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Harnack Inequality for The Heat Equation on A Complete Riemannian Manifold

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摘 要

設 M 是一個光滑且連通的完備非緊緻黎曼流形. 若 M 滿足體積倍增條件和弱 L^2 龐加萊不等式的話, 則針對底律雷特熱方程正解的哈納克不等式成立. 本論文主要探討這個定理. 基本上, 本論文可分成四個部分. 第一部分討論具有體積倍增條件和弱 L^2 龐加萊不等式的流形上的一些重要性質. 第二部分則利用這些性質證明納許不等式和索伯列夫不等式. 第三部分著重在底律雷特熱方程的 subsolutions 和 supersolutions 並分別從這兩種類型的解中萃取出均值不等式及逆赫爾德不等式. 最後一部分則運用了在前三個部份中所獲得的工具來完成本論文主要定理的證明.

關鍵字: 哈納克不等式, 體積倍增條件, s-型覆蓋, 惠特尼型覆蓋, 弱 L^2 龐加萊不等式, 加權龐加萊不等式, 納許不等式, 索伯列夫不等式, 底律雷特熱方程, 均值不等式, 反向赫爾德不等式, 莫澤迭代法.

Abstract

Let M be a smooth connected complete non-compact Riemannian manifold. If M satisfies the volume doubling condition (VDC) and the weak L^2 Poincaré inequality (WPI), then the Harnack inequality for positive solutions to the Dirchlet heat equation holds on M. This is the main theorem in this paper. Basically, This paper can be seperated into four part. The first part discusses some important and useful properties on the manifold equipped with both VDC and WPI. The second part utilizes those properties to establish the Nash inequality and the Sobolev inequality. The third part focuses on subsolutions and supersolutions to the Dirichlet heat equation, and extracts the mean value inequality and the reverse Hölder inequality from them respectively. The last part applies all the tools obtained in previous parts to show the proof of the main theorem in this paper.

Key Words: Harnack inequality, volume doubling condition, s-packing covering, Whitney-type covering, weak L^2 Poincaré inequality, weighted Poincaré inequality, Nash inequality, Sobolev inequality, Dirichlet heat equation, mean value inequality, reverse Hölder inequality, Moser's iteration.

Harnack Inequality for The Heat Equation on A Complete Riemannian Manifold

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July 23, 2012

Abstract

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Contents

0 Introduction

This paper is a survey mainly about the Harnack inequality for positive solutions to the Dirichlet heat equation on a smooth connected complete non-compact Riemannian manifold which satisfies both the volume doubling condition and the weak L^2 Poincaré inequality.

In the beginning of this paper, we introduce the definition and give some propositions of the volume doubing condition, the weak L^2 Poincaré inequality and two types of covering, the the s-packing covering and the Whitney type covering. The propositions of these two types of covering are very useful since we can get the Nash inequality and the weighted Poincaré inequality from them. Once we get the Nash inequality, then, applying the propositions of the Dirichlet heat kenel and the the spectrum theory for elliptic operators, we can further obtain the Sobolev inequality.

The Sobolev inequality plays an extremely important role in Geometry Analysis. If the Sobolev inequality holds, we can get the mean value inequlity and the reverse Hölder inequality by applying Moser's cerebrated iteration on the positive subsolution and supersolution to the Dirichlet heat equation respectively.

Finally, by a powerful lemma, we can get the Harnarck ineaulaity for positive solutions to the Dirichlet heat equation from the mean value inequlity and the reverse Hölder inequality. Since that lemma is provided by the volume doubling condition and the weak L^2 Poincaré inequality, and so are the last two inequalities, we can conclude that the Harnarck inequality for positive solutions to the Dirichlet heat equation holds on a complete non-compact Riemannian manifold which satisfies both the volume doubling condition and the weak L^2 Poincaré inequality. This conclusion is the main goal of this paper.

1 Volume doubling condition and two types of covering

Let M be a smooth connected complete non-compact Riemannian manifold. Let d be the canonical metric associated to the Riemannian structure of M. That is, $d : M \times M \to \mathbb{R}^+ \cup \{0\}$ is defined by

$$d(x,y) \equiv \text{the length of a geodesic curve joining } x \text{ to } y$$

for any $x, y \in M$. In this case, by Hopf and Rinow's theorem, we know that (M, d) forms a complete metric space and any bounded subset of M is precompact since M is complete.

On the other hand, we let $B_x(r)$ be the geodesic ball of center $x \in M$ and radius $r > 0$. That is,

$$B_x(r) \equiv \{y \in M : d(y,x) < r\}.$$

Also, we denote by $|B_x(r)|$ the volume of $B_x(r)$.

1.1 Volume doubling condition

Definition 1.1. *M is said to have the volume doubling condition (VDC) if there exists a constant $d_0 \geq 1$ such that*

$$|B_x(2r)| \leq d_0 |B_x(r)| \tag{1.1}$$

for any $x \in M$ and $r > 0$. In this case, d_0 is called the controlling constant in VDC.

Proposition 1.1. *Suppose that M has VDC. For any $x, y \in M$, for any $r \geq s > 0$, if $d(x,y) < r$, then*

$$\frac{|B_x(r)|}{|B_y(s)|} \leq d_0^2 \left(\frac{r}{s}\right)^{\log_2 d_0}, \tag{1.2}$$

where d_0 is the controlling constant in VDC.

Proof. Since $d(x,y) \leq r$, for any $z \in B_x(r)$,

$$\begin{aligned} d(z,y) &\leq d(z,x) + d(x,y) \\ &< r + r \\ &= 2r. \end{aligned}$$

That is, $z \in B_y(2r)$. Since z is arbitrary, we have $B_x(r) \subseteq B_y(2r)$ and thus

$$\begin{aligned} |B_x(r)| &\leq |B_y(2r)| \\ &\leq d_0 |B_y(r)|. \quad \text{(by (1.1))} \end{aligned} \tag{1}$$

On the other hand, because $r \geq s > 0$, there exists a positive integer k such that $2^{k-1}s \leq r \leq 2^k s$. Therefore,

$$\begin{aligned} |B_y(r)| &\leq \left|B_y\left(2^k s\right)\right| \\ &\leq d_0^k |B_y(s)| \quad \text{(by (1.1))} \\ &\leq d_0^{1+\log_2 \frac{r}{s}} |B_y(s)| \quad (\text{since } 2^{k-1}s \leq r) \\ &= d_0 \left(\frac{r}{s}\right)^{\log_2 d_0} |B_y(s)|. \end{aligned} \tag{2}$$

Finally, pluging (2) into (1), we obtain

$$|B_x(r)| \leq d_0^2 \left(\frac{r}{s}\right)^{\log_2 d_0} |B_y(s)|.$$

This finishes the proof. □

1.2 s-packing covering

Definition 1.2. *Let Λ be an index set. For any $s > 0$, a collection of balls*

$$\left\{E_\alpha \equiv B_{x_\alpha}\left(\frac{s}{2}\right) : x_\alpha \in M, \alpha \in \Lambda\right\}$$

is said to be an s-packing covering (s-PC) for M if it satisfies the following two conditions:

1. *For any $\alpha, \beta \in \Lambda$, if $\alpha \neq \beta$, then $E_\alpha \cap E_\beta = \emptyset$;*
2. *$M \subseteq \bigcup_{\alpha \in \Lambda} 2E_\alpha$, where $kE_\alpha \equiv B_{x_\alpha}\left(\frac{ks}{2}\right)$ for any $k > 0$.*

Proposition 1.2. *For any $s > 0$, there exists an s-PC for M.*

Proof. First we fix a point $x_0 \in M$ and let $E_0 = B_{x_0}\left(\frac{s}{2}\right)$. If there exists $x_1 \in M$ such that $B_{x_1}\left(\frac{s}{2}\right) \cap B_{x_0}\left(\frac{s}{2}\right) = \emptyset$, then we let $E_1 = B_{x_1}\left(\frac{s}{2}\right)$; otherwise we let $S = \{E_0\}$. In the former case, if there exists $x_2 \in M$ such that $B_{x_2}\left(\frac{s}{2}\right) \cap \left(B_{x_0}\left(\frac{s}{2}\right) \cup B_{x_1}\left(\frac{s}{2}\right)\right) = \emptyset$, then we let $E_2 = B_{x_2}\left(\frac{s}{2}\right)$; otherwise we let $S = \{E_0, E_1\}$. Continue this process, we obtain a collection of balls

$$S = \{E_0, E_1, \ldots, E_n, \ldots\}.$$

For such S, it is clear that any two distinct balls in S are disjoint by construction. So S satisfies the first condition of an s-PC for M. On the other hand, if M is not contained in $\bigcup_{E_\alpha \in S} 2E_\alpha$, then there exists some $x \in M$ such that $x \notin \bigcup_{E_\alpha \in S} 2E_\alpha$ and thus $x \notin 2E_\alpha = B_{x_\alpha}(s)$ for all $E_\alpha \in S$. That is, $d(x, x_\alpha) \geq s$ for all $E_\alpha \in S$. In this case, we consider the ball $B_x\left(\frac{s}{2}\right)$. If there exists some $E_\alpha \in S$ such that $B_x\left(\frac{s}{2}\right) \cap E_\alpha \neq \emptyset$, then there exist some $y \in B_x\left(\frac{s}{2}\right) \cap E_\alpha$ for such E_α and thus

$$\begin{aligned} d(x, x_\alpha) &\leq d(x, y) + d(y, x_\alpha) \\ &< \frac{s}{2} + \frac{s}{2} \\ &= s. \end{aligned}$$

So we get a contradiction since $d(x, x_\alpha) \geq s$ for all $E_\alpha \in S$. Hence, $B_x\left(\frac{s}{2}\right) \cap E_\alpha = \emptyset$ for all $E_\alpha \in S$. By construction of S, $B_x\left(\frac{s}{2}\right)$ must be collected in S. Thus we get that $x \in B_x\left(\frac{s}{2}\right) \subseteq B_x(s) \subseteq \bigcup_{E_\alpha \in S} 2E_\alpha$. This yields a contradiction again, hence M must be contained in $\bigcup_{E_\alpha \in S} 2E_\alpha$. Therefore, S also satisfies the second condition of an s-PC for M and thus S forms an s-PC for M. □

Proposition 1.3. *For any $s > 0$, let $S = \left\{E_\alpha \equiv B_{x_\alpha}\left(\frac{s}{2}\right) : x_\alpha \in M, \alpha \in \Lambda\right\}$, Λ is an index set, be an s-PC for M. For any $z \in M$, define*

$$N(z) \equiv \#\{\alpha \in \Lambda : z \in 8E_\alpha\}.$$

If M has VDC, then

$$\sup_{z \in M} N(z) \leq d_0^6,$$

where d_0 is the controlling constant in VDC.

Proof. For each $z \in M$, let

$$\Lambda_z \equiv \{\alpha \in \Lambda : z \in 8E_\alpha\}.$$

Then, for any $\alpha \in \Lambda_z$, we have $z \in 8E_\alpha = B_{x_\alpha}(4s)$ and thus

$$d(z, x_\alpha) < 4s.$$

So, for any $y \in B_{x_\alpha}\left(\frac{s}{2}\right)$,

$$\begin{aligned} d(y, z) &\leq d(y, x_\alpha) + d(x_\alpha, z) \\ &< \frac{s}{2} + 4s \\ &< 8s. \end{aligned}$$

That is, $y \in B_z(8s)$. Since y is arbitrary, we get

$$B_{x_\alpha}\left(\frac{s}{2}\right) \subseteq B_z(8s). \tag{1}$$

This induces two results. First, since α is arbitrary, we have

$$\bigcup_{\alpha\in\Lambda_z} B_{x_\alpha}\left(\frac{s}{2}\right) \subseteq B_z(8s)$$

and thus

$$\begin{aligned} |B_z(8s)| &\geq \left|\bigcup_{\alpha\in\Lambda_z} B_{x_\alpha}\left(\frac{s}{2}\right)\right| \\ &= \left|\bigcup_{\alpha\in\Lambda_z} E_\alpha\right| \\ &= \sum_{\alpha\in\Lambda_z} |E_\alpha|. \quad \text{(by Definition 1.2)} \end{aligned} \tag{2}$$

Second, for any $\alpha \in \Lambda_z$, by (1), we have $x_\alpha \in B_{x_\alpha}\left(\frac{s}{2}\right) \subseteq B_z(8s)$. So

$$d(x_\alpha, z) < 8s$$

and thus

$$\begin{aligned} \frac{|B_z(8s)|}{|E_\alpha|} &= \frac{|B_z(8s)|}{\left|B_{x_\alpha}\left(\frac{s}{2}\right)\right|} \\ &\leq d_0^2\left(\frac{8s}{\frac{s}{2}}\right)^{\log_2 d_0} \quad \text{(by (1.2))} \\ &= d_0^6. \end{aligned}$$

That is,

$$|E_\alpha| \geq d_0^{-6}|B_z(8s)|.$$

Since α is arbitrary, we get

$$\begin{aligned} \sum_{\alpha\in\Lambda_z}|E_\alpha| &\geq \sum_{\alpha\in\Lambda_z} d_0^{-6}|B_z(8s)| \\ &= d_0^{-6}|B_z(8s)|\sum_{\alpha\in\Lambda_z} 1 \\ &= d_0^{-6}|B_z(8s)|\,\#\{\alpha\in\Lambda : z\in 8E_\alpha\} \\ &= d_0^{-6}|B_z(8s)|\,N(z). \end{aligned} \tag{3}$$

Pluging (3) into (2), we obtain that

$$|B_z(8s)| \geq d_0^{-6}|B_z(8s)|\,N(z).$$

That is,

$$N(z) \leq d_0^6.$$

Finally, since z is arbitrary, the proof of this proposition is finished. □

1.3 Whitney-type covering

Definition 1.3. *Let Λ be an index set. For any ball E of center $x_0 \in M$ and radius $r > 0$, a collection of balls*

$$\left\{E_\alpha \equiv B_{x_\alpha}(r_\alpha) : x_\alpha \in E, r_\alpha = 10^{-3} d(E_\alpha, \partial E), \alpha \in \Lambda\right\}$$

is said to be a Whitney-type covering (WC) for E if it satisfies the following two conditions:

1. *For any $\alpha, \beta \in \Lambda$, if $\alpha \neq \beta$, then $E_\alpha \cap E_\beta = \emptyset$;*
2. *$E \subseteq \bigcup_{\alpha \in \Lambda} 2E_\alpha$, where $kE_\alpha \equiv B_{x_\alpha}(kr_\alpha)$ for any $k > 0$.*

Proposition 1.4. *For any ball E of center $x_0 \in M$ and radius $r > 0$, there exists a WC for E.*

Proof. Let

$$\overline{S} = \left\{B_z(r_z) : z \in E, r_z = \frac{d(z, \partial E)}{1 + 10^3}\right\}.$$

Note that

$$r_z = \frac{d(z, \partial E)}{1 + 10^3}$$

is equivalent to

$$r_z = 10^{-3} d(E_z, \partial E).$$

So, to construct a WC for E, we can pick balls from $\overline{S}$. We start by picking a ball, say E_0, which has the largest possible radius in $\overline{S}$. Actually, E_0 must be the ball of center x_0 and radius $\frac{r}{1+10^3}$. That is,

$$E_0 = B_{x_0}\left(\frac{r}{1 + 10^3}\right).$$

Indeed, for any ball in $\overline{S}$ of center $y \in E \backslash \{x_0\}$, the radius

$$\begin{aligned} r_y &= \frac{d(y, \partial E)}{1 + 10^3} \\ &= \frac{r - d(y, x_0)}{1 + 10^3} \\ &< \frac{r}{1 + 10^3}. \end{aligned}$$

Also, since $r = d(x_0, \partial E)$, the radius of E_0 equals to $\frac{d(x_0, \partial E)}{1+10^3}$ and thus $E_0 \in \overline{S}$.

Next, we show that there exists at least a ball which does not intersect E_0 and has the maximal radius among all the radii of balls in $\overline{S} \backslash E_0$. To do so, we let ρ be the least upper bound of the radii of balls that does not intersect E_0 in $\overline{S} \backslash E_0$. Then, there exists an increasing sequence, say $\{\rho_j\}_{j \in \mathbb{N}}$, of radii of balls that does not intersect E_0 in $\overline{S} \backslash E_0$ such that $\rho_j \to \rho$ as $j \to \infty$. Note that, for each $j \in \mathbb{N}$, there exists at least a point, say z_j, in E such that $B_{z_j}(\rho_j) \in \overline{S}$. Since E is bounded, we can without lossing of generality (by extracting a subsequence if necessary) assume that $z_j \to z$ as $j \to \infty$ for some $z \in E$.

