-(b_1-c_1))d_k]\right\}.$$

Therefore, the term $(****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$, for $q \gg 0$.

*Case 4* $(m \neq s)$ *:* Suppose $u = c_{m+1} - b_{m+1}$, that is, $l_u = \frac{q-v-\epsilon}{c_{m+1}-b_{m+1}}$ .
If $c_{m+1} - b_{m+1} = d_1$, then $\prod_{k=1}^{t} [q-(l_u+1)d_k]_+ = 0$, and $(****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_ud_k)\right\}.$$

Similarly, the term $(****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$.

If $c_{m+1} - b_{m+1} > d_1$, then $q - (l_u+1)d_k > 0$ for $q > \frac{(c_{m+1}-1-2b_{m+1})d_k}{(c_{m+1}-b_{m+1})-d_k}$ .
It follows that for $q \gg 0$, $(****)$ is equal to

$$\left\{ \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j) \right\} \times \left\{ \prod_{h=m+1}^{s} [q - l_u(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-l_ud_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k] \right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{c_{m+1}-b_{m+1}}$ , we obtain the following expression for $(****)$ :

$$\frac{v+\epsilon-b_{m+1}}{(c_{m+1}-b_{m+1})^{\,s-m-1+t}} \left\{ \prod_{i=1}^{r}(q-a_i)\prod_{j=1}^{m}(q-b_j) - q^r\prod_{j=1}^{m}(q-c_j) \right\} \times$$
$$\left\{ \prod_{h=m+2}^{s} [(c_{m+1}-b_{m+1}-(c_h-b_h))q + (v+\epsilon)(c_h-b_h) - (c_{m+1}-b_{m+1})b_h] \right\} \times \left\{ \prod_{k=1}^{t} [(c_{m+1}-\right.$$
$$\left. b_{m+1}-d_k)q + (v+\epsilon)d_k] - \prod_{k=1}^{t}[(c_{m+1}-b_{m+1}-d_k)q + (v+\epsilon-c_{m+1}+b_{m+1})d_k] \right\}.$$

Therefore, the term $(****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$, for $q \gg 0$. $\square$

**Lemma 2.9.**

$$\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{i=1}^{r}(q-\alpha a_i) - \prod_{i=1}^{r}[q-(\alpha+1)a_i] \right\} \times \left\{ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)] \right\} \times$$
$$\left\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k] \right\}$$
$$= \left[ \sum_{i=1,\ j=0,\ k=1}^{r,\ s-m,\ t} (-1)^{i+j+k} S_{ir}(a) S_{j(s-m)}(c''-b'') S_{kt}(d) \frac{i\ k}{(i+j+k-1)\ u^{i+j+k-1}} \right] q^{r+s-m+t-1}$$
$$+ \ (\ \textit{terms of degree} \le r+s-m+t-2 \ \textit{in } q \ \textit{over } Q[\epsilon]\ ),$$

*where* $a=(a_1,\ldots,a_r)$, $d=(d_1,\ldots,d_t)$, $c''-b''=(c_{m+1}-b_{m+1},\ldots,c_s-b_s)$, $S_{0(s-m)}(c''-b'') := 1$, *and* $Q[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$.

**Proof :** Let

$$A=\prod_{i=1}^{r}(q-\alpha a_i)=q^r-\alpha S_{1r}(a)q^{r-1}\ +\ \sum_{i=2}^{r}(-1)^i\alpha^iS_{ir}(a)q^{r-i},$$

$$B=\prod_{i=1}^{r}\left[q-(\alpha+1)a_i\right]=q^r-(\alpha+1)S_{1r}(a)q^{r-1}\ +\ \sum_{i=2}^{r}(-1)^i(\alpha+1)^iS_{ir}(a)q^{r-i},$$

$$C=\prod_{k=1}^{t}(q-\alpha d_k)=q^t-\alpha S_{1t}(d)q^{t-1}\ +\ \sum_{k=2}^{t}(-1)^k\alpha^kS_{kt}(d)q^{t-k},$$

$$D=\prod_{k=1}^{t}\left[q-(\alpha+1)d_k\right]=q^t-(\alpha+1)S_{1t}(d)q^{t-1}\ +\ \sum_{k=2}^{t}(-1)^k(\alpha+1)^kS_{kt}(d)q^{t-k},$$

$$E=\prod_{h=m+1}^{s}\left[q-\alpha(c_h-b_h)\right]$$

$$=q^{s-m}-\alpha S_{1(s-m)}(c''-b'')q^{s-m-1}\ +\ \sum_{j=2}^{s-m}(-1)^j\alpha^jS_{j(s-m)}(c''-b'')q^{s-m-j}\ .$$

Then

$$\begin{aligned}A-B\ &=\ -\left[\ \alpha-(\alpha+1)\ \right]S_{1r}(a)q^{r-1}+\left[\ \alpha^2-(\alpha+1)^2\ \right]S_{2r}(a)q^{r-2}-\cdots\\ &\quad+(-1)^r\left[\ \alpha^r-(\alpha+1)^r\ \right]S_{rr}(a),\qquad\qquad \text{and}\\ C-D\ &=\ -\left[\ \alpha-(\alpha+1)\ \right]S_{1t}(d)q^{t-1}+\left[\ \alpha^2-(\alpha+1)^2\ \right]S_{2t}(d)q^{t-2}-\cdots\\ &\quad+(-1)^t\left[\ \alpha^t-(\alpha+1)^t\ \right]S_{tt}(d).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{\alpha=1}^{l_u-1}(A-B)(C-D)E\\ &=\sum_{\alpha=1}^{l_u-1}\Big\{\ (-1)^{1+0+1}S_{1r}(a)S_{1t}(d)q^{r+s-m+t-1-0-1}\cdot W_{101}(\alpha)\\ &\qquad+\Big[\ (-1)^{1+0+2}S_{1r}(a)S_{2t}(d)q^{r+s-m+t-1-0-2}\cdot W_{102}(\alpha)\\ &\qquad\quad+(-1)^{2+0+1}S_{2r}(a)S_{1t}(d)q^{r+s-m+t-2-0-1}\cdot W_{201}(\alpha)\\ &\qquad\quad+(-1)^{1+1+1}S_{1r}(a)S_{1(s-m)}(c''-b'')S_{1t}(d)q^{r+s-m+t-1-1-1}\cdot W_{111}(\alpha)\ \Big]\\ &\qquad+\cdots+(-1)^{r+s-m+t}S_{rr}(a)S_{(s-m)(s-m)}(c''-b'')S_{tt}(d)\cdot W_{r(s-m)t}(\alpha)\ \Big\}\\ &=\sum_{i=1,\ j=0,\ k=1}^{r,\ s-m,\ t}\left\{(-1)^{i+j+k}S_{ir}(a)S_{j(s-m)}(c''-b'')S_{kt}(d)q^{r+s-m+t-i-j-k}\ \sum_{\alpha=1}^{l_u-1}W_{ijk}(\alpha)\right\},\end{aligned}$$

where $S_{0(s-m)}(c'' - b'') := 1$, and $W_{ijk}(\alpha) = \alpha^j [\ \alpha^i - (\alpha+1)^i\ ]\ [\ \alpha^k - (\alpha+1)^k\ ]$, $1 \le i \le r$, $0 \le j \le s-m$, $1 \le k \le t$.

Since

$$\begin{aligned} W_{ijk}(\alpha) &= \alpha^j \left[\ \alpha^i - (\alpha+1)^i\ \right] \left[\ \alpha^k - (\alpha+1)^k\ \right] \\ &= i\ k\ \alpha^{i+j+k-2} + \left(\ i\ C_2^k + k\ C_2^i\ \right) \alpha^{i+j+k-3} + \cdots , \end{aligned}$$