Now we consider the ball $B_z(\rho)$. For such $B_z(\rho)$, first, since $B_{z_j}(\rho_j) \cap E_0 = \emptyset$ for all $j \in \mathbb{N}$, by convergency, $B_z(\rho) \cap E_0 = \emptyset$. Second, $B_z(\rho) \in \overline{S} \backslash E_0$. To see this, first we prove that $B_z(\rho) \in \overline{S}$. That is, we need to show that $\rho = \frac{d(z, \partial E)}{1+10^3}$. For each $j \in \mathbb{N}$, let y_j be a point such that $d(z_j, y_j) = \rho_j$ and $d\left(B_{z_j}\left(\rho_j\right), \partial E\right) = d(y_j, \partial E)$. Then, again, since E is bounded, we can without lossing of generality (by extracting a subsequence if necessary) assume that $y_j \to y$ as $j \to \infty$ for some $y \in E$. For such y, note that

$$\begin{aligned} d(z, y) &= \lim_{j\to\infty} d(z_j, y_j) \\ &= \lim_{j\to\infty} \rho_j \\ &= \rho, \end{aligned}$$

so $y \in \partial B_z(\rho) \subseteq \overline{B_z(\rho)}$ and thus we obtain that

$$\begin{aligned} d(B_z(\rho), \partial E) &= \inf_{p\in B_z(\rho)} d(p, \partial E) \\ &= \inf_{p\in \overline{B_z(\rho)}} d(p, \partial E) \\ &\leq d(y, \partial E) \ \text{(since } y \in \overline{B_z(\rho)}\text{)} \\ &= \lim_{j\to\infty} d(y_j, \partial E) \\ &= \lim_{j\to\infty} d\left(B_{z_j}\left(\rho_j\right), \partial E\right) \\ &= \lim_{j\to\infty} 10^3 \rho_j \\ &= 10^3 \rho. \end{aligned} \tag{1}$$

On the other hand, for any $\varepsilon > 0$, there exists some $J \in \mathbb{N}$ such that $B_z(\rho) \subseteq B_{z_j}\left(\rho_j + \varepsilon\right)$ for all $j \geq J$. Indeed, since $z_j \to z$ and $\rho_j \to \rho$ as $j \to \infty$, there exist $J \in \mathbb{N}$ large enough such that

$$\begin{cases} d(z, z_j) < \frac{\varepsilon}{2} \\ \rho - \rho_j < \frac{\varepsilon}{2} \end{cases}$$

for all $j \geq J$. So, for any $y \in B_z(\rho)$, we have

$$\begin{aligned} d(y, z_j) &\leq d(y, z) + d(z, z_j) \\ &< \rho + \frac{\varepsilon}{2} \\ &< \left(\rho_j + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \\ &= \rho_j + \varepsilon \end{aligned}$$

which indicates that $y \in B_{z_j}\left(\rho_j + \varepsilon\right)$. Since y is arbitrary, we get $B_z(\rho) \subseteq B_{z_j}\left(\rho_j + \varepsilon\right)$ as desired. Now, for the $J \in \mathbb{N}$, because

$$\begin{aligned} d(B_z(\rho), \partial E) &= \inf_{p\in B_z(\rho)} d(p, \partial E) \\ &\geq \inf_{p\in B_{z_j}\left(\rho_j+\varepsilon\right)} d(p, \partial E) \ \text{(since } B_z(\rho) \subseteq B_{z_j}\left(\rho_j + \varepsilon\right)\text{)} \\ &= d\left(B_{z_j}\left(\rho_j + \varepsilon\right), \partial E\right) \\ &= d(z_j, \partial E) - \left(\rho_j + \varepsilon\right) \\ &= d\left(B_{z_j}\left(\rho_j\right), \partial E\right) - \varepsilon \\ &= 10^3 \rho_j - \varepsilon \end{aligned}$$

for all $j \geq J$. So, as j tends to ∞, we get $d(B_z(\rho), \partial E) \geq 10^3 \rho - \varepsilon$. Since ε is arbitrary,

$$d(B_z(\rho), \partial E) \geq 10^3 \rho. \tag{2}$$

Finally, by (1) and (2), we obtain that $d\left(B_z\left(\rho\right), \partial E\right) = 10^3\rho$ which is equivalent to

$$\rho = \frac{d\left(z, \partial E\right)}{1+10^3}.$$

Thus $B_z\left(\rho\right) \in \overline{S}$. On the other hand, since $B_z\left(\rho\right) \cap E_0 = \emptyset$, we can conclude that $B_z\left(\rho\right)$ is a ball which does not intersect E_0 in $\overline{S}\setminus\left\{E_0\right\}$. We pick such $B_z\left(\rho\right)$ and thus, in this case, $\left\{E_0, B_z\left(\rho\right)\right\} \subseteq S$. For convenience, we write $E_1 = B_z\left(\rho\right)$.

Continue this process, we establish a collection of balls

$$S = \left\{E_0, E_1, \ldots, E_n, \ldots\right\}.$$

Such S must satisfies the first condition of a WC for E by its construction. So, to see that S is a WC for E, we have to show that $E \subseteq \bigcup_{E_\alpha \in S} 2E_\alpha$. For any $z \in E$, let $\rho = \frac{d(z,\partial E)}{1+10^3}$. Then, it is clear that $B_z\left(\rho\right) \in \overline{S}$ and $d\left(B_z\left(\rho\right), \partial E\right) = 10^3\rho$. Let N be the largest integer such that $E_N \in S$ has radius $r_N \geq \rho$. If $B_z\left(\rho\right) \cap \left(\bigcup_{n=0}^{N} E_n\right) = \emptyset$, then, because the radius of E_{N+1}, say r_{N+1}, is the least upper bound of radii of balls that does not intersect $\bigcup_{n=0}^{N} E_n$ in $\overline{S}\setminus\left\{E_0, E_1, \ldots, E_N\right\}$, we obtain that $r_{N+1} \geq \rho$. This yields a contradiction since N is the larest interger such that $r_N \geq \rho$. Hence, $B_z\left(\rho\right) \cap \left(\bigcup_{n=0}^{N} E_n\right) \neq \emptyset$. Let $y \in B_z\left(\rho\right) \cap \left(\bigcup_{n=0}^{N} E_n\right)$, then $y \in B_z\left(\rho\right)$ and $y \in E_{n_0}$ for some $0 \leq n_0 \leq N$. Note that $\left\{r_n\right\}_{n\in\mathbb{N}}$ forms a decreasing sequence of radii by the construction of S, we have $r_{n_0} \geq r_N \geq \rho$ and thus

$$\begin{aligned} d\left(z, x_{n_0}\right) &\leq d\left(z, y\right) + d\left(y, x_{n_0}\right) \\ &< \rho + r_{n_0} \\ &\leq 2r_{n_0}. \end{aligned}$$

It implies that $z \in B_{x_{n_0}}\left(2r_{n_0}\right)$. because $B_{x_{n_0}}\left(2r_{n_0}\right) \subseteq \bigcup_{E_\alpha \in S} 2E_\alpha$, we get $z \in \bigcup_{E_\alpha \in S} 2E_\alpha$ immediately. Fincally, since z is arbitrary, we obtain that $E \subseteq \bigcup_{E_\alpha \in S} 2E_\alpha$ as desired. Therefore, S forms a WC for E. □

Let E be a ball of center $x_0 \in M$ and radius $r > 0$. Also, let $S = \{E_\alpha \equiv B_{x_\alpha}\left(r_\alpha\right) : x_\alpha \in E,$ $r_\alpha = 10^{-3} d\left(E_\alpha, \partial E\right), \alpha \in \Lambda\}$ be a WC for E, where Λ is an index set. Then we have some important propositions as follows.

Proposition 1.5. *For any $z \in E$, define*

$$M\left(z\right) \equiv \#\left\{\alpha \in \Lambda : z \in 10^2 E_\alpha\right\}.$$

If M has VDC, then

$$\sup_{z\in E} M\left(z\right) \leq d_0^{13},$$

where d_0 is the controlling constant in VDC.

Proof. For each $z \in E$, let

$$\Lambda_z = \left\{\alpha \in \Lambda : z \in 10^2 E_\alpha\right\}.$$

Then, for any $\alpha \in \Lambda_z$, since $z \in 10^2 E_\alpha = B_{x_\alpha}\left(10^2 r_\alpha\right)$, we have

$$d\left(z, x_\alpha\right) < 10^2 r_\alpha. \tag{1}$$

This gives us two results. First, because

$$\begin{aligned} 10^3 r_\alpha &= d\left(E_\alpha, \partial E\right) \\ &= \inf_{p \in E_\alpha} d\left(p, \partial E\right) \\ &\leq d\left(x_\alpha, \partial E\right) \\ &\leq d\left(x_\alpha, z\right) + d\left(z, \partial E\right) \\ &< 10^2 r_\alpha + d\left(z, \partial E\right), \quad \text{(by (1))} \end{aligned}$$

we have

$$\begin{aligned} d\left(z, \partial E\right) &> 10^3 r_\alpha - 10^2 r_\alpha \\ &= 900 r_\alpha. \end{aligned}$$

That is,

$$r_\alpha < \frac{1}{900} d\left(z, \partial E\right).$$

Thus, for any $x \in E_\alpha = B_{x_\alpha}\left(r_\alpha\right)$,

$$\begin{aligned} d\left(x, z\right) &\leq d\left(x, x_\alpha\right) + d\left(x_\alpha, z\right) \\ &< r_\alpha + 10^2 r_\alpha \text{ (by (1))} \\ &= 101 r_\alpha \\ &< \frac{101}{900} d\left(z, \partial E\right) \\ &< d\left(z, \partial E\right). \end{aligned}$$

That is, $x \in B_z\left(d\left(z, \partial E\right)\right)$. Since x is arbitrary, we get

$$E_\alpha \subseteq B_z\left(d\left(z, \partial E\right)\right). \tag{2}$$

Second, if we let $y \in \partial E$ such that $d\left(x_\alpha, y\right) = d\left(x_\alpha, \partial E\right)$, then

$$\begin{aligned} d\left(z, \partial E\right) &= \inf_{p \in \partial E} d\left(z, p\right) \\ &\leq d\left(z, y\right) \\ &\leq d\left(z, x_\alpha\right) + d\left(x_\alpha, y\right) \\ &< 10^2 r_\alpha + d\left(x_\alpha, \partial E\right) \quad \text{(by (1))} \\ &= 10^2 r_\alpha + \left(10^3 + 1\right) r_\alpha \\ &= 1101 r_\alpha \\ &< 2^{11} r_\alpha. \end{aligned}$$

That is,

$$r_\alpha > 2^{-11} d\left(z, \partial E\right). \tag{3}$$

Note that $\alpha \in \Lambda_z$ is arbitrary in both (2) and (3), we can conclude that, for each $\alpha \in \Lambda$ such that $10^2 E_\alpha$ contains z, the corresponding ball of α has radius at least $2^{-11} d\left(z, \partial E\right)$ and is contained in $B_z\left(d\left(z, \partial E\right)\right)$. Also, since any two different balls in a WC are disjoint, the size of Λ_z is finite. In fact, if we let $\alpha_0 \in \Lambda_z$ be such that its corresponding ball $E_{\alpha_0} = B_{x_{\alpha_0}}\left(r_{\alpha_0}\right)$

reaches the smallest volume among all balls in the collection $\{E_\alpha : \alpha \in \Lambda_z\}$, then an upper bound of the size of Λ_z can be estimated as follows.

$$\begin{aligned}
|\Lambda_z| &\le \frac{|B_z(d(z,\partial E))|}{\left|B_{x_{\alpha_0}}(r_{\alpha_0})\right|} \\
&\le d_0^2 \left(\frac{d(z,\partial E)}{r_{\alpha_0}}\right)^{\log_2 d_0} \quad \text{(by (1.2))} \\
&\le d_0^2 \left(2^{11}\right)^{\log_2 d_0} \quad \text{(by (3))} \\
&= d_0^{13}.
\end{aligned}$$

Finally, since

$$\begin{aligned}
|\Lambda_z| &= \#\left\{\alpha \in \Lambda : z \in 10^2 E_\alpha\right\} \\
&= M(z)
\end{aligned}$$

and z is arbitrary, the proof of this proposition is finished. □

Proposition 1.6. *For any $E_\alpha \in S$, let γ_α be a geodesic curve joining x_0 to x_α. For any $E_\beta \in S$, if $2E_\beta \cap \gamma_\alpha \neq \emptyset$, then*

$$r_\alpha < 2r_\beta. \tag{1.3}$$

Proof. First we note that, since γ_α is a geodesic curve joining x_0 to x_α and x_0 is the center of $E = B_{x_0}(r)$,

$$d(\gamma_\alpha, \partial E) = d(x_\alpha, \partial E). \tag{1}$$

Now, to see the relation between r_β and r_α, we let $y \in 2E_\beta \cap \gamma_\alpha$. For such y, since y is contained in both $2E_\beta$ and γ_α, we get

$$d(y, x_\beta) < 2r_\beta \tag{2}$$

and

$$\begin{aligned}
d(y, \partial E) &> \inf_{p \in \gamma_\alpha} d(p, \partial E) \\
&= d(\gamma_\alpha, \partial E) \\
&= d(x_\alpha, \partial E) \quad \text{(by (1))} \\
&= \left(10^3 + 1\right) r_\alpha \\
&> 10^3 r_\alpha.
\end{aligned} \tag{3}$$

Therefore, by (2) and (3), we have

$$\begin{aligned}
10^3 r_\alpha - 2r_\beta &< d(y, \partial E) - d(y, x_\beta) \\
&\le d(x_\beta, \partial E) \\
&= \left(1 + 10^3\right) r_\beta.
\end{aligned}$$

It implies that

$$\begin{aligned}
r_\alpha &< \frac{3 + 10^3}{10^3} r_\beta \\
&< 2r_\beta
\end{aligned}$$

which finishes the proof. □

Note that, by the construction of a WC for E, the first ball in S is the ball of center x_0 and radius $r_0 = \frac{1}{10^3+1}r$. For such ball, we call it the central ball in S and denote it by E_0.

For any E_α in S with $E_\alpha \neq E_0$, let γ_α be a geodesic curve joining x_0 to x_α. Since S is a WC, we get $E_0 \cap E_\alpha = \emptyset$ and thus $x_\alpha \notin E_0$. In this case, we let p_1 be the first point along γ_α (starting from x_0) which does not belong to $2E_0$. For such p_1, since $p_1 \in E$ and S is a WC for E, there exists a ball, say E_{α_1}, in S such that $p_1 \in 2E_{\alpha_1}$. If $2E_{\alpha_1}$ also contains x_α, then we define a string of balls

$$L_\alpha = \{E_{\alpha_0} = E_0, E_{\alpha_1}, E_{\alpha_2} = E_\alpha\}.$$

If $2E_{\alpha_1}$ does not contain x_α, then we let p_2 be the first point along γ_α (starting from x_0) which does not belong to $2E_{\alpha_1}$. For such p_2, similarly there exists a ball, say E_{α_2}, in S such that $p_2 \in 2E_{\alpha_2}$. If $2E_{\alpha_2}$ contains x_α, then we define a string of balls

$$L_\alpha = \{E_{\alpha_0} = E_0, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_3} = E_\alpha\}.$$

If $2E_{\alpha_2}$ does not contain x_α, then we continue this process until $2E_{\alpha_{n(\alpha)-2}}$ contains both $p_{n(\alpha)-2}$ and x_α for some $n(\alpha) \in \mathbb{N}$. The exsitence of $n(\alpha)$ shall be proved in the next paragraph. In this case, we define a string of balls

$$L_\alpha = \left\{E_{\alpha_0} = E_0, E_{\alpha_1}, \ldots, E_{\alpha_{n(\alpha)-2}}, E_{\alpha_{n(\alpha)-1}} = E_\alpha\right\}.$$

To see $n(\alpha)$ is finite, we note that, for each $E_{\alpha_i} \in L_\alpha$, since $2E_{\alpha_i} \cap \gamma_\alpha \neq \emptyset$,

$$r_{\alpha_i} > \frac{1}{2} r_\alpha. \text{ (by (1.3))}$$

Thus

$$\begin{aligned} n(\alpha) &= |L_\alpha| \\ &\leq \#\left\{\beta \in \Lambda : r_\beta > \frac{1}{2} r_\alpha\right\}. \end{aligned}$$

Since $\frac{1}{2}r_\alpha > 0$, by the construction of a WC for E, we know that the number of balls which are of radius larger than $\frac{1}{2}r_\alpha$ is finite. That is,

$$\#\left\{\beta \in \Lambda : r_\beta > \frac{1}{2} r_\alpha\right\} < \infty.$$

So $n(\alpha)$ is finite.

Now, for any $E_\alpha \in S \backslash \{E_0\}$, if we consider a string of balls L_α for E_α, then we obtain two useful propositions as follows.

Proposition 1.7. *For any $E_\beta \in L_\alpha$, we have*

$$E_\alpha \subseteq 10^4 E_\beta. \tag{1.4}$$

Proof. Let γ_α be the geodesic curve which induces L_α. Then, for any $p \in \gamma_\alpha$, since x_0 is the center of E,

$$d(p, \partial E) = d(p, x_\alpha) + d(x_\alpha, \partial E). \tag{1}$$

Because $E_\beta \in L_\alpha$, $2E_\beta$ must contains at least one point, say y, on γ_α. For such y, since y is contained in both $2E_\beta = B_{x_\beta}(2r_\beta)$ and γ_α, we have

$$d(y, x_\beta) < 2r_\beta$$

and thus

$$\begin{aligned} d(y, x_\alpha) &= d(y, \partial E) - d(x_\alpha, \partial E) \quad \text{(by (1))} \\ &< d(y, \partial E) \\ &\le d(y, x_\beta) + d(x_\beta, \partial E) \\ &< 2r_\beta + \left(10^3 + 1\right) r_\beta \\ &= \left(10^3 + 3\right) r_\beta. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_\alpha, x_\beta) &\le d(x_\alpha, y) + d(y, x_\beta) \\ &< \left(10^3 + 3\right) r_\beta + 2r_\beta \\ &= \left(10^3 + 5\right) r_\beta. \end{aligned} \tag{3}$$

Now, for any $z \in E_\alpha$, because

$$\begin{aligned} d(z, x_\beta) &\le d(z, x_\alpha) + d(x_\alpha, x_\beta) \\ &< r_\alpha + \left(10^3 + 5\right) r_\beta \\ &< 2r_\beta + \left(10^3 + 5\right) r_\beta \quad \text{(by (1.3))} \\ &= \left(10^3 + 7\right) r_\beta \\ &< 10^4 r_\beta, \end{aligned}$$

we have $z \in B_{x_\beta}(10^4 r_\beta) = 10^4 E_\beta$. Finally, since z is arbitrary, we get $E_\alpha \subseteq 10^4 E_\beta$ as desired. □

Proposition 1.8. *For any $E_{\alpha_i}, E_{\alpha_{i+1}} \in L_\alpha$, we have*

1. $2^{-1} r_{\alpha_i} \le r_{\alpha_{i+1}} \le 2r_{\alpha_i}$;
2. $8^{-1} E_{\alpha_i} \subseteq E_{\alpha_{i+1}} \subseteq 8E_{\alpha_i}$;
3. $\max\left\{|E_{\alpha_i}|, \left|E_{\alpha_{i+1}}\right|\right\} \le \left|8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}\right|$.