it follows that

$$\begin{aligned} \sum_{\alpha=1}^{l_u-1} W_{ijk}(\alpha) &= i\ k \sum_{\alpha=1}^{l_u-1} \alpha^{i+j+k-2} + \left(\ i\ C_2^k + k\ C_2^i\ \right) \sum_{\alpha=1}^{l_u-1} \alpha^{i+j+k-3} + \cdots \\ &= \frac{i\ k}{i+j+k-1}\ l_u^{\,i+j+k-1} + (\text{ terms of degree } \le\ i+j+k-2 \text{ in } l_u \text{ over } Q\ ). \end{aligned}$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{u}$, we obtain the following expression

$$\begin{aligned} \sum_{\alpha=1}^{l_u-1} W_{ijk}(\alpha) &= \frac{i\ k}{i+j+k-1} \left( \frac{q-v-\epsilon}{u} \right)^{i+j+k-1} \\ &\quad + \left( \text{terms of degree } \le\ i+j+k-2 \text{ in } \frac{q-v-\epsilon}{u} \text{ over } Q \right) \\ &= \frac{i\ k}{(i+j+k-1)\ u^{i+j+k-1}}\ q^{i+j+k-1} \\ &\quad + (\text{ terms of degree } \le\ i+j+k-2 \text{ in } q \text{ over } Q[\epsilon]\ ), \end{aligned}$$

where $Q[\epsilon]$ is the polynomial ring in $\epsilon$ over $Q$. Now, we have

$$\begin{aligned} &\sum_{\alpha=1}^{l_u-1} (A-B)E(C-D) \\ &= \left[ \sum_{i=1,\ j=0,\ k=1}^{r,\ s-m,\ t} (-1)^{i+j+k} S_{ir}(a) S_{j(s-m)}(c''-b'') S_{kt}(d)\ \frac{i\ k}{(i+j+k-1)\ u^{i+j+k-1}} \right] q^{r+s-m+t-1} \\ &\quad + (\text{ terms of degree } \le\ r+s-m+t-2 \text{ in } q \text{ over } Q[\epsilon]\ ). \qquad \square \end{aligned}$$

**Lemma 2.10.**

$$\begin{aligned} &\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{i=1}^{r} (q - \alpha a_i) - \prod_{i=1}^{r} [q - (\alpha+1)a_i] \right\} \times \left\{ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right\} \times \\ &\qquad \left\{ \prod_{k=1}^{t} (q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1)d_k] \right\} \\ &= \left[ \sum_{i=1,\ j=0,\ k=1}^{r,\ s-m,\ t} (-1)^{i+j+k} S_{ir}(a) S_{j(s-m)}(c''-b'') S_{kt}(d) \frac{i\ k}{(i+j+k-1)\ u^{i+j+k-1}} \right] q^{r+s-m+t-1} \\ &\quad + (\ \textit{terms of degree} \le\ r+s-m+t-2 \textit{ in } q \textit{ over } Q[\epsilon]\ ), \end{aligned}$$

*where* $a = (a_1, \ldots, a_r)$, $d = (d_1, \ldots, d_t)$, $c^{''} - b^{''} = (c_{m+1} - b_{m+1}, \ldots, c_s - b_s)$, $S_{0(s-m)}(c^{''} - b^{''}) := 1$, *and* $Q\,[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$.

**Proof :** Since

$$\begin{aligned}
&\prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \\
&= \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h)] - \sum_{m+1 \,\le\, h_1 \,\le\, s} \left[ b_{h_1} \prod_{h \ne h_1} [q - \alpha(c_h - b_h)] \right] \\
&\quad + \sum_{m+1 \,\le\, h_1 \,<\, h_2 \,\le\, s} \left[ b_{h_1} b_{h_2} \prod_{h \,\ne\, h_1,\, h_2} [q - \alpha(c_h - b_h)] \right] - \cdots + (-1)^{s-m}\, b_{m+1} \cdots b_s,
\end{aligned}$$

by applying Lemma 2.9, this lemma can be easily proved. □

**Lemma 2.11.**

$$\begin{aligned}
&\sum_{\alpha=1}^{l_u - 1} \left\{ \prod_{i=1}^{r} (q - \alpha a_i) \prod_{j=1}^{m} [q - \alpha(b_j - c_j) - c_j] - \prod_{i=1}^{r} [q - (\alpha+1) a_i] \prod_{j=1}^{m} [q - (\alpha+1)(b_j - c_j) - c_j] \right\} \\
&\quad \times \left\{ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t} (q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1) d_k] \right\} \\
&= \left[ \sum_{i=1,\ j=0,\ k=1}^{r+m,\ s-m,\ t} (-1)^{i+j+k} S_{i(r+m)}(e^{'}) S_{j(s-m)}(c^{''} - b^{''}) S_{kt}(d)\ \frac{i\ k}{(i+j+k-1)\ u^{i+j+k-1}} \right] q^{r+s+t-1} \\
&\qquad\qquad + \ (\ \textit{terms of degree} \ \le\ r+s+t-2 \ \textit{ in } \ q \ \textit{ over } \ Q\,[\epsilon]\ )\,,
\end{aligned}$$

*where* $e^{'} = (a_1, \ldots, a_r, b_1 - c_1, \ldots, b_m - c_m)$, $d = (d_1, \ldots, d_t)$, $S_{0(s-m)}(c^{''} - b^{''}) := 1$, $c^{''} - b^{''} = (c_{m+1} - b_{m+1}, \ldots, c_s - b_s)$, *and* $Q\,[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$.

**Proof :** Since

$$\begin{aligned}
&\prod_{j=1}^{m} [q - \alpha(b_j - c_j) - c_j\ ] \\
&= \prod_{j=1}^{m} [q - \alpha(b_j - c_j)] - \sum_{1 \,\le\, j_1 \,\le\, m} \left[ c_{j_1} \prod_{j \ne j_1} [q - \alpha(b_j - c_j)] \right] \\
&\quad + \sum_{1 \,\le\, j_1 \,<\, j_2 \,\le\, m} \left[ c_{j_1} c_{j_2} \prod_{j \,\ne\, j_1,\, j_2} [q - \alpha(b_j - c_j)] \right] - \cdots + (-1)^{m}\, c_1 \cdots c_m, \qquad \text{and}
\end{aligned}$$

$$\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]$$
$$=\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)]-\sum_{1\ \le\ j_1\ \le\ m}\left[c_{j_1}\prod_{j\neq j_1}[q-(\alpha+1)(b_j-c_j)]\right]$$
$$+\sum_{1\ \le\ j_1\ <\ j_2\ \le\ m}\left[c_{j_1}c_{j_2}\prod_{j\ \neq\ j_1,\ j_2}[q-(\alpha+1)(b_j-c_j)]\right]-\cdots+(-1)^m\ c_1\cdots c_m,$$

by applying Lemma 2.10, this lemma can be proved . □

**Lemma 2.12.**

$$\left\{\prod_{i=1}^{r}(q-l_ua_i)\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j]-\prod_{i=1}^{r}[q-(l_u+1)a_i]_+\prod_{j=1}^{m}[q-(l_u+1)(b_j-c_j)-c_j]_+\right\}$$
$$\times\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-l_ud_k)-\prod_{k=1}^{t}[q-(l_u+1)d_k\ ]_+\right\}\qquad(*****)$$

$=$ *terms of degree* $\le\ r+s+t-2$ *in* $q$ *over* $Q\,[\epsilon]$, *for* $q\gg0.$

**Proof :** We prove this lemma by discussing on $u$.

*Case 1 :* Suppose $u=d_1$, that is, $\prod_{k=1}^{t}[q-(l_u+1)d_k]_+=0$, and $l_u=\frac{q-v-\epsilon}{d_1}$ .