Proof. Let p be the first point along γ_α (starting from x_0) which does not belong to $2E_{\alpha_i}$, then we have

$$d(p, x_{\alpha_i}) = 2r_{\alpha_i}.$$

Since, by the construction of L_α, $p \in 2E_{\alpha_{i+1}}$, we have

$$d\left(p, x_{\alpha_{i+1}}\right) < 2r_{\alpha_{i+1}}.$$

Therefore,

$$\begin{aligned} d\left(x_{\alpha_i}, x_{\alpha_{i+1}}\right) &\le d(x_{\alpha_i}, p) + d\left(p, x_{\alpha_{i+1}}\right) \\ &< 2r_{\alpha_i} + 2r_{\alpha_{i+1}}. \end{aligned} \tag{1}$$

It implies that

$$\begin{aligned}\left(1+10^3\right) r_{\alpha_{i+1}} &= d\left(x_{\alpha_{i+1}}, \partial E\right) \\ &\leq d\left(x_{\alpha_{i+1}}, x_{\alpha_i}\right) + d\left(x_{\alpha_i}, \partial E\right) \\ &< 2r_{\alpha_i} + 2r_{\alpha_{i+1}} + \left(1+10^3\right) r_{\alpha_i} \quad \text{(by (1))} \\ &= 2r_{\alpha_{i+1}} + \left(3+10^3\right) r_{\alpha_{i+1}}\end{aligned}$$

and thus

$$\begin{aligned}r_{\alpha_{i+1}} &< \frac{3+10^3}{-1+10^3} r_{\alpha_i} \\ &= \frac{1003}{999} r_{\alpha_i} \\ &< 2r_{\alpha_i}.\end{aligned}$$

Similarly,

$$r_{\alpha_i} < 2r_{\alpha_{i+1}}.$$

Combining the last two inequalities, we get

$$\frac{1}{2} r_{\alpha_i} \leq r_{\alpha_{i+1}} \leq 2r_{\alpha_i} \tag{2}$$

as desired.

Next, for any $y \in E_{\alpha_{i+1}} = B_{x_{\alpha_{i+1}}}\left(r_{\alpha_{i+1}}\right)$, because

$$\begin{aligned}d\left(y, x_{\alpha_i}\right) &\leq d\left(y, x_{\alpha_{i+1}}\right) + d\left(x_{\alpha_{i+1}}, x_{\alpha_i}\right) \\ &< r_{\alpha_{i+1}} + 2r_{\alpha_i} + 2r_{\alpha_{i+1}} \quad \text{(by (1))} \\ &= 2r_{\alpha_i} + 3r_{\alpha_{i+1}} \\ &< 2r_{\alpha_i} + 6r_{\alpha_i} \quad \text{(by (2))} \\ &= 8r_{\alpha_i},\end{aligned}$$

we have $y \in B_{x_{\alpha_i}}\left(8r_{\alpha_i}\right) = 8E_{\alpha_i}$. Since y is arbitrary,

$$E_{\alpha_{i+1}} \subseteq 8E_{\alpha_i}.$$

Similarly,

$$E_{\alpha_i} \subseteq 8E_{\alpha_{i+1}}.$$

Combining the last two results, we get

$$\frac{1}{8} E_{\alpha_i} \subseteq E_{\alpha_{i+1}} \subseteq 8E_{\alpha_i} \tag{3}$$

as desired.

Finally, applying (3), we obtain that

$$\begin{cases} E_{\alpha_i} \subseteq 8E_{\alpha_i} \cap 8E_{\alpha_{i+1}} \\ E_{\alpha_{i+1}} \subseteq 8E_{\alpha_i} \cap 8E_{\alpha_{i+1}} \end{cases}$$

and thus

$$\max\left\{\left|E_{\alpha_i}\right|, \left|E_{\alpha_{i+1}}\right|\right\} \leq \left|8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}\right|,$$

which finished the proof. □

2 Poincaré inequality and weighted Poincaré inequality

Let $u \in L^1(M)$. For any $x \in M$ and $r > 0$, we denote $u_{B_x(r)}$ the average of u on $B_x(r)$. That is,

$$u_{B_x(r)} \equiv \frac{1}{|B_x(r)|} \int_{B_x(r)} u.$$

Definition 2.1. *M is said to have the weak L_2 Poincaré inequality (WPI) if there exists a constant $P_2 > 0$ such that*

$$\int_{B_x(r)} \left|u - u_{B_x(r)}\right|^2 \leq P_2 r^2 \int_{B_x(2r)} |\nabla u|^2 \tag{2.1}$$

for any $x \in M$, $r > 0$ and $u \in C^\infty(M)$. In this case, P_2 is called the controlling constant in WPI.

Lemma 2.1. *For any $x_0 \in M$ and $r > 0$, let $E = B_{x_0}(r)$. Let S be a WC for E with the central ball $E_0 = B_{x_0}\left(\frac{r}{10^3+1}\right)$. If M has WPI, then, for any $E_\alpha \in S$ and any $E_{\alpha_i}, E_{\alpha_{i+1}} \in L_\alpha$, we have*

$$\left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right| \leq 8\sqrt{10} P_2^{\frac{1}{2}} \frac{r_{\alpha_i}}{|E_{\alpha_i}|^{\frac{1}{2}}} \left(\int_{64E_{\alpha_i}} |\nabla u|^2\right)^{\frac{1}{2}}, \tag{2.2}$$

for any $u \in C^\infty(M)$, where P_2 is the controlling constant in WPI and L_α is a string of balls joining E_0 to E_α (which is obtained by the process in page 10).

Proof. Let

$$I = \left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \left|8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}\right|,$$

then, since $\left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2$ is constant on M,

$$\begin{aligned}
I &= \left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \int_{8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}} 1 \\
&= \int_{8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}} \left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \\
&= \int_{8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}} \left|u_{8E_{\alpha_i}} - u + u - u_{8E_{\alpha_{i+1}}}\right|^2 \\
&\leq 2 \int_{8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}} \left(\left|u_{8E_{\alpha_i}} - u\right|^2 + \left|u - u_{8E_{\alpha_{i+1}}}\right|^2\right) \\
&\leq 2 \left(\int_{8E_{\alpha_i}} \left|u - u_{8E_{\alpha_i}}\right|^2 + \int_{8E_{\alpha_{i+1}}} \left|u - u_{8E_{\alpha_{i+1}}}\right|^2\right) \\
&\leq 2P_2 \left[(8r_{\alpha_i})^2 \int_{16E_{\alpha_i}} |\nabla u|^2 + \left(8r_{\alpha_{i+1}}\right)^2 \int_{16E_{\alpha_{i+1}}} |\nabla u|^2\right] \quad \text{(by (2.1))} \\
&= 128P_2 \left(r_{\alpha_i}^2 \int_{16E_{\alpha_i}} |\nabla u|^2 + r_{\alpha_{i+1}}^2 \int_{16E_{\alpha_{i+1}}} |\nabla u|^2\right).
\end{aligned} \tag{1}$$

Note that, following the proof in Proposition 1.8, we have

$$d\left(x_{\alpha_{i+1}}, x_{\alpha_i}\right) < 2r_{\alpha_i} + 2r_{i+1}$$

and thus, for any $y \in 16E_{\alpha_{i+1}} = B_{x_{\alpha_{i+1}}}\left(16r_{\alpha_{i+1}}\right)$,

$$\begin{aligned} d\left(y, x_{\alpha_i}\right) &\leq d\left(y, x_{\alpha_{i+1}}\right) + d\left(x_{\alpha_{i+1}}, x_{\alpha_i}\right) \\ &< 16r_{\alpha_{i+1}} + 2r_{\alpha_i} + 2r_{i+1} \\ &= 18r_{\alpha_{i+1}} + 2r_{\alpha_i} \\ &\leq 36r_{\alpha_i} + 2r_{\alpha_i} \text{ (by Proposition 1.8)} \\ &= 38r_{\alpha_i} \\ &< 64r_{\alpha_i}. \end{aligned}$$

That is, $y \in B_{x_{\alpha_i}}\left(64r_{\alpha_i}\right) = 64E_{\alpha_i}$. Since y is arbitrary,

$$16E_{\alpha_{i+1}} \subseteq 64E_{\alpha_i}.$$

Pluging this into (1), we obtain that

$$\begin{aligned} I &\leq 128P_2\left(r_{\alpha_i}^2 \int_{64E_{\alpha_i}} |\nabla u|^2 + r_{\alpha_{i+1}}^2 \int_{64E_{\alpha_i}} |\nabla u|^2\right) \\ &= 128P_2\left(r_{\alpha_i}^2 + r_{\alpha_{i+1}}^2\right) \int_{64E_{\alpha_i}} |\nabla u|^2 \\ &\leq 128P_2\left(r_{\alpha_i}^2 + 4r_{\alpha_i}^2\right) \int_{64E_{\alpha_i}} |\nabla u|^2 \quad \text{(by Proposition 1.8)} \\ &= 640P_2 r_{\alpha_i}^2 \int_{64E_{\alpha_i}} |\nabla u|^2. \end{aligned} \tag{2}$$

On the other hand, applying Proposition 1.8 again, we have

$$\begin{aligned} I &= \left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \left|8E_{\alpha_i} \cap 8E_{\alpha_{i+1}}\right| \\ &\geq \left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \max\left\{\left|E_{\alpha_i}\right|, \left|E_{\alpha_{i+1}}\right|\right\} \\ &\geq \left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \left|E_{\alpha_i}\right|. \end{aligned} \tag{3}$$

Combining (2) and (3), we finally arrive at

$$\left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right|^2 \leq 640P_2 \frac{r_{\alpha_i}^2}{\left|E_{\alpha_i}\right|} \int_{64E_{\alpha_i}} |\nabla u|^2$$

which is equivalent to

$$\left|u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i+1}}}\right| \leq 8\sqrt{10} P_2^{\frac{1}{2}} \frac{r_{\alpha_i}}{\left|E_{\alpha_i}\right|^{\frac{1}{2}}} \left(\int_{64E_{\alpha_i}} |\nabla u|^2\right)^{\frac{1}{2}}.$$

So we get the proof. □

Lemma 2.2. *For any $\delta \in (0,1)$, let $f : [0,\infty) \to \mathbb{R}$ be defined by*

$$f(l) = \begin{cases} 1 & , l \in [0,\delta] \\ \frac{1-l}{1-\delta} & , l \in [\delta, 1] \\ 0 & , l \in [1,\infty) \end{cases} .$$

Fix an $x_0 \in M$ and $r > 0$. Let $E = B_{x_0}(r)$ and $S = \{E_\alpha\}_{\alpha\in\Lambda}$, Λ is an index set, be a WC for E. Let $\varphi : M \to \mathbb{R}$ be defined by

$$\varphi(x) = f\left(\frac{d(x,x_0)}{r}\right)$$

for all $x \in M$. Then, for any $E_\alpha \in S$ and $\kappa \in (1, 10^3]$, we have

$$\sup_{\kappa E_\alpha} \varphi^2 \leq 2^{22} \inf_{\kappa E_\alpha} \varphi^2. \tag{2.3}$$

Proof. For any $E_\alpha \in S$ and $\kappa \in (1, 10^3]$, take y_{α_1} and y_{α_2} on $\partial \kappa E_\alpha$ so that

$$\begin{cases} d(y_{\alpha_1}, x_0) = d(x_\alpha, x_0) - \kappa r_\alpha \\ d(y_{\alpha_2}, x_0) = d(x_\alpha, x_0) + \kappa r_\alpha \end{cases} . \tag{1}$$

For any $x \in \kappa E_\alpha = B_{x_\alpha}(\kappa r_\alpha)$, since $d(x, x_\alpha) < \kappa r_\alpha$, we have

$$\begin{aligned} d(x,x_0) &\leq d(x_0,x_\alpha) + d(x_\alpha,x) \\ &< d(x_0,x_\alpha) + \kappa r_\alpha \\ &= d(y_{\alpha_2},x_0) \ \text{(by (1))} \end{aligned}$$

and

$$\begin{aligned} d(x,x_0) &\geq d(x_0,x_\alpha) - d(x_\alpha,x) \\ &> d(x_0,x_\alpha) - \kappa r_\alpha \\ &= d(y_{\alpha_1},x_0). \ \text{(by (1))} \end{aligned}$$

That is,

$$d(y_{\alpha_1},x_0) < d(x,x_0) < d(y_{\alpha_2},x_0).$$

Since f is positive and decreasing on $[0,\infty)$, the last inequalities implies that

$$\begin{aligned} \sup_{x\in\kappa E_\alpha} [\varphi(x)]^2 &= \sup_{x\in\kappa E_\alpha} \left[f\left(\frac{d(x,x_0)}{r}\right)\right]^2 \\ &= \left[\sup_{x\in\kappa E_\alpha} f\left(\frac{d(x,x_0)}{r}\right)\right]^2 \\ &\leq \left[f\left(\frac{d(y_{\alpha_1},x_0)}{r}\right)\right]^2 \\ &= [\varphi(y_{\alpha_1})]^2 \end{aligned} \tag{2}$$

and

$$\begin{aligned} \inf_{x\in\kappa E_\alpha} [\varphi(x)]^2 &= \inf_{x\in\kappa E_\alpha} \left[f\left(\frac{d(x,x_0)}{r}\right)\right]^2 \\ &= \left[\inf_{x\in\kappa E_\alpha} f\left(\frac{d(x,x_0)}{r}\right)\right]^2 \\ &\geq \left[f\left(\frac{d(y_{\alpha_2},x_0)}{r}\right)\right]^2 \\ &= [\varphi(y_{\alpha_2})]^2. \end{aligned} \tag{3}$$

Now, if $d\left(y_{\alpha_2}, x_0\right) < \delta r$, then $d\left(y_{\alpha_1}, x_0\right) < d\left(y_{\alpha_2}, x_0\right) < \delta r$. So, by the definition of f, we obtain that

$$\begin{aligned} \varphi\left(y_{\alpha_1}\right) &= f\left(\frac{d\left(y_{\alpha_1}, x_0\right)}{r}\right) \\ &= 1 \\ &= f\left(\frac{d\left(y_{\alpha_2}, x_0\right)}{r}\right) \\ &= \varphi\left(y_{\alpha_2}\right). \end{aligned}$$

It implies that

$$\begin{aligned} \sup_{x\in\kappa E_\alpha} \left[\varphi(x)\right]^2 &\leq \left[\varphi\left(y_{\alpha_1}\right)\right]^2 \text{ (by (2))} \\ &= \left[\varphi\left(y_{\alpha_2}\right)\right]^2 \\ &\leq \inf_{x\in\kappa E_\alpha} \left[\varphi(x)\right]^2. \text{ (by (3))} \end{aligned} \tag{4}$$

On the other hand, if $d\left(y_{\alpha_2}, x_0\right) \geq \delta r$, we claim that

$$\sup_{x\in\kappa E_\alpha} \varphi(x) \leq 2^{22} \inf_{x\in\kappa E_\alpha} \varphi(x).$$

To do so, we note that, for any $l \in [0, 1]$,

$$f(l) \leq \frac{1-l}{1-\delta}. \tag{5}$$

Thus

$$\begin{aligned} \varphi\left(y_{\alpha_1}\right) &= f\left(\frac{d\left(y_{\alpha_1}, x_0\right)}{r}\right) \\ &= f\left(\frac{d\left(x_\alpha, x_0\right) - \kappa r_\alpha}{r}\right) \text{ (by (1))} \\ &= f\left(\frac{r - d\left(x_\alpha, \partial E\right) - \kappa r_\alpha}{r}\right) \\ &= f\left(1 - \frac{\left(10^3+1\right) r_\alpha + \kappa r_\alpha}{r}\right) \\ &= f\left(1 - \frac{\left(1001+\kappa\right) r_\alpha}{r}\right) \\ &\leq \frac{1}{1-\delta} \cdot \frac{\left(1001+\kappa\right) r_\alpha}{r} \text{ (by (5))} \\ &\leq \frac{1}{1-\delta} \cdot \frac{2001 r_\alpha}{r} \text{ (since } \kappa \leq 10^3\text{)} \\ &\leq \frac{1}{1-\delta} \cdot \frac{2^{11} r_\alpha}{r}. \end{aligned}$$

Also, by a similar calculation, we have

$$\begin{aligned}\varphi\left(y_{\alpha_2}\right) &= f\left(\frac{d\left(y_{\alpha_1}, x_0\right)}{r}\right) \\ &= f\left(1-\frac{(1001-\kappa)\, r_\alpha}{r}\right) \\ &= \frac{1}{1-\delta}\cdot\frac{(1001-\kappa)\, r_\alpha}{r} \text{ (since } \frac{d\left(y_{\alpha_1}, x_0\right)}{r}\geq\delta) \\ &\geq \frac{1}{1-\delta}\cdot\frac{r_\alpha}{r}. \text{ (since } \kappa\leq 10^3)\end{aligned}$$

Applying the last two estimates, we get

$$\begin{aligned}\left[\varphi\left(y_{\alpha_1}\right)\right]^2 &\leq \frac{1}{(1-\delta)^2}\cdot\frac{2^{22} r_\alpha^2}{r^2} \\ &\leq 2^{22}\left[\varphi\left(y_{\alpha_2}\right)\right]^2\end{aligned}$$

and thus

$$\begin{aligned}\sup_{x\in\kappa E_\alpha}\left[\varphi(x)\right]^2 &\leq \left[\varphi\left(y_{\alpha_1}\right)\right]^2 \text{ (by (2))} \\ &\leq 2^{22}\left[\varphi\left(y_{\alpha_2}\right)\right]^2 \\ &\leq 2^{22}\inf_{x\in\kappa E_\alpha}\left[\varphi(x)\right]^2. \text{ (by (3))}\end{aligned} \tag{6}$$