If $d_1>a_1$ and $d_1>b_1-c_1$ $(m\neq0)$, then

$$q-(l_u+1)a_i>0\quad\text{for}\ q>\frac{(d_1-1)a_i}{d_1-a_i}\ ,\ i=1,2,\ldots,r\qquad\text{and}$$
$$q-(l_u+1)(b_j-c_j)-c_j>0\quad\text{for}\ q>\frac{d_1b_j-(b_j-c_j)}{d_1-(b_j-c_j)}\ ,\ j=1,2,\ldots,m.$$

It follows that for $q\gg0$, the term $(*****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-l_ua_i)\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j]-\prod_{i=1}^{r}[q-(l_u+1)a_i]\prod_{j=1}^{m}[q-(l_u+1)(b_j-c_j)-c_j\ ]\right\}\times$$
$$\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-l_ud_k)\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{d_1}$ , we obtain the following expression for $(*****)$ :

$$\frac{v+\epsilon}{d_1^{\,r+s+t-1}}\left\{\prod_{i=1}^{r}[(d_1-a_i)q+(v+\epsilon)a_i\ ]\prod_{j=1}^{m}[(d_1-(b_j-c_j))q+(v+\epsilon)(b_j-c_j)-d_1c_j\ ]\right.$$
$$\left.-\prod_{i=1}^{r}[(d_1-a_i)q+(v+\epsilon-d_1)a_i\ ]\prod_{j=1}^{m}[(d_1-(b_j-c_j))q+(v+\epsilon-d_1)(b_j-c_j)-d_1c_j\ ]\right\}\times$$
$$\left\{\prod_{h=m+1}^{s}[(d_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-d_1b_h]\right\}\times\left\{\prod_{k=2}^{t}[(d_1-d_k)q+(v+\epsilon)d_k]\right\}.$$

Therefore, the term $(*****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$, for $q \gg 0$.

If $d_1 = a_1$, then $\prod_{i=1}^{r}[q-(l_u+1)a_i]_+ = 0$, and $(*****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-l_u a_i)\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j]\right\}\times\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-l_u d_k)\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{d_1}$, we obtain the following expression for $(*****)$ :

$$\frac{(v+\epsilon)^2}{d_1^{\,r+s+t-2}}\left\{\prod_{i=2}^{r}[(d_1-a_i)q+(v+\epsilon)a_i\,]\prod_{j=1}^{m}[(d_1-(b_j-c_j))q+(v+\epsilon)(b_j-c_j)-d_1c_j\,]\right\}\times$$
$$\left\{\prod_{h=m+1}^{s}[(d_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-d_1b_h]\right\}\times\left\{\prod_{k=2}^{t}[(d_1-d_k)q+(v+\epsilon)d_k]\right\}.$$

which is a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$.

If $d_1 = b_1 - c_1$ $(m \neq 0)$, then $\prod_{j=1}^{m}[q-(l_u+1)(b_i-c_j)-c_j\,]_+ = 0$, and $(*****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-l_u a_i)\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j]\right\}\times\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-l_u d_k)\right\}.$$

Similarly, the term $(*****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$.

*Case 2 :* Suppose $u = a_1$, that is, $\prod_{i=1}^{r}[\,q-(l_u+1)a_i\,]_+ = 0$, and $l_u = \frac{q-v-\epsilon}{a_1}$ .

If $a_1 = d_1$, then by the argument as above, we are done.

If $a_1 > d_1$, then $q-(l_u+1)d_k > 0$ for $q > \frac{(a_1-1)d_k}{a_1-d_k}$ , $k = 1, 2, \ldots, t$.

It follows that for $q \gg 0$, the term $(*****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-l_u a_i)\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j\,]\right\}\times\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times$$
$$\left\{\prod_{k=1}^{t}(q-l_u d_k)-\prod_{k=1}^{t}[q-(l_u+1)d_k]\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{a_1}$, we obtain the following expression for $(*****)$ :

$$\frac{v+\epsilon}{a_1^{\,r+s+t-1}}\left\{\prod_{i=2}^{r}[(a_1-a_i)q+(v+\epsilon)a_i\,]\prod_{j=1}^{m}[(a_1-(b_j-c_j))q+(v+\epsilon)(b_j-c_j)-a_1c_j\,]\right\}\times$$
$$\left\{\prod_{h=m+1}^{s}[(a_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-a_1b_h]\right\}\times$$
$$\left\{\prod_{k=1}^{t}[(a_1-d_k)q+(v+\epsilon)d_k]-\prod_{k=1}^{t}[(a_1-d_k)q+(v+\epsilon-a_1)d_k]\right\}.$$

Therefore, the term $(*****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$, for $q \gg 0$.

*Case 3* $(m \neq 0)$ *:* Suppose $u = b_1 - c_1$, that is, $\prod_{j=1}^{m} [\, q-(l_u+1)(b_j-c_j)]_+ = 0$, and $l_u = \frac{q-v-\epsilon}{b_1-c_1}$ .

If $b_1 - c_1 = d_1$, then by the argument as above, we are done.

If $b_1 - c_1 > d_1$, then $q-(l_u+1)d_k > 0$ for $q > \frac{(b_1-1-2c_1)d_k}{(b_1-c_1)-d_k}$ , $k = 1, 2, \ldots, t$.

It follows that for $q \gg 0$, the term $(*****)$ is equal to

$$\left\{ \prod_{i=1}^{r}(q-l_u a_i) \prod_{j=1}^{m} [q-l_u(b_j-c_j)-c_j\ ] \right\} \times \left\{ \prod_{h=m+1}^{s} [q-l_u(c_h-b_h)-b_h] \right\} \times$$
$$\left\{ \prod_{k=1}^{t}(q-l_u d_k) - \prod_{k=1}^{t} [q-(l_u+1)d_k] \right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{b_1-c_1}$ , we obtain the following expression for $(*****)$ :

$$\frac{v+\epsilon-c_1}{(b_1-c_1)^{r+s+t-1}} \left\{ \prod_{i=1}^{r} [(b_1-c_1-a_i)q+(v+\epsilon)a_i] \prod_{j=2}^{m} [(b_1-c_1-(b_j-c_j))q+(v+\epsilon)(b_j-c_j) \right.$$
$$\left. -(b_1-c_1)c_j] \right\} \times \left\{ \prod_{h=m+1}^{s} [(b_1-c_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-(b_1-c_1)b_h] \right\} \times$$
$$\left\{ \prod_{k=1}^{t} [(b_1-c_1-d_k)q+(v+\epsilon)d_k] - \prod_{k=1}^{t} [(b_1-c_1-d_k)q+(v+\epsilon-(b_1-c_1))d_k] \right\}.$$

Therefore, the term $(*****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q[\epsilon]$, for $q \gg 0$.

*Case 4* $(m \neq s)$ *:* Suppose $u = c_{m+1} - b_{m+1}$, that is, $l_u = \frac{q-v-\epsilon}{c_{m+1}-b_{m+1}}$ .

If $c_{m+1} - b_{m+1} > a_1$, $c_{m+1} - b_{m+1} > d_1$, and $c_{m+1} - b_{m+1} > b_1 - c_1$ $(m \neq 0)$, then

$$q-(l_u+1)a_i > 0 \quad \text{for } q > \frac{(c_{m+1}-b_{m+1})-a_i(1+b_{m+1})}{(c_{m+1}-b_{m+1})-a_i}\ , \quad i = 1, 2, \ldots, r\ ,$$

$$q-(l_u+1)d_k > 0 \quad \text{for } q > \frac{(c_{m+1}-b_{m+1})-d_k(1+b_{m+1})}{(c_{m+1}-b_{m+1})-d_k}\ , \quad k = 1, 2, \ldots, t\ , \qquad \text{and}$$

$$q-(l_u+1)(b_j-c_j)-c_j > 0 \quad \text{for } q > \frac{(b_j-c_j)(c_{m+1}-1-2b_{m+1})+c_j(c_{m+1}-b_{m+1})}{(c_{m+1}-b_{m+1})-(b_j-c_j)}, \quad j = 1, 2, \ldots, m.$$