Combining (4) and (6), we arrive at

$$\sup_{x\in\kappa E_\alpha}\left[\varphi(x)\right]^2\leq 2^{22}\inf_{x\in\kappa E_\alpha}\left[\varphi(x)\right]^2.$$

So we get the proof. □

Proposition 2.1. *Fix $x_0\in M$ and $r>0$. For any $\delta\in(0,1)$, let f and φ be defined as the functions in Lemma 2.2. If M has both VDC and WPI, then there exists a constant $P_w=P_w(d_0,P_2)>0$ such that*

$$\int_M\left|u-u_\varphi\right|^2\varphi^2\leq P_w r^2\int_M|\nabla u|^2\varphi^2, \tag{2.4}$$

for any $u\in C^\infty(M)$, where d_0 and P_2 are the controlling constants in VDC and WPI, respectively. Also, in this inequality,

$$u_\varphi\equiv\frac{1}{\int_M\varphi^2}\int_M u\varphi^2.$$

Proof. By the setting in Lemma 2.2, since $S=\{E_\alpha\}_{\alpha\in\Lambda}$ is a WC for $E=B_{x_0}(r)$, we have

$$E\subseteq\bigcup_{\alpha\in\Lambda}2E_\alpha$$

and thus

$$\begin{aligned}
\int_E |u-u_{8E_0}|^2\varphi^2 &\le \sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{8E_0}|^2\varphi^2\\
&\le \sum_{\alpha\in\Lambda}\int_{2E_\alpha}2\left(|u-u_{8E_\alpha}|^2+|u_{8E_\alpha}-u_{8E_0}|^2\right)\varphi^2\\
&= 2\left(\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{8E_\alpha}|^2\varphi^2+\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u_{8E_\alpha}-u_{8E_0}|^2\varphi^2\right)\\
&= 2\left(I+II\right), \qquad (1)
\end{aligned}$$

where E_0 is the central ball of S and

$$\begin{cases} I=\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{8E_\alpha}|^2\varphi^2\\ II=\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u_{8E_\alpha}-u_{8E_0}|^2\varphi^2\end{cases}.$$

First, we estimate I. Note that, for any $\alpha\in\Lambda$, we have

$$\begin{aligned}
\int_{8E_\alpha}|u-u_{8E_\alpha}|^2\varphi^2 &\le \left(\sup_{8E_\alpha}\varphi^2\right)\int_{8E_\alpha}|u-u_{8E_\alpha}|^2\\
&\le \left(\sup_{16E_\alpha}\varphi^2\right)\int_{8E_\alpha}|u-u_{8E_\alpha}|^2\\
&\le 2^{22}\left(\inf_{16E_\alpha}\varphi^2\right)\int_{8E_\alpha}|u-u_{8E_\alpha}|^2 \text{ (by 2.3)}\\
&\le 2^{22}\left(\inf_{16E_\alpha}\varphi^2\right)P_2\left(8r_\alpha\right)^2\int_{16E_\alpha}|\nabla u|^2 \text{ (by (2.1))}\\
&\le 2^{28}P_2r_\alpha^2\int_{16E_\alpha}|\nabla u|^2\varphi^2\\
&\le 2^{28}P_2r^2\int_{16E_\alpha}|\nabla u|^2\varphi^2.
\end{aligned}$$

Since α is arbitrary, we get

$$\begin{aligned}
I &= \sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{8E_\alpha}|^2\varphi^2\\
&\le \sum_{\alpha\in\Lambda}\int_{8E_\alpha}|u-u_{8E_\alpha}|^2\varphi^2\\
&\le \sum_{\alpha\in\Lambda}2^{28}P_2r^2\int_{16E_\alpha}|\nabla u|^2\varphi^2\\
&= 2^{28}P_2r^2\sum_{\alpha\in\Lambda}\int_{16E_\alpha}|\nabla u|^2\varphi^2. \qquad (2)
\end{aligned}$$

By Proposition 1.5, we know that the overlapping number of 10^2E_α with $E_\alpha\in S$ is uniformly bounded. More precisely, accroding to that proposition, if we let

$$M(z)=\#\left\{\alpha\in\Lambda: z\in 10^2E_\alpha\right\}$$

for all $z \in E$, then, since M has VDC with the controlling constant d_0, we have

$$\sup_{z \in E} M(z) \le d_0^{13}.$$

Thus,

$$\sum_{\alpha \in \Lambda} \int_{10^2 E_\alpha} |\nabla u|^2 \varphi^2 \le d_0^{13} \int_E |\nabla u|^2 \varphi^2. \tag{3}$$

Pluging this into (2), we obtain that

$$\begin{aligned} I &\le 2^{28} P_2 r^2 \sum_{\alpha \in \Lambda} \int_{16 E_\alpha} |\nabla u|^2 \varphi^2 \\ &\le 2^{28} P_2 r^2 \sum_{\alpha \in \Lambda} \int_{10^2 E_\alpha} |\nabla u|^2 \varphi^2 \\ &\le 2^{28} d_0^{13} P_2 r^2 \int_E |\nabla u|^2 \varphi^2. \end{aligned} \tag{4}$$

Next we estimate II. Note that, for any $\alpha \in \Lambda$, $|u_{8E_\alpha} - u_{8E_0}|$ is constant on $8E_\alpha$. So we have

$$\begin{aligned} II &= \sum_{\alpha \in \Lambda} \int_{2E_\alpha} |u_{8E_\alpha} - u_{8E_0}|^2 \varphi^2 \\ &\le \sum_{\alpha \in \Lambda} \int_{8E_\alpha} |u_{8E_\alpha} - u_{8E_0}|^2 \varphi^2 \\ &= \sum_{\alpha \in \Lambda} |u_{8E_\alpha} - u_{8E_0}|^2 \int_{8E_\alpha} \varphi^2 \\ &= \sum_{\alpha \in \Lambda} |u_{8E_\alpha} - u_{8E_0}|^2 \varphi(8E_\alpha), \end{aligned} \tag{5}$$

where $\varphi(8E_\alpha)$ is viewed as a notation which represents the integral of φ^2 on $8E_\alpha$. For any $A \subseteq M$, let $\chi_A : M \to \mathbb{R}$ be defined by

$$\chi_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}.$$

Then

$$\begin{aligned} (5) &= \sum_{\alpha \in \Lambda} |u_{8E_\alpha} - u_{8E_0}|^2 \varphi(8E_\alpha) \left(\int_M \frac{1}{|E_\alpha|} \chi_{E_\alpha} \right) \\ &= \int_M \sum_{\alpha \in \Lambda} |u_{8E_\alpha} - u_{8E_0}|^2 \frac{\varphi(8E_\alpha)}{|E_\alpha|} \chi_{E_\alpha}. \end{aligned} \tag{6}$$

Now, for any $E_\alpha \in S$, let

$$L_\alpha = \left\{ E_{\alpha_0} = E_0, E_{\alpha_1}, \ldots, E_{\alpha_{n(\alpha)-2}}, E_{\alpha_{n(\alpha)-1}} = E_\alpha \right\}$$

be a string of balls joining E_0 to E_α (which is obtained by the method in page 10). Then we have

$$\begin{aligned} |u_{8E_\alpha} - u_{8E_0}| &= \left| u_{8E_{\alpha_{n(\alpha)-1}}} - u_{8E_{\alpha_0}} \right| \\ &\le \sum_{i=1}^{n(\alpha)-1} \left| u_{8E_{\alpha_i}} - u_{8E_{\alpha_{i-1}}} \right|. \end{aligned} \tag{7}$$

Also, for any $E_{\alpha_i} \in L_\alpha$, we have

$$\begin{aligned}
\inf_{p\in 64E_{\alpha_i}} d(p,\partial E) &= d(64E_{\alpha_i},\partial E) \\
&= d(x_{\alpha_i},\partial E) - 64r_{\alpha_i} \\
&= \left(10^3+1\right) r_{\alpha_i} - 64r_{\alpha_i} \\
&= \left(10^3-63\right) r_{\alpha_i} \\
&> 2r_{\alpha_i} \\
&> r_\alpha \text{ (by (1.3))} \\
&= \left(10^3+1\right) r_\alpha - 10^3 r_\alpha \\
&= d(x_\alpha,\partial E) - 10^3 r_\alpha \\
&= d\left(10^3 E_\alpha,\partial E\right) \\
&= \inf_{p\in 10^3 E_\alpha} d(p,\partial E).
\end{aligned}$$

So, if we let $y_\alpha \in \partial(10^3 E_\alpha)$ and $y_{\alpha_i} \in \partial(64E_{\alpha_i})$ such that

$$\begin{cases} d(y_\alpha,\partial E) = \inf_{p\in 10^3 E_\alpha} d(p,\partial E) \\ d(y_{\alpha_i},\partial E) = \inf_{p\in 64E_{\alpha_i}} d(p,\partial E) \end{cases},$$

then

$$\begin{aligned}
d(y_\alpha,x_0) &= r - d(y_\alpha,\partial E) \\
&= r - \inf_{p\in 10^3 E_\alpha} d(p,\partial E) \\
&> r - \inf_{p\in 64E_{\alpha_i}} d(p,\partial E) \\
&= r - d(y_{\alpha_i},\partial E) \\
&= d(y_{\alpha_i},x_0).
\end{aligned}$$

Since f is decreasing on $[0,\infty)$, we get

$$\begin{aligned}
\varphi(y_\alpha) &= f\left(\frac{d(y_\alpha,x_0)}{r}\right) \\
&< f\left(\frac{d(y_{\alpha_i},x_0)}{r}\right) \\
&= \varphi(y_{\alpha_i}). \qquad (8)
\end{aligned}$$

Note that y_α is a point in $\overline{10^3E_\alpha}$ such that y_α is closest to ∂E, that is, y_α is a piont in $\overline{10^3E_\alpha}$ such that y_α is farest from x_0. Thus $\varphi(x) = f\left(\frac{d(x,x_0)}{r}\right)$ must arrive at its infimum at y_α on $\overline{10^3E_\alpha}$ since f is decreasing on $[0,\infty)$. That is,

$$\begin{aligned}
\varphi(y_\alpha) &= \inf_{x\in\overline{10^3E_\alpha}} \varphi(x) \\
&= \inf_{x\in 10^3E_\alpha} \varphi(x). \text{ (since } f \text{ and } d \text{ are continuous)}
\end{aligned}$$

Similarly, we have

$$\varphi(y_{\alpha_i}) = \inf_{x\in 64E_{\alpha_i}} \varphi(x).$$

So

$$\begin{aligned}\inf_{x\in 10^3 E_\alpha}\varphi(x) &= \varphi(y_\alpha)\\ &< \varphi(y_{\alpha_i})\ \text{(by (8))}\\ &= \inf_{x\in 64E_{\alpha_i}}\varphi(x)\end{aligned}$$

and thus

$$\inf_{10^3 E_\alpha}\varphi^2 < \inf_{64E_{\alpha_i}}\varphi^2$$

since f is positive on $[0,\infty)$. Applying this inequality, we get

$$\begin{aligned}\frac{\varphi(8E_\alpha)}{|E_\alpha|} &= \frac{1}{|E_\alpha|}\int_{8E_\alpha}\varphi^2\\ &\le \frac{1}{|E_\alpha|}\left(\sup_{8E_\alpha}\varphi^2\right)\int_{8E_\alpha}1\\ &= 8\sup_{8E_\alpha}\varphi^2\\ &\le 8\cdot 2^{22}\inf_{8E_\alpha}\varphi^2\ \text{(by (2.3))}\\ &= 2^{25}\inf_{8E_\alpha}\varphi^2\\ &\le 2^{25}\inf_{10^3E_\alpha}\varphi^2\\ &< 2^{25}\inf_{64E_{\alpha_i}}\varphi^2.\end{aligned}$$

Note that E_{α_i} is arbitrary, we can conclude the last inequality in the following form

$$\frac{\varphi(8E_\alpha)}{|E_\alpha|} < 2^{25}\inf_{64E_{\alpha_{i-1}}}\varphi^2 \tag{9}$$

for all $i=1,2,\ldots,n(\alpha)$.

Now, let

$$II_\alpha = |u_{8E_\alpha}-u_{8E_0}|\left(\frac{\varphi(8E_\alpha)}{|E_\alpha|}\right)^{\frac{1}{2}}, \tag{10}$$

then we get

$$\begin{aligned}II_\alpha &< 2^{13}\sum_{i=1}^{n(\alpha)-1}\left|u_{8E_{\alpha_i}}-u_{8E_{\alpha_{i-1}}}\right|\left(\inf_{64E_{\alpha_{i-1}}}\varphi^2\right)^{\frac{1}{2}}\ \text{(by (7) and (9))}\\ &\le 2^{13}\sum_{i=1}^{n(\alpha)-1}8\sqrt{10}P_2^{\frac{1}{2}}\frac{r_{\alpha_{i-1}}}{\left|E_{\alpha_{i-1}}\right|^{\frac{1}{2}}}\left(\int_{64E_{\alpha_{i-1}}}|\nabla u|^2\right)^{\frac{1}{2}}\left(\inf_{64E_{\alpha_{i-1}}}\varphi^2\right)^{\frac{1}{2}}\ \text{(by (2.2))}\\ &\le 2^{20}P_2^{\frac{1}{2}}\sum_{i=1}^{n(\alpha)-1}\frac{r_{\alpha_{i-1}}}{\left|E_{\alpha_{i-1}}\right|^{\frac{1}{2}}}\left(\int_{64E_{\alpha_{i-1}}}|\nabla u|^2\varphi^2\right)^{\frac{1}{2}}\\ &\le 2^{20}P_2^{\frac{1}{2}}r\sum_{i=1}^{n(\alpha)-1}\frac{1}{\left|E_{\alpha_{i-1}}\right|^{\frac{1}{2}}}\left(\int_{64E_{\alpha_{i-1}}}|\nabla u|^2\varphi^2\right)^{\frac{1}{2}}\end{aligned}$$

and thus

$$
\begin{aligned}
II_{\alpha}^{2}\chi_{E_{\alpha}} &< \left[2^{20}P_{2}^{\frac{1}{2}}r\sum_{i=1}^{n(\alpha)-1}\frac{1}{\left|E_{\alpha_{i-1}}\right|^{\frac{1}{2}}}\left(\int_{64E_{\alpha_{i-1}}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}\right]^{2}\chi_{E_{\alpha}} \\
&= 2^{40}P_{2}r^{2}\left[\sum_{i=1}^{n(\alpha)-1}\frac{1}{\left|E_{\alpha_{i-1}}\right|^{\frac{1}{2}}}\left(\int_{64E_{\alpha_{i-1}}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}\chi_{10^{4}E_{\alpha_{i-1}}}\right]^{2}\chi_{E_{\alpha}}\ \text{(by (1.4))} \\
&\leq 2^{40}P_{2}r^{2}\left[\sum_{E_{\beta}\in S}\frac{1}{|E_{\beta}|^{\frac{1}{2}}}\left(\int_{64E_{\beta}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}\chi_{10^{4}E_{\beta}}\right]^{2}\chi_{E_{\alpha}}.
\end{aligned}
$$

Since the last inequality holds for any $E_{\alpha}\in S$, we have

$$
\begin{aligned}
\sum_{E_{\alpha}\in S}II_{\alpha}^{2}\chi_{E_{\alpha}} &< 2^{40}P_{2}r^{2}\sum_{E_{\alpha}\in S}\left[\sum_{E_{\beta}\in S}\frac{1}{|E_{\beta}|^{\frac{1}{2}}}\left(\int_{64E_{\beta}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}\chi_{10^{4}E_{\beta}}\right]^{2}\chi_{E_{\alpha}} \\
&= 2^{40}P_{2}r^{2}\left[\sum_{E_{\beta}\in S}\frac{1}{|E_{\beta}|^{\frac{1}{2}}}\left(\int_{64E_{\beta}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}\chi_{10^{4}E_{\beta}}\right]^{2}\sum_{E_{\alpha}\in S}\chi_{E_{\alpha}}
\end{aligned}
$$

Note S is a WC for E, so any two different balls in S are disjoint. Therefore,

$$
\sum_{E_{\alpha}\in S}\chi_{E_{\alpha}}\leq 1
$$

and thus the last inequality becomes

$$
\begin{aligned}
\sum_{E_{\alpha}\in S}II_{\alpha}^{2}\chi_{E_{\alpha}} &< 2^{40}P_{2}r^{2}\left[\sum_{E_{\beta}\in S}\frac{1}{|E_{\beta}|^{\frac{1}{2}}}\left(\int_{64E_{\beta}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}\chi_{10^{4}E_{\beta}}\right]^{2} \\
&= 2^{40}P_{2}r^{2}\left(\sum_{E_{\beta}\in S}J_{\beta}\chi_{10^{4}E_{\beta}}\right)^{2}, \qquad (11)
\end{aligned}
$$

where

$$
J_{\beta}\equiv\frac{1}{|E_{\beta}|^{\frac{1}{2}}}\left(\int_{64E_{\beta}}|\nabla u|^{2}\varphi^{2}\right)^{\frac{1}{2}}. \qquad (12)
$$

Let $g=\sum_{E_{\beta}\in S}J_{\beta}\chi_{10^{4}E_{\beta}}$. Since $J_{\beta}\geq 0$ for all $E_{\beta}\in S$, we have $g\geq 0$. If $g>0$, note that

$$
\left\|\frac{g}{\left(\int_{M}g^{2}\right)^{\frac{1}{2}}}\right\|_{2}=1
$$

and $\frac{g}{\left(\int_M g^2\right)^{\frac{1}{2}}} > 0$, we have

$$\begin{aligned}\left(\int_M g^2\right)^{\frac{1}{2}} &= \frac{\int_M g^2}{\left(\int_M g^2\right)^{\frac{1}{2}}} \\ &= \int_M g \frac{g}{\left(\int_M g^2\right)^{\frac{1}{2}}} \\ &\leq \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \int_M g\rho.\end{aligned}$$