It follows that for $q \gg 0$, the term $(*****)$ is equal to

$$\left\{ \prod_{i=1}^{r}(q-l_u a_i) \prod_{j=1}^{m} [q-l_u(b_j-c_j)-c_j] - \prod_{i=1}^{r} [q-(l_u+1)a_i] \prod_{j=1}^{m} [q-(l_u+1)(b_j-c_j)-c_j] \right\} \times$$
$$\left\{ \prod_{h=m+1}^{s} [q-l_u(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-l_u d_k) - \prod_{k=1}^{t} [q-(l_u+1)d_k] \right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{c_{m+1}-b_{m+1}}$ , we obtain the following expression for $(*****)$ :

$$\frac{v+\epsilon-b_{m+1}}{(c_{m+1}-b_{m+1})^{\,r+s+t-1}}\left\{\prod_{i=1}^{r}[(c_{m+1}-b_{m+1}-a_i)q+(v+\epsilon)a_i]\prod_{j=1}^{m}[(c_{m+1}-b_{m+1}-(b_j-c_j))q\right.$$

$$+(v+\epsilon)(b_j-c_j)-(c_{m+1}-b_{m+1})c_j]-\prod_{i=1}^{r}[(c_{m+1}-b_{m+1}-a_i)q+(v+\epsilon-(c_{m+1}-b_{m+1}))a_i]$$

$$\left.\prod_{j=1}^{m}[(c_{m+1}-b_{m+1}-(b_j-c_j))q+(v+\epsilon-(c_{m+1}-b_{m+1}))(b_j-c_j)-(c_{m+1}-b_{m+1})c_j]\right\}\times$$

$$\left\{\prod_{h=m+2}^{s}[(c_{m+1}-b_{m+1}-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-(c_{m+1}-b_{m+1})b_h]\right\}\times\left\{\prod_{k=1}^{t}[(c_{m+1}-\right.$$

$$\left.b_{m+1}-d_k)q+(v+\epsilon)d_k]-\prod_{k=1}^{t}[(c_{m+1}-b_{m+1}-d_k)q+(v+\epsilon-(c_{m+1}-b_{m+1}))d_k]\right\}.$$

Therefore, the term $(*****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q\,[\epsilon]$, for $q \gg 0$.

If $c_{m+1}-b_{m+1}=a_1$ and $c_{m+1}-b_{m+1}=d_1$, then $\prod_{i=1}^{r}[q-(l_u+1)a_i\,]_+=0$, and $\prod_{k=1}^{t}[q-(l_u+1)d_k\,]_+=0$. Therefore, $(*****)$ is equal to

$$\left\{\prod_{i=1}^{r}(q-l_ua_i)\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j]\right\}\times\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-l_ud_k)\right\}.$$

Similarly, the term $(*****)$ can be expressed as a polynomial of degree less than $r+s+t-1$ in $q$ over $Q\,[\epsilon]$.

The arguments for the case $u=c_{m+1}-b_{m+1}=a_1>d_1$, $u=c_{m+1}-b_{m+1}=b_1-c_1=d_1$, $u=c_{m+1}-b_{m+1}=b_1-c_1>d_1$, $u=c_{m+1}-b_{m+1}=d_1>a_1$, and $u=c_{m+1}-b_{m+1}=d_1>b_1-c_1$ are similar. □

**Theorem 2.13.** *The Hilbert-Kunz function of the hypersurface*

$$X_1^{a_1}\dots X_r^{a_r}Y_1^{b_1}\dots Y_s^{b_s}+Y_1^{c_1}\dots Y_s^{c_s}Z_1^{d_1}\dots Z_t^{d_t}$$

*is*

$$n\quad\longmapsto\quad\lambda p^{\,(r+s+t-1)n}\;+\sum_{k=0}^{r+s+t-2}f_k\,(n)\,p^{\,kn}\qquad \textit{for}\;\; n\gg 0,$$

*where* $\lambda=S_{1r}(a)+S_{1s}(b)+$

$$\left[\sum_{i=2,\;j=0,\;k=1}^{r+m,\;s-m,\;t}(-1)^{i+j+k}S_{i(r+m)}(e^{'})S_{j(s-m)}(c^{''}-b^{''})S_{kt}(d)\frac{i\;k}{(i+j+k-1)u^{i+j+k-1}}\right],$$