If $g = 0$, then the last inequality holds automaticlly. Therefore, we can conclude that

$$\left(\int_M g^2\right)^{\frac{1}{2}} \leq \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \int_M g\rho.$$

That is,

$$\left[\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right)^2\right]^{\frac{1}{2}} \leq \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right) \rho.$$

Since

$$\begin{aligned}R.H.S. &= \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \sum_{E_\beta \in S} J_\beta \int_{10^4 E_\beta} \rho \\ &= \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \sum_{E_\beta \in S} J_\beta \left|10^4 E_\beta\right| \frac{1}{\left|10^4 E_\beta\right|} \int_{10^4 E_\beta} \rho \\ &\leq \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \sum_{E_\beta \in S} J_\beta d_0^{16} \left|E_\beta\right| \frac{1}{\left|10^4 E_\beta\right|} \int_{10^4 E_\beta} \rho \text{ (by (1.1))} \\ &= d_0^{16} \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \sum_{E_\beta \in S} J_\beta \left|E_\beta\right| \frac{1}{\left|10^4 E_\beta\right|} \int_{10^4 E_\beta} \rho\end{aligned}$$

we get

$$\left[\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right)^2\right]^{\frac{1}{2}} \leq d_0^{16} \sup_{\substack{\|\rho\|_2=1, \\ \rho>0}} \sum_{E_\beta \in S} J_\beta \left|E_\beta\right| \frac{1}{\left|10^4 E_\beta\right|} \int_{10^4 E_\beta} \rho. \tag{13}$$

Now, for any $z \in E_\beta = B_{x_\beta}(r_\beta)$, if $y \in 10^4 E_\beta = B_{x_\beta}(10^4 r_\beta)$, then

$$\begin{aligned}d(y,z) &= d(y,x_\beta) + d(x_\beta,z) \\ &< 10^4 r_\beta + r_\beta \\ &< 10^5 r_\beta.\end{aligned}$$

This implies $y \in B_z(10^5 r_\beta)$. Since y is arbitrary, we get

$$10^4 E_\beta \subseteq B_z\left(10^5 r_\beta\right)$$

and thus, for any positive $\rho \in C(M)$,

$$\begin{aligned}
\frac{1}{|10^4 E_\beta|}\int_{10^4 E_\beta}\rho &\le \frac{1}{|10^4 E_\beta|}\int_{B_z\left(10^5 r_\beta\right)}\rho \\
&= \frac{|B_z(10^5 r_\beta)|}{|10^4 E_\beta|}\cdot\frac{1}{|B_z(10^5 r_\beta)|}\int_{B_z\left(10^5 r_\beta\right)}\rho \\
&= \frac{|B_z(10^5 r_\beta)|}{\left|B_{x_\beta}(10^4 r_\beta)\right|}\cdot\frac{1}{|B_z(10^5 r_\beta)|}\int_{B_z\left(10^5 r_\beta\right)}\rho \\
&\le d_0^2\left(\frac{10^5 r_\beta}{10^4 r_\beta}\right)^{\log_2 d_0}\frac{1}{|B_z(10^5 r_\beta)|}\int_{B_z\left(10^5 r_\beta\right)}\rho \text{ (by (1.2))} \\
&= d_0^2 10^{\log_2 d_0}\frac{1}{|B_z(10^5 r_\beta)|}\int_{B_z\left(10^5 r_\beta\right)}\rho \\
&< d_0^6\frac{1}{|B_z(10^5 r_\beta)|}\int_{B_z\left(10^5 r_\beta\right)}\rho \\
&\le d_0^6\sup_{s>0}\frac{1}{|B_z(s)|}\int_{B_z(s)}\rho \\
&= d_0^6 M\rho(z),
\end{aligned} \tag{14}$$

where $M\rho : M \to \mathbb{R}$ is the maximal function defined by

$$M\rho(z) = \sup_{s>0}\frac{1}{|B_z(s)|}\int_{B_z(s)}\rho$$

for all $z \in M$. Note that, in (14), $z \in E_\beta$ is arbitrary, we have

$$\begin{aligned}
\int_{E_\beta}\left(\frac{1}{|10^4 E_\beta|}\int_{10^4 E_\beta}\rho\right)dz &< \int_{E_\beta} d_0^6 M\rho(z)\,dz \\
&= d_0^6\int_{E_\beta} M\rho.
\end{aligned}$$

Since

$$\begin{aligned}
L.H.S. &= \left(\frac{1}{|10^4 E_\beta|}\int_{10^4 E_\beta}\rho\right)\left(\int_{E_\beta} 1dz\right) \\
&= \frac{|E_\beta|}{|10^4 E_\beta|}\int_{10^4 E_\beta}\rho,
\end{aligned}$$

we get

$$\frac{|E_\beta|}{|10^4 E_\beta|}\int_{10^4 E_\beta}\rho < d_0^6\int_{E_\beta} M\rho.$$

Pluging this into (13), we arrive at

$$\begin{aligned}
\left[\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right)^2\right]^{\frac{1}{2}} &< d_0^{16} \sup_{\substack{\|\rho\|_2=1,\\ \rho>0}} \sum_{E_\beta \in S} J_\beta d_0^6 \int_{E_\beta} M\rho \\
&= d_0^{22} \sup_{\substack{\|\rho\|_2=1,\\ \rho>0}} \int_M \sum_{E_\beta \in S} J_\beta \chi_{E_\beta} M\rho \\
&\leq d_0^{22} \sup_{\substack{\|\rho\|_2=1,\\ \rho>0}} \left[\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{E_\beta}\right)^2\right]^{\frac{1}{2}} \left[\int_M (M\rho)^2\right]^{\frac{1}{2}} \\
&\leq d_0^{22} \sup_{\substack{\|\rho\|_2=1,\\ \rho>0}} \left[\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{E_\beta}\right)^2\right]^{\frac{1}{2}} \cdot C \left(\int_M \rho^2\right)^{\frac{1}{2}} \\
&= C d_0^{22} \left[\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{E_\beta}\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

for some constant $C = C(d_0) > 0$, where we apply the Hölder inequality and the maximal function property in the third and the fourth inequality, respectively. Note that the last inequality is equivalent to

$$\int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right)^2 \leq C^2 d_0^{44} \int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{E_\beta}\right)^2. \tag{15}$$

Thus,

$$\begin{aligned}
II &\leq \int_M \sum_{\alpha \in \Lambda} |u_{8E_\alpha} - u_{8E_0}|^2 \frac{\varphi(8E_\alpha)}{|E_\alpha|} \chi_{E_\alpha} \text{ (by (6))} \\
&= \int_M \sum_{\alpha \in \Lambda} II_\alpha^2 \chi_{E_\alpha} \text{ (by (10))} \\
&< \int_M 2^{40} P_2 r^2 \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right)^2 \text{ (by (11))} \\
&= 2^{40} P_2 r^2 \int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{10^4 E_\beta}\right)^2 \\
&\leq 2^{40} P_2 r^2 C^2 d_0^{44} \int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{E_\beta}\right)^2 \text{ (by (15))} \\
&= 2^{40} C^2 d_0^{44} P_2 r^2 \int_M \left(\sum_{E_\beta \in S} J_\beta \chi_{E_\beta}\right)^2.
\end{aligned} \tag{16}$$

Again, since S is a WC for E, the intersection of any two distinct balls in E is empty. Thus

$$\begin{aligned}\left(\sum_{E_\beta\in S} J_\beta \chi_{E_\beta}\right)^2 &= \sum_{E_\beta\in S}\left(J_\beta\chi_{E_\beta}\right)^2\\ &= \sum_{E_\beta\in S} J_\beta^2\chi_{E_\beta}.\end{aligned}$$

Also, note that $J_\beta = \frac{1}{|E_\beta|^{\frac{1}{2}}}\left(\int_{64E_\beta}|\nabla u|^2\varphi^2\right)^{\frac{1}{2}}$ is constant on M, so

$$\begin{aligned}\int_M\sum_{E_\beta\in S} J_\beta^2\chi_{E_\beta} &= \sum_{E_\beta\in S}\int_M J_\beta^2\chi_{E_\beta}\\ &= \sum_{E_\beta\in S} J_\beta^2\int_M \chi_{E_\beta}\\ &= \sum_{E_\beta\in S} J_\beta^2\,|E_\beta|\,.\end{aligned}$$

Applying the last two results in (16), we get

$$\begin{aligned}(16) &= 2^{40}C^2 d_0^{44}P_2r^2\int_M\sum_{E_\beta\in S}J_\beta^2\chi_{E_\beta}\\ &= 2^{40}C^2 d_0^{44}P_2r^2\sum_{E_\beta\in S}J_\beta^2\,|E_\beta|\\ &= 2^{40}C^2 d_0^{44}P_2r^2\sum_{E_\beta\in S}\frac{1}{|E_\beta|}\left(\int_{64E_\beta}|\nabla u|^2\varphi^2\right)|E_\beta| \ \text{ (by (12))}\\ &= 2^{40}C^2 d_0^{44}P_2r^2\sum_{E_\beta\in S}\int_{64E_\beta}|\nabla u|^2\varphi^2\\ &\leq 2^{40}C^2 d_0^{44}P_2r^2\sum_{E_\beta\in S}\int_{10^2E_\beta}|\nabla u|^2\varphi^2\\ &\leq 2^{40}C^2 d_0^{57}P_2r^2\int_E|\nabla u|^2\varphi^2. \text{ (by (3))}\end{aligned}$$

That is, we have

$$II \leq 2^{40}C^2 d_0^{57}P_2r^2\int_E|\nabla u|^2\varphi^2. \tag{17}$$

Now, combining (4) and (17), we arrive at

$$\begin{aligned}\int_E|u-u_{8E_0}|^2\varphi^2 &\leq 2\,(I+II)\\ &\leq 2^{28}d_0^{13}P_2r^2\int_E|\nabla u|^2\varphi^2 + 2^{40}C^2d_0^{57}P_2r^2\int_E|\nabla u|^2\varphi^2\\ &\leq 2^{40}d_0^{57}P_2r^2\int_E|\nabla u|^2\varphi^2 + 2^{40}C^2d_0^{57}P_2r^2\int_E|\nabla u|^2\varphi^2 \text{ (since } d_0\geq 1)\\ &\leq 2^{40}d_0^{57}\left(1+C^2\right)P_2r^2\int_E|\nabla u|^2\varphi^2.\end{aligned}$$

Note that u_φ is constant on M, so $u_\varphi \in C^\infty(M)$. Thus, by (18), we have

$$\begin{aligned}\int_E |u_\varphi - u_{8E_0}|^2 \varphi^2 &\le 2^{40} d_0^{57} \left(1 + C^2\right) P_2 r^2 \int_E |\nabla u_\varphi|^2 \varphi^2 \\ &= 0. \text{ (since } u_\varphi \text{ is constant on } M) \end{aligned} \tag{19}$$

Finally, applying the last inequalities, we get

$$\begin{aligned}\int_M |u - u_\varphi|^2 \varphi^2 &= \int_E |u - u_\varphi|^2 \varphi^2 \text{ (since } \varphi \in C_0(E)\text{)} \\ &\le \int_E 2\left(|u - u_{8E_0}|^2 + |u_{8E_0} - u_\varphi|^2\right)\varphi^2 \\ &= 2\left(\int_E |u - u_{8E_0}|^2 \varphi^2 + \int_E |u_{8E_0} - u_\varphi|^2 \varphi^2\right) \\ &\le 2\left[2^{40} d_0^{57}\left(1 + C^2\right) P_2 r^2 \int_E |\nabla u|^2 \varphi^2 + 0\right] \text{ (by (18) and (19))} \\ &= 2^{41} d_0^{57}\left(1 + C^2\right) P_2 r^2 \int_E |\nabla u|^2 \varphi^2 \\ &= 2^{41} d_0^{57}\left(1 + C^2\right) P_2 r^2 \int_M |\nabla u|^2 \varphi^2 \\ &= P_w r^2 \int_M |\nabla u|^2 \varphi^2, \end{aligned}$$

where

$$P_w = 2^{41} d_0^{57}\left(1 + C^2\right) P_2$$

is a positive constant dependent only on d_0 and P_2. Thus the proof is finished. □

3 Nash inequality and Sobolev inequality

3.1 Nash inequality

For any $s > 0$, let $u_s : M \to \mathbb{R}$ be defined by

$$u_s(y) \equiv u_{B_y(s)}$$

for all $y \in M$. Also, we denote by $C_0^\infty(\Omega)$ the set of all C^∞ functions defined on M and supported on Ω, where Ω is a domain of M.

Lemma 3.1. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. If M has both VDC, then*

$$\int_M u_s^2 \le \frac{d_0^5 \left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|} \left(\int_M |u|\right)^2$$

holds for any $s \in (0, r]$ and $u \in C_0^\infty(E)$, where d_0 is the controlling constant in VDC.

Proof. First we estimate $\sup_M |u_s|$. For any $y \in M$, if $E \cap B_y(s) = \emptyset$, then

$$B_y(s) \subseteq E^C,$$

where E^C is the complement of E in M. Therefore,

$$\begin{aligned}
|u_s(y)| &= \left|\frac{1}{|B_y(s)|}\int_{B_y(s)} u\right| \\
&\leq \frac{1}{|B_y(s)|}\int_{B_y(s)} |u| \\
&\leq \frac{1}{|B_y(s)|}\int_{E^C} |u| \\
&= 0. \text{ (since } u \in C_0^\infty(E)\text{)}
\end{aligned}$$

Note that y is arbitrary with $E \cap B_y(s) = \emptyset$, so we get

$$\sup_{\substack{y\in M \\ E\cap B_y(s)=\emptyset}} |u_s(y)| = 0. \tag{1}$$

On the other hand, if $E \cap B_y(s) \neq \emptyset$, then there exists some $z \in E \cap B_y(s)$. By such z, we have

$$\begin{aligned}
d(y, x_0) &\leq d(y,z) + d(z, x_0) \\
&< r + s
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{|E|}{|B_y(s)|} &= \frac{|B_{x_0}(r)|}{|B_y(s)|} \\
&\leq \frac{|B_{x_0}(r+s)|}{|B_y(s)|} \\
&\leq d_0^2\left(\frac{r+s}{s}\right)^{\log_2 d_0} \qquad \text{(by (1.2))} \\
&\leq d_0^2\left(\frac{2r}{s}\right)^{\log_2 d_0} \qquad \text{(since } s \leq r\text{)} \\
&= d_0^2 \cdot 2^{\log_2 d_0}\left(\frac{r}{s}\right)^{\log_2 d_0} \\
&= d_0^3\left(\frac{r}{s}\right)^{\log_2 d_0}.
\end{aligned} \tag{2}$$

It implies that

$$\begin{aligned}
|u_s(y)| &= \left|\frac{1}{|B_y(s)|}\int_{B_y(s)} u\right| \\
&\leq \frac{1}{|B_y(s)|}\int_{B_y(s)} |u| \\
&= \frac{1}{|E|}\cdot\frac{|E|}{|B_y(s)|}\int_{B_y(s)} |u| \\
&\leq \frac{1}{|E|}\cdot d_0^3\left(\frac{r}{s}\right)^{\log_2 d_0}\int_{B_y(s)} |u|. \text{ (by (2))}
\end{aligned}$$

So, in this case, since y is arbitrary with $E \cap B_y(s) \neq \emptyset$, we get

$$\sup_{\substack{y \in M \\ E \cap B_y(s) \neq \emptyset}} |u_s(y)| \leq \frac{d_0^3 \left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|} \int_{B_y(s)} |u| . \tag{3}$$

Combining (1) and (3), we get

$$\begin{aligned} \sup_{y \in M} |u_s(y)| &\leq \frac{d_0^3 \left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|} \int_{B_y(s)} |u| \\ &\leq \frac{d_0^3 \left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|} \int_M |u| . \end{aligned} \tag{4}$$

Next we estimate $\int_M u_s^2$. Note that

$$\begin{aligned} \int_M |u_s| &= \int_M \left| \frac{1}{|B_y(s)|} \int_{B_y(s)} u(z)\, dz \right| dy \\ &\leq \int_M \frac{\int_{B_y(s)} |u(z)|\, dz}{|B_y(s)|} dy \\ &= \int_M \left(\int_{B_z(s)} \frac{1}{|B_y(s)|} dy \right) |u(z)|\, dz \\ &= \int_M \left(\int_{B_z(s)} \frac{|B_z(s)|}{|B_y(s)|} \cdot \frac{1}{|B_z(s)|} dy \right) |u(z)|\, dz \\ &\leq \int_M \left(\int_{B_z(s)} d_0^2 \left(\frac{s}{s}\right)^{\log_2 d_0} \frac{1}{|B_z(s)|} dy \right) |u(z)|\, dz \text{ (by (1.2))} \\ &= d_0^2 \int_M \left(\int_{B_z(s)} \frac{1}{|B_z(s)|} dy \right) |u(z)|\, dz \\ &= d_0^2 \int_M |u(z)|\, dz \\ &= d_0^2 \int_M |u| . \end{aligned}$$

That is,

$$\int_M |u_s| \leq d_0^2 \int_M |u| . \tag{5}$$

Finally, by previous arguments, we obtain that

$$\begin{aligned} \int_M u_s^2 &= \int_M |u_s|\, |u_s| \\ &\leq \left(\sup_M |u_s| \right) \int_M |u_s| \\ &\leq \left(\frac{d_0^3 \left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|} \int_M |u| \right) \cdot \left(d_0^2 \int_M |u| \right) \\ &= \frac{d_0^5 \left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|} \left(\int_M |u| \right)^2 . \end{aligned}$$

So the proof is finished. □

Lemma 3.2. *If M has both VDC and WPI, then*

$$\int_M |u-u_s|^2 \le 8d_0^6\left(1+d_0^2\right)P_2 s^2 \int_M |\nabla u|^2$$

holds for any $s>0$ and $u \in C_0^\infty(M)$, where d_0 and P_2 are the controlling constants in VDC and WPI, respectively.