*$a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_s)$, $d = (d_1, \ldots, d_t)$, $c^{''} - b^{''} = (c_{m+1} - b_{m+1}, \ldots, c_s - b_s)$, $e^{'} = (a_1, \ldots, a_r, b_1 - c_1, \ldots, b_m - c_m)$, $0 \leq m \leq s$, and $f_k(n)$ is an eventually periodic function of $n$ for each $k$.*

**Proof :** Let $q = p^n$. By Proposition 2.2, $HK_R(q)$ can be written as the sum of five terms :

$$q^{r+s+t} - q^t \prod_{i=1}^{r}(q - a_i) \prod_{j=1}^{s}(q - b_j), \tag{1}$$

$$\left\{ \prod_{i=1}^{r}(q - a_i) \prod_{j=1}^{m}(q - b_j) - q^r \prod_{j=1}^{m}(q - c_j) \right\} \times \left\{ \sum_{\alpha=1}^{l_u-1} \left[ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right] \times \left[ \prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1)d_k] \right] \right\}, \tag{2}$$

$$\left\{ \prod_{i=1}^{r}(q - a_i) \prod_{j=1}^{m}(q - b_j) - q^r \prod_{j=1}^{m}(q - c_j) \right\} \times \left\{ \prod_{h=m+1}^{s} [q - l_u(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q - l_u d_k) - \prod_{k=1}^{t} [q - (l_u+1)d_k]_+ \right\}, \tag{3}$$

$$\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{i=1}^{r}(q - \alpha a_i) \prod_{j=1}^{m} [q - \alpha(b_j - c_j) - c_j] - \prod_{i=1}^{r} [q - (\alpha+1)a_i] \prod_{j=1}^{m} [q - (\alpha+1)(b_j - c_j) - c_j] \right\} \times \left\{ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1)d_k] \right\}, \tag{4}$$

$$\left\{ \prod_{i=1}^{r}(q - l_u a_i) \prod_{j=1}^{m} [q - l_u(b_j - c_j) - c_j] - \prod_{i=1}^{r} [q - (l_u+1)a_i]_+ \prod_{j=1}^{m} [q - (l_u+1)(b_j - c_j) - c_j]_+ \right\} \times \left\{ \prod_{h=m+1}^{s} [q - l_u(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q - l_u d_k) - \prod_{k=1}^{t} [q - (l_u+1)d_k]_+ \right\}. \tag{5}$$

By applying Lemma 2.3 , Lemma 2.7 , Lemma 2.8 , Lemma 2.11 , and Lemma 2.12 to the five terms, we obtain that

$$HK_R(q) = \lambda q^{r+s+t-1} + \triangle_{r+s+t-2}(\epsilon) q^{r+s+t-2} + \cdots + \triangle_1(\epsilon) q + \triangle_0(\epsilon),$$

where $\lambda = S_{1r}(a) + S_{1s}(b) +$

$$\left[ \sum_{i=2,\ j=0,\ k=1}^{r+m,\ s-m,\ t} (-1)^{i+j+k} S_{i(r+m)}(e^{'}) S_{j(s-m)}(c^{''} - b^{''}) S_{kt}(d) \frac{i\ k}{(i+j+k-1)u^{i+j+k-1}} \right],$$

$a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_s)$, $d = (d_1, \ldots, d_t)$, $c^{''} - b^{''} = (c_{m+1} - b_{m+1}, \ldots, c_s - b_s)$, $e^{'} = (a_1, \ldots, a_r, b_1 - c_1, \ldots, b_m - c_m)$, $0 \leq m \leq s$, $u$ is the maximal integers among all $a_i$'s, $(b_j - c_j)$'s, $(c_h - b_h)$'s, and $d_k$'s, and $\triangle_{r+s+t-2}(\epsilon)$, $\ldots$, $\triangle_1(\epsilon)$, $\triangle_0(\epsilon)$ are polynomials in $\epsilon$ over $Q$.

Let $f_k(n) := \triangle_k(\epsilon)$, $k = 0, 1, 2, \ldots, r + s + t - 2$.

Since $\epsilon$ is the remainder of $q - v$ divided by $u$, $f_k(n)$ is an eventually periodic function of $n$ for $n \gg 0$. $\square$