Proof. Let $S=\{E_\alpha\}_{\alpha\in\Lambda}$ be an s-PC for M. Since

$$M \subseteq \bigcup_{\alpha\in\Lambda} 2E_\alpha,$$

we have

$$\begin{aligned}
\int_M |u-u_s|^2 &\le \int_{\bigcup_{\alpha\in\Lambda}2E_\alpha} |u-u_s|^2 \\
&\le \sum_{\alpha\in\Lambda}\int_{2E_\alpha} |u-u_s|^2 \\
&\le \sum_{\alpha\in\Lambda}\int_{2E_\alpha} 2\left(|u-u_{4E_\alpha}|^2+|u_{4E_\alpha}-u_s|^2\right) \\
&= 2\left(\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{4E_\alpha}|^2+\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u_{4E_\alpha}-u_s|^2\right) \\
&= 2\left(I+II\right), \qquad (1)
\end{aligned}$$

where

$$\begin{cases} I=\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{4E_\alpha}|^2 \\ II=\sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u_{4E_\alpha}-u_s|^2 \end{cases}.$$

First we estimate I. By a direct calculation, we have

$$\begin{aligned}
I &= \sum_{\alpha\in\Lambda}\int_{2E_\alpha}|u-u_{4E_\alpha}|^2 \\
&\le \sum_{\alpha\in\Lambda}\int_{4E_\alpha}|u-u_{4E_\alpha}|^2 \\
&= \sum_{\alpha\in\Lambda}\int_{B_{x_\alpha}(2s)}\left|u-u_{B_{x_\alpha}(2s)}\right|^2 \\
&\le \sum_{\alpha\in\Lambda}P_2(2s)^2\int_{B_{x_\alpha}(4s)}|\nabla u|^2 \ \text{ (by (2.1))} \\
&= 4P_2s^2\sum_{\alpha\in\Lambda}\int_{B_{x_\alpha}(4s)}|\nabla u|^2 \\
&= 4P_2s^2\sum_{\alpha\in\Lambda}\int_{8E_\alpha}|\nabla u|^2.
\end{aligned}$$

That is,

$$I \le 4P_2s^2\sum_{\alpha\in\Lambda}\int_{8E_\alpha}|\nabla u|^2. \qquad (2)$$

Next, we estimate II. For any $y \in 2E_\alpha = B_{x_\alpha}(s)$, we have, for any $z \in B_y(s)$,

$$\begin{aligned} d(z, x_\alpha) &\leq d(z, y) + d(y, x_\alpha) \\ &< s + s \\ &= 2s. \end{aligned}$$

That is,

$$z \in B_{x_\alpha}(2s).$$

Since z is arbitrary, we get

$$\begin{aligned} B_y(s) &\subseteq B_{x_\alpha}(2s) \\ &= 4E_\alpha. \end{aligned} \tag{3}$$

On the other hand, for the given y, note that

$$\begin{aligned} |B_y(s)| \left|u_{4E_\alpha} - u_s(y)\right|^2 &= |B_y(s)| \left| u_{4E_\alpha} - \frac{1}{|B_y(s)|} \int_{B_y(s)} u \right|^2 \\ &= \frac{1}{|B_y(s)|} \left| u_{4E_\alpha} |B_y(s)| - \int_{B_y(s)} u \right|^2 \\ &= \frac{1}{|B_y(s)|} \left| u_{4E_\alpha} \int_{B_y(s)} 1 - \int_{B_y(s)} u \right|^2 \\ &= \frac{1}{|B_y(s)|} \left| \int_{B_y(s)} (u_{4E_\alpha} - u) \right|^2. \end{aligned} \tag{4}$$

Since, by Hölder's inequality,

$$\begin{aligned} \left| \int_{B_y(s)} (u_{4E_\alpha} - u) \right| &\leq \left| \left(\int_{B_y(s)} 1 \right)^{\frac{1}{2}} \left(\int_{B_y(s)} |u_{4E_\alpha} - u|^2 \right)^{\frac{1}{2}} \right| \\ &= |B_y(s)|^{\frac{1}{2}} \left(\int_{B_y(s)} |u_{4E_\alpha} - u|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{|B_y(s)|} \left| \int_{B_y(s)} (u_{4E_\alpha} - u) \right|^2 &\leq \frac{1}{|B_y(s)|} |B_y(s)| \int_{B_y(s)} |u_{4E_\alpha} - u|^2 \\ &= \int_{B_y(s)} |u_{4E_\alpha} - u|^2. \end{aligned}$$

Pluging this into (4), we get

$$|B_y(s)| \left|u_{4E_\alpha} - u_s(y)\right|^2 \leq \int_{B_y(s)} |u_{4E_\alpha} - u|^2$$

and thus

$$
\begin{aligned}
\left|u_{4E_\alpha}-u_s(y)\right|^2 &\le \frac{1}{|B_y(s)|}\int_{B_y(s)}\left|u_{4E_\alpha}-u\right|^2\\
&\le \frac{1}{|B_y(s)|}\int_{4E_\alpha}\left|u-u_{4E_\alpha}\right|^2 \text{ (by (3))}\\
&= \frac{1}{|B_y(s)|}\int_{B_{x_\alpha}(2s)}\left|u-u_{B_{x_\alpha}(2s)}\right|^2\\
&\le \frac{1}{|B_y(s)|}P_2(2s)^2\int_{B_{x_\alpha}(4s)}|\nabla u|^2 \text{ (by (2.1))}\\
&= \frac{4P_2s^2}{|B_y(s)|}\int_{8E_\alpha}|\nabla u|^2. \qquad (5)
\end{aligned}
$$

Now, since y is arbitrary, we get

$$
\begin{aligned}
\sum_{\alpha\in\Lambda}\int_{2E_\alpha}\left|u_{4E_\alpha}-u_s\right|^2 &= \sum_{\alpha\in\Lambda}\int_{2E_\alpha}\left|u_{4E_\alpha}-u_s(y)\right|^2dy\\
&\le \sum_{\alpha\in\Lambda}\int_{2E_\alpha}\frac{4P_2s^2}{|B_y(s)|}\left(\int_{8E_\alpha}|\nabla u|^2\right)dy \text{ (by (5))}\\
&= 4P_2s^2\sum_{\alpha\in\Lambda}\left(\int_{8E_\alpha}|\nabla u|^2\right)\int_{2E_\alpha}\frac{1}{|B_y(s)|}dy. \qquad (6)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{2E_\alpha}\frac{1}{|B_y(s)|}dy &= \int_{B_{x_\alpha}(s)}\frac{1}{|B_y(s)|}dy\\
&= \int_{B_{x_\alpha}(s)}\frac{1}{|B_{x_\alpha}(s)|}\cdot\frac{|B_{x_\alpha}(s)|}{|B_y(s)|}dy\\
&\le \int_{B_{x_\alpha}(s)}\frac{1}{|B_{x_\alpha}(s)|}\cdot d_0^2\left(\frac{s}{s}\right)^{\log_2 d_0}dy \text{ (by (1.2))}\\
&= \frac{d_0^2}{|B_{x_\alpha}(s)|}\int_{B_{x_\alpha}(s)}1dy\\
&= d_0^2. \qquad (7)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{\alpha\in\Lambda}\int_{2E_\alpha}\left|u_{4E_\alpha}-u_s\right|^2 &\le 4P_2s^2\sum_{\alpha\in\Lambda}\left(\int_{8E_\alpha}|\nabla u|^2\right)\int_{2E_\alpha}\frac{1}{|B_y(s)|}dy \text{ (by (6))}\\
&\le 4P_2s^2\sum_{\alpha\in\Lambda}\left(\int_{8E_\alpha}|\nabla u|^2\right)d_0^2 \text{ (by (7))}\\
&= 4d_0^2P_2s^2\sum_{\alpha\in\Lambda}\int_{8E_\alpha}|\nabla u|^2.
\end{aligned}
$$

That is,

$$
II \le 4d_0^2P_2s^2\sum_{\alpha\in\Lambda}\int_{8E_\alpha}|\nabla u|^2. \qquad (8)
$$

So, by previous arguments, we obtain that

$$\begin{aligned}\int_M |u-u_s|^2 &\le 2\,(I+II) \ \text{(by (1))}\\ &\le 8P_2 s^2 \sum_{\alpha\in\Lambda}\int_{8E_\alpha} |\nabla u|^2 + 8d_0^2 P_2 s^2 \sum_{\alpha\in\Lambda}\int_{8E_\alpha} |\nabla u|^2 \ \text{(by (2) and (8))}\\ &= 8\left(1+d_0^2\right) P_2 s^2 \sum_{\alpha\in\Lambda}\int_{8E_\alpha} |\nabla u|^2 . \qquad (9)\end{aligned}$$

Note that, by Proposition 1.3, we know that the overlapping number of $8E_\alpha$ is uniformly bounded. More precisely, accroding to that proposition, if we let

$$N(z) = \#\{\alpha\in\Lambda : z\in 8E_\alpha\}$$

for all $z\in M$, then, since M has VDC with the controlling constant d_0, we have

$$\sup_{z\in M} N(z) \le d_0^6.$$

Thus,

$$\sum_{\alpha\in\Lambda}\int_{8E_\alpha} |\nabla u|^2 \le d_0^6 \int_M |\nabla u|^2 .$$

Pluging this into (9), we finally get

$$\begin{aligned}\int_M |u-u_s|^2 &\le 8\left(1+d_0^2\right) P_2 s^2 \sum_{\alpha\in\Lambda}\int_{8E_\alpha} |\nabla u|^2 \ \text{(by (9))}\\ &\le 8d_0^6\left(1+d_0^2\right) P_2 s^2 \int_M |\nabla u|^2\end{aligned}$$

as desired. □

Theorem 3.1. *Let E be a ball of center $x_0\in M$ and raius $r>0$ in M. If M has both VDC and WPI, then there exists a constant $C_N = C_N(d_0, P_2) > 0$ such that*

$$\left(\int_M u^2\right)^{1+\frac{2}{v}} \le \frac{C_N r^2}{|E|^{\frac{2}{v}}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)\left(\int_M |u|\right)^{\frac{4}{v}} \qquad (3.1)$$

holds for any $u\in C_0^\infty(E)$, where $v=\log_2 d_0 > 0$, d_0 and P_2 are the controlling constant in VDC and WPI, respectively. The inequality in this theorem is called the Nash inequality (NI) and C_N is called the controlling constant in NI.

Proof. For any $s\in(0,r]$, by Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}\int_M u^2 &\le \int_M 2\left(|u-u_s|^2 + |u_s|^2\right)\\ &= 2\left(\int_M |u-u_s|^2 + \int_M u_s^2\right)\\ &\le 16d_0^6\left(1+d_0^2\right)P_2 s^2\int_M |\nabla u|^2 + \frac{2d_0^5\left(\frac{r}{s}\right)^{\log_2 d_0}}{|E|}\left(\int_M |u|\right)^2\\ &\le cs^2\int_M |\nabla u|^2 + \frac{2d_0^5\left(\frac{r}{s}\right)^v}{|E|}\left(\int_M |u|\right)^2\\ &= cs^2\int_M |\nabla u|^2 + \frac{2d_0^5 r^v}{|E|}s^{-v}\left(\int_M |u|\right)^2, \qquad (1)\end{aligned}$$

where

$$c = 16d_0^6\left(1+d_0^2\right)P_2+1.$$

On the other hand, for any $s\in(r,\infty)$, we have $s^2r^{-2}>1$. Since $c\geq 1$, in this case we have

$$\int_M u^2 \leq cs^2r^{-2}\int_M u^2. \tag{2}$$

Combining (1) and (2), we obtain that, for any $s>0$,

$$\begin{aligned}\int_M u^2 &\leq cs^2\int_M |\nabla u|^2+\frac{2d_0^5r^v}{|E|}s^{-v}\left(\int_M |u|\right)^2+cs^2r^{-2}\int_M u^2\\ &= cs^2\left(\int_M |\nabla u|^2+\frac{1}{r^2}\int_M u^2\right)+\frac{2d_0^5r^v}{|E|}s^{-v}\left(\int_M |u|\right)^2.\end{aligned}$$

Note that $\int_M u^2$ is independent on s, so we can minimize the R.H.S. in the last inequality to get a better upper bound for $\int_M u^2$. To do so, we let

$$\begin{cases} A = c\left(\int_M |\nabla u|^2+\frac{1}{r^2}\int_M u^2\right)\\ B=\frac{2d_0^5r^v}{|E|}\left(\int_M |u|\right)^2\end{cases} \tag{3}$$

and consider

$$h(s) = As^2+Bs^{-v}$$

on $(0,\infty)$. Then we have

$$h'(s) = 2As-Bvs^{-v-1}$$

and

$$h''(s) = 2A+Bv(v+1)s^{-v-2}>0$$

for all $s>0$. Since $h(s)$ tends to ∞ as s tends to 0^+ and ∞, the minima of $h(s)$ appears at s_0 with $h'(s_0)=0$. To find such s_0, we solve $h'(s)=0$. Note that

$$0=h'(s)=2As-Bvs^{-v-1}$$

implies that

$$2As = Bvs^{-v-1},$$

so

$$s=\left(\frac{vB}{2A}\right)^{\frac{1}{v+2}}$$

and thus we can choose $h\left(\left(\frac{vB}{2A}\right)^{\frac{1}{v+2}}\right)$ as a (better) upper bound for $\int_M u^2$. Note that

$$\begin{aligned} h\left(\left(\frac{vB}{2A}\right)^{\frac{1}{v+2}}\right) &= A\left(\frac{vB}{2A}\right)^{\frac{2}{v+2}}+B\left(\frac{vB}{2A}\right)^{\frac{-v}{v+2}}\\ &= A\left(\frac{vB}{2A}\right)^{\frac{v+2-v}{v+2}}+B\left(\frac{vB}{2A}\right)^{\frac{-v}{v+2}}\\ &= \frac{vB}{2}\left(\frac{vB}{2A}\right)^{\frac{-v}{v+2}}+B\left(\frac{vB}{2A}\right)^{\frac{-v}{v+2}}\\ &= B\left(\frac{vB}{2A}\right)^{\frac{-v}{v+2}}\left(\frac{v}{2}+1\right)\\ &= B\left(\frac{2A}{vB}\right)^{\frac{v}{v+2}}\left(\frac{v}{2}+1\right),\end{aligned}$$

so

$$\begin{aligned}
\int_M u^2 &\le h\left(\left(\frac{vB}{2A}\right)^{\frac{1}{v+2}}\right) \\
&= B\left(\frac{2A}{vB}\right)^{\frac{v}{v+2}}\left(\frac{v}{2}+1\right)
\end{aligned}$$

and thus

$$\begin{aligned}
\left(\int_M u^2\right)^{\frac{v+2}{v}} &\le B^{\frac{v+2}{v}}\left(\frac{2A}{vB}\right)\left(\frac{v}{2}+1\right)^{\frac{v+2}{v}} \\
&= B^{1+\frac{2}{v}}\frac{A}{B}\cdot\frac{2}{v}\left(\frac{v}{2}+1\right)^{\frac{v+2}{v}} \\
&= B^{\frac{2}{v}}A\cdot\frac{2}{v}\left(\frac{v}{2}+1\right)^{\frac{v+2}{v}} \\
&= \left[\frac{2d_0^5 r^v}{|E|}\left(\int_M |u|\right)^2\right]^{\frac{2}{v}} c\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)\frac{2}{v}\left(\frac{v}{2}+1\right)^{\frac{v+2}{v}} \quad \text{(by (3))} \\
&= \frac{2c}{v}\left(\frac{v}{2}+1\right)^{1+\frac{2}{v}}\frac{2^{\frac{2}{v}}d_0^{\frac{10}{v}}r^2}{|E|^{\frac{2}{v}}}\left(\int_M |u|\right)^{\frac{4}{v}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right) \\
&= \frac{C_N r^2}{|E|^{\frac{2}{v}}}\left(\int_M |u|\right)^{\frac{4}{v}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right),
\end{aligned}$$

where

$$C_N = \frac{2c}{v}\left(\frac{v}{2}+1\right)^{1+\frac{2}{v}} 2^{\frac{2}{v}} d_0^{\frac{10}{v}}.$$

Recall that c and v are constants only involving d_0 and P_2, so C_N is also a constant involving nothing but d_0 and P_2. Therefore, in conclusion, we get

$$\left(\int_M u^2\right)^{1+\frac{2}{v}} \le \frac{C_N r^2}{|E|^{\frac{2}{v}}}\left(\int_M |u|\right)^{\frac{4}{v}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)$$

with $C_N = C_N(d_0, P_2)$ is constant as desired. □

3.2 Heat kernel upper bound and Sobolev inequality

Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $H = H(x,t;y)$ be the Dirichlet heat kernel for E. That is, for each $y \in E$, H is of $C^\infty(E \times \mathbb{R}^+)$ and satisfies the following conditions:

$$\begin{cases} \Delta H(x,t;y) - \partial_t H(x,t;y) = 0 & \text{on } E \times \mathbb{R}^+ \\ H(x,t;y) = 0 & \text{on } \partial E \times \mathbb{R}^+ \\ \lim_{t\to 0^+} H(x,t;y) = \delta_y(x) & \text{for any } x \in E \end{cases},$$

where Δ is the Laplace-Beltrami operator which acts only on the x variable and δ_y is the Dirac delta funtion concentrated at y.

For such H, for any $y \in E$ and $t \in \mathbb{R}^+$, the domain of $H(\cdot,t;y)$ can be extended by assigning $H(x,t;y) = 0$ for all $x \in E^C$. Thus, in this case, we have $H(\cdot,t;y) \in C_0^\infty(E)$ for any $y \in E$ and $t \in \mathbb{R}^+$. Also, we can define $H(\cdot,\cdot;y) = 0$ if $y \in E^C$, so H can be viewed as a function defined on $M \times \mathbb{R}^+ \times M$.

On the other hand, by elliptic theory, there exists eigenfunctions $\eta_1, \eta_2, \eta_3, \ldots$ in $C_0^\infty(E) \cap C^1(\overline{E})$ and corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ such that

$$\Delta \eta_i = -\lambda_i \eta_i$$

for all $i \in \mathbb{N}$. Also, $\{\eta_i\}_{i\in\mathbb{N}}$ forms an orthonormal basis for $L^2(E)$. Applying the eigenfunction expansion, we can express the Dirichlet heat kernel in the following form:

$$H(x,t;y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \eta_i(x)\, \eta_i(y).$$

There are some significant properties about the Dirichlet heat kernel. We shall just state them without proof.

Proposition 3.1. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $H = H(x,t;y)$ be the Dirichlet heat kernel for E, then*

1. *$H(x,t;y) \ge 0$ for any $x, y \in M$ and $t > 0$;*
2. *$\int_M H(x,t;y)\,dy \le 1$ for any $x \in M$ and $t > 0$;*
3. *$H(x,t;y) = H(y,t;x)$ for any $x, y \in M$ and $t > 0$;*
4. *$H(x,t+s;y) = \int_M H(x,t;z)\,H(z,s;y)\,dz$ for any $x, y \in M$ and $t, s > 0$.*

Theorem 3.2. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $H = H(x,t;y)$ be the Dirichlet heat kernel for E. If NI holds for all functions in $C_0^\infty(E)$ on M, then*

$$H \le \frac{r^v}{|E|}\left(\frac{vC_N}{t}\right)^{\frac{v}{2}} e^{\frac{t}{r^2}}$$

holds for any $x, y \in M$ and $t > 0$, where v is a positive constant and C_N is the controlling contant in NI. The R.H.S. of the inequality in this theorem is called a Dirichlet heat kernel upper bound at $t > 0$.

Proof. Fix $y \in M$ and define $u : M \times \mathbb{R}^+ \to \mathbb{R}$ by

$$u(x,t) = H(x,t;y)$$

for all $x \in M$ and $t > 0$. For any $t > 0$, since $u \in C_0^\infty(E)$, we have

$$\left(\int_M u^2\right)^{1+\frac{2}{v}} \le \frac{C_N r^2}{|E|^{\frac{2}{v}}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)\left(\int_M |u|\right)^{\frac{4}{v}}, \text{ (by (3.1))} \tag{1}$$

where v is a positive constant and C_N is the controlling contant in NI. Note that, by Proposition 3.1,

$$\begin{aligned}
\int_M |u| &= \int_M |u(x,t)|\,dx \\
&= \int_M |H(x,t;y)|\,dx \\
&= \int_M H(x,t;y)\,dx \\
&= \int_M H(y,t;x)\,dx \\
&\le 1.
\end{aligned}$$

Pluging this into (1), we get

$$\left(\int_M u^2\right)^{1+\frac{2}{v}} \leq \frac{C_N r^2}{|E|^{\frac{2}{v}}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)$$

which is equivalent to

$$-\int_M |\nabla u|^2 \leq -\frac{|E|^{\frac{2}{v}}}{C_N r^2}\left(\int_M u^2\right)^{1+\frac{2}{v}} + \frac{1}{r^2}\int_M u^2. \qquad (2)$$

Since the last inequality holds and $u \in C_0^\infty(E)$ for all $t > 0$,

$$\begin{aligned}
\partial_t\left(\int_M u^2\right) &= \partial_t\left(\int_E u^2\right) \\
&= \int_E \partial_t u^2 \\
&= 2\int_E u\partial_t u \\
&= 2\int_E u\Delta u \\
&= -2\int_E \langle \nabla u, \nabla u\rangle \text{ (by Green's identity)} \\
&= -2\int_E |\nabla u|^2 \\
&= -2\int_M |\nabla u|^2 \\
&\leq -\frac{2\,|E|^{\frac{2}{v}}}{C_N r^2}\left(\int_M u^2\right)^{1+\frac{2}{v}} + \frac{2}{r^2}\int_M u^2 \text{ (by (2))} \\
&= -A\left(\int_M u^2\right)^{1+\frac{2}{v}} + B\int_M u^2, \qquad (3)
\end{aligned}$$

where

$$\begin{cases} A = \frac{2|E|^{\frac{2}{v}}}{C_N r^2} \\ B = \frac{2}{r^2} \end{cases}.$$

Now, define $f, g : (0,\infty) \to \mathbb{R}$ by

$$f(t) = \int_M u^2$$

and

$$g(t) = e^{-Bt} f(t)$$

for all $t \in (0,\infty)$. Then we have

$$\begin{aligned}
\partial_t f(t) &= \partial_t\left(\int_M u^2\right) \\
&\leq -A\left(\int_M u^2\right)^{1+\frac{2}{v}} + B\int_M u^2 \text{ (by (3))} \\
&= -A\left[f(t)\right]^{1+\frac{2}{v}} + Bf(t) \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
\partial_t g(t) &= \partial_t \left[e^{-Bt} f(t) \right] \\
&= -Be^{-Bt} f(t) + e^{-Bt} \partial_t f(t) \\
&\leq -Be^{-Bt} f(t) + e^{-Bt} \left[-A [f(t)]^{1+\frac{2}{v}} + Bf(t) \right] \text{ (by (4))} \\
&= -Ae^{-Bt} [f(t)]^{1+\frac{2}{v}} \\
&= -Ae^{-Bt} \left[e^{Bt} g(t) \right]^{1+\frac{2}{v}} \\
&= -Ae^{\frac{2Bt}{v}} [g(t)]^{1+\frac{2}{v}}.
\end{aligned}$$

That is,

$$-[g(t)]^{-1-\frac{2}{v}} \partial_t g(t) \geq Ae^{\frac{2Bt}{v}}. \tag{5}$$

So

$$\begin{aligned}
\partial_t [g(t)]^{-\frac{2}{v}} &= -\frac{2}{v} [g(t)]^{-\frac{2}{v}-1} \partial_t g(t) \\
&\geq \frac{2A}{v} e^{\frac{2Bt}{v}}. \text{ (since } v > 0 \text{ and by (5))}
\end{aligned}$$

Note that the last inequality holds for any $t > 0$. So, for any $t > 0$, by the last inequality, we have

$$\begin{aligned}
\int_{\frac{t}{2}}^{t} \partial_s [g(s)]^{-\frac{2}{v}} ds &\geq \int_{\frac{t}{2}}^{t} \frac{2A}{v} e^{\frac{2Bs}{v}} ds \\
&= \frac{2A}{v} \cdot \frac{v}{2B} e^{\frac{2Bs}{v}} \Big|_{\frac{t}{2}}^{t} \\
&= \frac{A}{B} \left(e^{\frac{2Bt}{v}} - e^{\frac{Bt}{v}} \right) \\
&= \frac{A}{B} e^{\frac{Bt}{v}} \left(e^{\frac{Bt}{v}} - 1 \right) \\
&\geq \frac{A}{B} e^{\frac{Bt}{v}} \frac{Bt}{v} \\
&= \frac{At}{v} e^{\frac{Bt}{v}}.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\int_{\frac{t}{2}}^{t} \partial_s [g(s)]^{-\frac{2}{v}} ds &= [g(t)]^{-\frac{2}{v}} - \left[g\left(\frac{t}{2} \right) \right]^{-\frac{2}{v}} \\
&= \left[e^{-Bt} f(t) \right]^{-\frac{2}{v}} - \left[e^{\frac{-Bt}{2}} f\left(\frac{t}{2} \right) \right]^{-\frac{2}{v}} \\
&= e^{\frac{2Bt}{v}} [f(t)]^{-\frac{2}{v}} - e^{\frac{Bt}{v}} \left[f\left(\frac{t}{2} \right) \right]^{-\frac{2}{v}} \\
&\leq e^{\frac{2Bt}{v}} [f(t)]^{-\frac{2}{v}},
\end{aligned}$$

together with the previous result, we get

$$e^{\frac{2Bt}{v}} [f(t)]^{-\frac{2}{v}} \geq \frac{At}{v} e^{\frac{Bt}{v}}$$

and thus

$$f(t) \leq \left(\frac{v}{At}\right)^{\frac{v}{2}} e^{\frac{Bt}{2}}. \tag{6}$$

Note that, by Proposition 3.1,

$$\begin{aligned} f(t) &= \int_M u^2 \\ &= \int_M [u(x,t)]^2 \, dx \\ &= \int_M H(x,t;y) H(x,t;y) \, dx \\ &= \int_M H(y,t;x) H(x,t;y) \, dx \\ &= H(y,2t;y). \end{aligned}$$

Pluging this into (6), we obtain that

$$H(y,2t;y) \leq \left(\frac{v}{At}\right)^{\frac{v}{2}} e^{\frac{Bt}{2}}$$

and thus

$$\begin{aligned} H(y,t;y) &\leq \left(\frac{2v}{At}\right)^{\frac{v}{2}} e^{\frac{Bt}{4}} \\ &\leq \left(\frac{2v}{At}\right)^{\frac{v}{2}} e^{\frac{Bt}{2}}. \text{ (since } B > 0) \end{aligned} \tag{7}$$

Note that $y \in M$ and $t > 0$ are both arbitrary in the last inequality.

Finally, for any $x, y \in M$ and $t > 0$, by Proposition 3.1 again,

$$\begin{aligned} H(x,t;y) &= \int_M H\left(x,\frac{t}{2};z\right) H\left(z,\frac{t}{2};y\right) dz \\ &\leq \left(\int_M \left[H\left(x,\frac{t}{2};z\right)\right]^2 dz\right)^{\frac{1}{2}} \left(\int_M \left[H\left(z,\frac{t}{2};y\right)\right]^2 dz\right)^{\frac{1}{2}} \\ &= [H(x,t;x)]^{\frac{1}{2}} [H(y,t;y)]^{\frac{1}{2}} \\ &\leq \left(\frac{2v}{At}\right)^{\frac{v}{2}} e^{\frac{Bt}{2}} \text{ (by (7))} \\ &= \left(\frac{2v}{t} \cdot \frac{C_N r^2}{2|E|^{\frac{2}{v}}}\right)^{\frac{v}{2}} e^{\frac{2r^{-2}t}{2}} \\ &= \frac{r^v}{|E|} \left(\frac{vC_N}{t}\right)^{\frac{v}{2}} e^{\frac{t}{r^2}}, \end{aligned}$$

where the second inequality holds by Hölder's inequality. So we get the proof. □

Let $\eta_1, \eta_2, \eta_3, \ldots$ be eigenfunctions in $C_0^\infty(E) \cap C^1(\overline{E})$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be correspoding eigenvalues such that

$$\Delta \eta_i = -\lambda_i \eta_i$$

for all $i \in \mathbb{N}$. Since $\{\eta_i\}_{i\in\mathbb{N}}$ forms an orthonormal basis for $L^2(E)$, for any $u \in L^2(E)$, there exists $\{a_i = a_i(u)\}_{i\in\mathbb{N}}$ such that

$$u = \sum_{i=1}^{\infty} a_i \eta_i.$$

In fact,

$$a_i = \int_M u\eta_i$$

for all $i \in \mathbb{N}$. In this case, we can define

$$\Delta u = \sum_{i=1}^{\infty} a_i (-\lambda_i) \eta_i$$

so that Δ can be viewed as an operator on $L^2(E)$. We write the last identity in the following form:

$$(-\Delta) \sum_{i=1}^{\infty} a_i \eta_i = \sum_{i=1}^{\infty} a_i \lambda_i \eta_i.$$

Then, by such point of view, for any $c, \zeta \in \mathbb{R}$, we can define an operator $(c-\Delta)^{\zeta}$ on $span\{\eta_1, \eta_2, \eta_3, \ldots\}$ over $\mathbb{R}$ by setting

$$(c-\Delta)^{\zeta} \sum_{i=1}^{\infty} b_i \eta_i \equiv \sum_{i=1}^{\infty} b_i (c+\lambda_i)^{\zeta} \eta_i$$

for any $\{b_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}$. Note that, when $c = 0$ and $\zeta = 1$, we get an "extended" Laplace-Beltrami operator as before. Also, $(c-\Delta)^{\zeta}$ and $(c-\Delta)^{-\zeta}$ are the "inverse" to each other since

$$\begin{aligned}
(c-\Delta)^{-\zeta} (c-\Delta)^{\zeta} \sum_{i=1}^{\infty} b_i \eta_i &= (c-\Delta)^{-\zeta} \sum_{i=1}^{\infty} b_i (c+\lambda_i)^{\zeta} \eta_i \\
&= \sum_{i=1}^{\infty} b_i (c+\lambda_i)^{\zeta} (c+\lambda_i)^{-\zeta} \eta_i \\
&= \sum_{i=1}^{\infty} b_i \eta_i
\end{aligned}$$

for any $\{b_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}$.

Theorem 3.3. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $H = H(x,t;y)$ be the Dirichlet heat kernel for E. Suppose that*

$$H \leq \frac{r^v}{|E|} \left(\frac{c}{t}\right)^{\frac{v}{2}} e^{\frac{t}{r^2}}$$

holds for any $x, y \in M$ and $t > 0$, where v and c are positive contants. If $v > 2$, then there exists a constant $C_S = C_S(v,c) > 0$ such that

$$\left(\int_M u^{\frac{2v}{v-2}}\right)^{1-\frac{2}{v}} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}} \left(\int_M |\nabla u|^2 + \frac{1}{r^2} \int_M u^2\right) \tag{3.2}$$

holds for any $u \in C_0^{\infty}(E)$. The last inequlaity in this theorem is called the Sobolev inequality (SI) and C_S is called the controlling constant in SI.

Proof. First, for convenience, we let

$$\begin{cases} c_1 = \frac{r^v c^{\frac{v}{2}}}{|E|} \\ c_2 = \frac{1}{r^2} \end{cases} . \tag{1}$$

In this case, the assumption can be rewritten in the following form:

$$H \leq c_1 t^{-\frac{v}{2}} e^{c_2 t}. \tag{2}$$

Now, let $\eta_1, \eta_2, \eta_3, \ldots$ be eigenfunctions in $C_0^\infty(E) \cap C^1(\overline{E})$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be correspoding eigenvalues such that

$$\Delta \eta_i = -\lambda_i \eta_i$$

for all $i \in \mathbb{N}$. For any $\zeta \in \mathbb{R}$, we let $(c_2 - \Delta)^\zeta$ be defined as the operator in page **??**. Then, since $u \in C_0^\infty(E) \subseteq L^2(E)$, we have

$$u = \sum_{i=1}^{\infty} a_i \eta_i,$$

where $a_i = a_i(u) = \int_M u\eta_i$ for all $i \in \mathbb{N}$, and thus

$$(c_2 - \Delta)^\zeta u = \sum_{i=1}^{\infty} a_i (c_2 + \lambda_i)^\zeta \eta_i.$$

Note that

$$\begin{aligned} \int_M |\nabla u|^2 + c_2 \int_M u^2 &= -\int_M u\Delta u + c_2 \int_M u^2 \text{ (by Green's identity)} \\ &= \int_M u (c_2 - \Delta) u \\ &= \int_M \left(\sum_{i=1}^{\infty} a_i \eta_i \right) \left[\sum_{j=1}^{\infty} a_j (c_2 + \lambda_j) \eta_j \right] \\ &= \int_M \sum_{i=1}^{\infty} a_i^2 (c_2 + \lambda_i) \text{ (since } \{\eta_i\}_{i \in \mathbb{N}} \text{ is orthonormal)} \\ &= \int_M \left[\sum_{i=1}^{\infty} a_i (c_2 + \lambda_i)^{\frac{1}{2}} \eta_i \right] \left[\sum_{j=1}^{\infty} a_j (c_2 + \lambda_j)^{\frac{1}{2}} \eta_j \right] \\ &= \int_M \left| (c_2 - \Delta)^{\frac{1}{2}} u \right|^2. \end{aligned}$$

So, to show

$$\left(\int_M u^{\frac{2v}{v-2}} \right)^{1-\frac{2}{v}} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}} \left(\int_M |\nabla u|^2 + c_2 \int_M u^2 \right)$$

is equivalent to show that

$$\left(\int_M u^{\frac{2v}{v-2}} \right)^{1-\frac{2}{v}} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}} \int_M \left| (c_2 - \Delta)^{\frac{1}{2}} u \right|^2.$$

Let

$$\widetilde{u} = (c_2 - \Delta)^{\frac{1}{2}} u,$$

then

$$\begin{aligned} u &= (c_2 - \Delta)^{-\frac{1}{2}} (c_2 - \Delta)^{\frac{1}{2}} u \\ &= (c_2 - \Delta)^{-\frac{1}{2}} \widetilde{u} \end{aligned}$$

since $(c_2 - \Delta)^{-\frac{1}{2}}$ and $(c_2 - \Delta)^{\frac{1}{2}}$ are the inverse to each other by definition. Therefore, to show the last inequality, it is enough to show that

$$\left(\int_M \left| (c_2 - \Delta)^{-\frac{1}{2}} \widetilde{u} \right|^{\frac{2v}{v-2}} \right)^{1-\frac{2}{v}} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}} \int_M |\widetilde{u}|^2 . \tag{3}$$

To see this, we apply the Marcinkiewicz interpolation theorem.

For any $x \in M$, we have

$$\begin{aligned} (c_2 - \Delta)^{-\frac{1}{2}} \widetilde{u}(x) &= \sum_{i=1}^{\infty} \left(\int_M \widetilde{u}(y) \eta_i(y) \, dy \right) (c_2 + \lambda_i)^{-\frac{1}{2}} \eta_i(x) \\ &= \int_M \sum_{i=1}^{\infty} (c_2 + \lambda_i)^{-\frac{1}{2}} \eta_i(x) \eta_i(y) \widetilde{u}(y) \, dy. \end{aligned} \tag{4}$$

Note that, by the property of the gamma distribution,

$$\int_0^{\infty} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \tau^{-\frac{1}{2}} e^{-\tau} d\tau = 1,$$

so

$$\begin{aligned} (c_2 + \lambda_i)^{-\frac{1}{2}} &= (c_2 + \lambda_i)^{-\frac{1}{2}} \int_0^{\infty} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \tau^{-\frac{1}{2}} e^{-\tau} d\tau \\ &= (c_2 + \lambda_i)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^{\infty} \left[(c_2 + \lambda_i) t\right]^{-\frac{1}{2}} e^{-(c_2+\lambda_i)t} (c_2 + \lambda_i) \, dt \\ &= \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^{\infty} t^{-\frac{1}{2}} e^{-(c_2+\lambda_i)t} dt \end{aligned}$$

holds for all $i \in \mathbb{N}$. Pluging this into (4), we obtain that

$$\begin{aligned} (4) &= \int_M \sum_{i=1}^{\infty} \left[\Gamma\left(\frac{1}{2}\right)^{-1} \int_0^{\infty} t^{-\frac{1}{2}} e^{-(c_2+\lambda_i)t} dt \right] \eta_i(x) \eta_i(y) \widetilde{u}(y) \, dy \\ &= \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^{\infty} \int_M \sum_{i=1}^{\infty} t^{-\frac{1}{2}} e^{-(c_2+\lambda_i)t} \eta_i(x) \eta_i(y) \widetilde{u}(y) \, dy dt \\ &= \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^{\infty} t^{-\frac{1}{2}} \int_M \left[\sum_{i=1}^{\infty} e^{-(c_2+\lambda_i)t} \eta_i(x) \eta_i(y) \right] \widetilde{u}(y) \, dy dt. \end{aligned} \tag{5}$$

Let $G = G(x,t;y) = e^{-c_2 t} H(x,t;y)$, then, by the eigenfunction expansion of H, we have

$$\begin{aligned} G(x,t;y) &= e^{-c_2 t}\sum_{i=1}^{\infty} e^{-\lambda_i t}\eta_i(x)\,\eta_i(y) \\ &= \sum_{i=1}^{\infty} e^{(-c_2-\lambda_i)t}\eta_i(x)\,\eta_i(y) \\ &= \sum_{i=1}^{\infty} e^{-(c_2+\lambda_i)t}\eta_i(x)\,\eta_i(y)\,. \end{aligned}$$

Pluging this into (5), we get

$$(5) = \Gamma\left(\frac{1}{2}\right)^{-1}\int_0^{\infty} t^{-\frac{1}{2}}\int_M G(x,t;y)\,\widetilde{u}(y)\,dydt.$$

That is,

$$(c_2-\Delta)^{-\frac{1}{2}}\,\widetilde{u}(x) = \Gamma\left(\frac{1}{2}\right)^{-1}\int_0^{\infty} t^{-\frac{1}{2}}\int_M G(x,t;y)\,\widetilde{u}(y)\,dydt. \tag{6}$$

Next, for any $T>0$, let

$$\left\{\begin{array}{l} L_1\widetilde{u}(x) = \Gamma\left(\frac{1}{2}\right)^{-1}\int_0^{T} t^{-\frac{1}{2}}\int_M G(x,t;y)\,\widetilde{u}(y)\,dydt \\ L_2\widetilde{u}(x) = \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^{\infty} t^{-\frac{1}{2}}\int_M G(x,t;y)\,\widetilde{u}(y)\,dydt \end{array}\right. . \tag{7}$$

Then

$$(c_2-\Delta)^{-\frac{1}{2}} = L_1 + L_2.$$

For any $\lambda>0$, if $z \in \left\{x\in M : |L_1\widetilde{u}(x)| < \frac{\lambda}{2}\right\} \cap \left\{x\in M : |L_2\widetilde{u}(x)| \le \frac{\lambda}{2}\right\}$, then

$$\begin{aligned} \left|(c_2-\Delta)^{-\frac{1}{2}}\,\widetilde{u}(z)\right| &= |L_1\widetilde{u}(z) + L_2\widetilde{u}(z)| \ \text{ (by (6))} \\ &\le |L_1\widetilde{u}(z)| + |L_2\widetilde{u}(z)| \\ &< \lambda. \end{aligned}$$

So, for such z, $z \in \left\{x\in M : \left|(c_2-\Delta)^{-\frac{1}{2}}\,\widetilde{u}(x)\right| < \lambda\right\}$. Since z is arbitrary, we get

$$\left\{x\in M : |L_1\widetilde{u}(x)| < \frac{\lambda}{2}\right\} \cap \left\{x\in M : |L_2\widetilde{u}(x)| \le \frac{\lambda}{2}\right\} \subseteq I^C,$$

where $I^C = \left\{x\in M : \left|(c_2-\Delta)^{-\frac{1}{2}}\,\widetilde{u}(x)\right| < \lambda\right\}$, which is equivalent to

$$I \subseteq \left\{x\in M : |L_1\widetilde{u}(x)| \ge \frac{\lambda}{2}\right\} \cup \left\{x\in M : |L_2\widetilde{u}(x)| > \frac{\lambda}{2}\right\}.$$

Thus,

$$\begin{aligned} |I| &\le \left|\left\{x\in M : |L_1\widetilde{u}(x)| \ge \frac{\lambda}{2}\right\}\right| + \left|\left\{x\in M : |L_2\widetilde{u}(x)| > \frac{\lambda}{2}\right\}\right| \\ &= A + B, \end{aligned} \tag{8}$$

where

$$\begin{cases} A = \left|\left\{x \in M : |L_1\widetilde{u}(x)| \geq \frac{\lambda}{2}\right\}\right| \\ B = \left|\left\{x \in M : |L_2\widetilde{u}(x)| > \frac{\lambda}{2}\right\}\right| \end{cases}.$$

Applying Hölder's inequality (with respect to dy), we have

$$\begin{aligned}
|L_2\widetilde{u}(x)| &= \left|\Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\int_M G(x,t;y)\,\widetilde{u}(y)\,dydt\right| \text{ (by (7))} \\
&\leq \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dydt \\
&\leq \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\left(\int_M [G(x,t;y)]^p\,dy\right)^{\frac{1}{p}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt \\
&\leq \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\sup_{y\in M}[G(x,t;y)]^{\frac{p-1}{p}}\left(\int_M G(x,t;y)\,dy\right)^{\frac{1}{p}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt \\
&\leq \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\sup_{y\in M}[G(x,t;y)]^{\frac{p-1}{p}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt \text{ (by Proposition 3.1)} \\
&= \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\left[\sup_{y\in M} G(x,t;y)\right]^{1-\frac{1}{p}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt \\
&= \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\left[\sup_{y\in M} G(x,t;y)\right]^{\frac{1}{q}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt, \qquad (8)
\end{aligned}$$

where $p, q \in (1, v)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\begin{aligned}
\sup_{y\in M} G(x,t;y) &= \sup_{y\in M} e^{-c_2 t} H(x,t;y) \\
&= e^{-c_2 t}\sup_{y\in M} H(x,t;y) \\
&\leq e^{-c_2 t} c_1 t^{-\frac{v}{2}} e^{c_2 t} \text{ (by (2))} \\
&= c_1 t^{-\frac{v}{2}} \qquad (9)
\end{aligned}$$

Thus

$$\begin{aligned}
|L_2\widetilde{u}(x)| &\leq \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\left[\sup_{y\in M} G(x,t;y)\right]^{\frac{1}{q}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt \text{ (by (8))} \\
&\leq \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\left(c_1 t^{-\frac{v}{2}}\right)^{\frac{1}{q}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}dt \text{ (by (9))} \\
&= \Gamma\left(\frac{1}{2}\right)^{-1} c_1^{\frac{1}{q}}\left(\int_M |\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}\int_T^\infty t^{-\frac{1}{2}-\frac{v}{2q}}\,dt. \qquad (10)
\end{aligned}$$

Note that $q < v$. So

$$\frac{1}{2} - \frac{v}{2q} < 0$$

and thus

$$\begin{aligned}\int_T^\infty t^{-\frac{1}{2}-\frac{v}{2q}}dt &= \lim_{s\to\infty}\left(\frac{t^{\frac{1}{2}-\frac{v}{2q}}}{\frac{1}{2}-\frac{v}{2q}}\Bigg|_T^s\right)\\ &= \frac{1}{\frac{1}{2}-\frac{v}{2q}}\left[\left(\lim_{s\to\infty}s^{\frac{1}{2}-\frac{v}{2q}}\right)-T^{\frac{1}{2}-\frac{v}{2q}}\right]\\ &= \frac{1}{\frac{v}{2q}-\frac{1}{2}}T^{\frac{1}{2}-\frac{v}{2q}}.\end{aligned}$$

Pluging this into (10), we get

$$\begin{aligned}(10) &= \Gamma\left(\frac{1}{2}\right)^{-1}c_1^{\frac{1}{q}}\left(\int_M|\widetilde{u}(y)|^q\,dy\right)^{\frac{1}{q}}\frac{1}{\frac{v}{2q}-\frac{1}{2}}T^{\frac{1}{2}-\frac{v}{2q}}\\ &= \Gamma\left(\frac{1}{2}\right)^{-1}c_1^{\frac{1}{q}}\left(\int_M|\widetilde{u}|^q\right)^{\frac{1}{q}}\frac{2q}{v-q}T^{\frac{1}{2}-\frac{v}{2q}}.\end{aligned}$$

That is,

$$|L_2\widetilde{u}(x)|\le\Gamma\left(\frac{1}{2}\right)^{-1}c_1^{\frac{1}{q}}\left(\int_M|\widetilde{u}|^q\right)^{\frac{1}{q}}\frac{2q}{v-q}T^{\frac{1}{2}-\frac{v}{2q}}.$$

Since the last inequality holds for any $T>0$, we can choose an appropriate T in the beginning such that the right hand side is equal to $\frac{\lambda}{2}$. Then, for such T, we get

$$\begin{aligned}B &= \left|\left\{x\in M:|L_2\widetilde{u}(x)|>\frac{\lambda}{2}\right\}\right|\\ &= 0\end{aligned}$$

since $\left\{x\in M:|L_2\widetilde{u}(x)|>\frac{\lambda}{2}\right\}$ is empty. Therefore,

$$\begin{aligned}|I| &\le A+B\ \text{(by (8))}\\ &= A\\ &= \left|\left\{x\in M:|L_1\widetilde{u}(x)|\ge\frac{\lambda}{2}\right\}\right|\\ &\le \left(\frac{\lambda}{2}\right)^{-q}\int_M|L_1\widetilde{u}(x)|^q\,dx.\ \text{(by Chebyshev's inequality)}\end{aligned}\tag{11}$$

Note that, by Hölder's inequality (with respect to dt),

$$\begin{aligned}
|L_1\widetilde{u}(x)|^q &= \left|\Gamma\left(\frac{1}{2}\right)^{-1}\int_0^T t^{-\frac{1}{2}}\int_M G(x,t;y)\,\widetilde{u}(y)\,dydt\right|^q \\
&\le \left[\Gamma\left(\frac{1}{2}\right)^{-1}\int_0^T t^{-\frac{1}{2}}\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dydt\right]^q \\
&= \Gamma\left(\frac{1}{2}\right)^{-q}\left[\int_0^T t^{-\frac{1}{2}}\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dydt\right]^q \\
&= \Gamma\left(\frac{1}{2}\right)^{-q}\left\{\int_0^T \left(t^{-\frac{1}{2}}\right)^{\frac{1}{p}}\left[\left(t^{-\frac{1}{2}}\right)^{\frac{1}{q}}\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dy\right]dt\right\}^q \\
&\le \Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_0^T t^{-\frac{1}{2}}dt\right)^{\frac{q}{p}}\int_0^T t^{-\frac{1}{2}}\left(\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dy\right)^q dt \\
&= \Gamma\left(\frac{1}{2}\right)^{-q} T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}\left(\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dy\right)^q dt. \qquad (12)
\end{aligned}$$

Since, by Hölder's inequality agian,

$$\begin{aligned}
\left(\int_M G(x,t;y)\,|\widetilde{u}(y)|\,dy\right)^q &= \left(\int_M [G(x,t;y)]^{\frac{1}{p}}\,[G(x,t;y)]^{\frac{1}{q}}\,|\widetilde{u}(y)|\,dy\right)^q \\
&\le \left(\int_M G(x,t;y)\,dy\right)^{\frac{q}{p}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dy \\
&= \left(\int_M e^{-c_2 t}H(x,t;y)\,dy\right)^{\frac{q}{p}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dy \\
&= e^{-\frac{qc_2t}{p}}\left(\int_M H(x,t;y)\,dy\right)^{\frac{q}{p}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dy \\
&\le e^{-\frac{qc_2t}{p}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dy \text{ (by Proposition 3.1)} \\
&\le \int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dy. \text{ (since } \frac{qc_2t}{p} > 0 \text{ for } t > 0)
\end{aligned}$$

Pluging this into (12), we get

$$(12) \le \Gamma\left(\frac{1}{2}\right)^{-q} T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dydt.$$

That is,

$$|L_1\widetilde{u}(x)|^q \le \Gamma\left(\frac{1}{2}\right)^{-q} T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dydt.$$

Since x is arbitrary in the last inequality,

$$\begin{aligned}
\int_M |L_1\widetilde{u}(x)|^q\,dx &\le \int_M \Gamma\left(\frac{1}{2}\right)^{-q} T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dydtdx \\
&= \Gamma\left(\frac{1}{2}\right)^{-q} T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}\int_M\int_M G(x,t;y)\,|\widetilde{u}(y)|^q\,dydxdt. \qquad (13)
\end{aligned}$$

Note that

$$\begin{aligned}\int_M\int_M G(x,t;y)\left|\widetilde{u}(y)\right|^q dydx &= \int_M\int_M G(x,t;y)\left|\widetilde{u}(y)\right|^q dxdy \\ &= \int_M\int_M e^{-c_2 t}H(x,t;y)\,dx\left|\widetilde{u}(y)\right|^q dy \\ &\leq \int_M\int_M H(x,t;y)\,dx\left|\widetilde{u}(y)\right|^q dy \text{ (since } c_2 t>0 \text{ for } t>0) \\ &\leq \int_M \left|\widetilde{u}(y)\right|^q dy. \text{ (by Proposition 3.1)}\end{aligned}$$

Pluging this into (13), we arrive at

$$\begin{aligned}\int_M \left|L_1\widetilde{u}(x)\right|^q dx &\leq \Gamma\left(\frac{1}{2}\right)^{-q} T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}\int_M \left|\widetilde{u}(y)\right|^q dydt \\ &= \Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_M \left|\widetilde{u}(y)\right|^q dy\right) T^{\frac{q}{2p}}\int_0^T t^{-\frac{1}{2}}dt \\ &= \Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_M \left|\widetilde{u}\right|^q\right) T^{\frac{q}{2p}}T^{\frac{1}{2}} \\ &= \Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_M \left|\widetilde{u}\right|^q\right) T^{\frac{q}{2p}+\frac{1}{2}} \\ &= \Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_M \left|\widetilde{u}\right|^q\right) T^{\frac{q}{2}}. \text{ (since } \frac{1}{p}+\frac{1}{q}=1)\end{aligned}$$

So

$$\begin{aligned}|I| &\leq \left(\frac{\lambda}{2}\right)^{-q}\int_M \left|L_1\widetilde{u}(x)\right|^q dx \text{ (by (11))} \\ &\leq \left(\frac{\lambda}{2}\right)^{-q}\Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_M \left|\widetilde{u}\right|^q\right) T^{\frac{q}{2}}. \qquad (14)\end{aligned}$$

Recall that T was choosen such that

$$\Gamma\left(\frac{1}{2}\right)^{-1} c_1^{\frac{1}{q}}\left(\int_M \left|\widetilde{u}\right|^q\right)^{\frac{1}{q}}\frac{2q}{v-q}T^{\frac{1}{2}-\frac{v}{2q}} = \frac{\lambda}{2},$$

thus

$$T^{\frac{1}{2}-\frac{v}{2q}} = \Gamma\left(\frac{1}{2}\right)\left(\frac{v-q}{2q}\right)\left(\frac{\lambda}{2}\right)\left(c_1\int_M \left|\widetilde{u}\right|^q\right)^{-\frac{1}{q}}.$$

It implies that

$$\begin{aligned}T^{\frac{q}{2}} &= \left(T^{\frac{q-v}{2q}}\right)^{\frac{q^2}{q-v}} \\ &= \left(T^{\frac{1}{2}-\frac{v}{2q}}\right)^{\frac{q^2}{q-v}} \\ &= \left[\Gamma\left(\frac{1}{2}\right)\left(\frac{v-q}{2q}\right)\left(\frac{\lambda}{2}\right)\right]^{\frac{q^2}{q-v}}\left(c_1\int_M \left|\widetilde{u}\right|^q\right)^{\frac{q}{v-q}}.\end{aligned}$$

Pluging this into (14), we get

$$\begin{aligned}(14) &\le \left(\frac{\lambda}{2}\right)^{-q}\Gamma\left(\frac{1}{2}\right)^{-q}\left(\int_M|\widetilde{u}|^q\right)\left[\Gamma\left(\frac{1}{2}\right)\left(\frac{v-q}{2q}\right)\left(\frac{\lambda}{2}\right)\right]^{\frac{q^2}{q-v}}\left(c_1\int_M|\widetilde{u}|^q\right)^{\frac{q}{v-q}}\\ &= C^{\frac{qv}{v-q}}\lambda^{\frac{qv}{q-v}}\left(\int_M|\widetilde{u}|^q\right)^{\frac{v}{v-q}}\\ &= \left[\frac{C\left(\int_M|\widetilde{u}|^q\right)^{\frac{1}{q}}}{\lambda}\right]^{\frac{qv}{v-q}},\end{aligned}$$

where

$$C = 2\Gamma\left(\frac{1}{2}\right)^{-1}\left(\frac{2q}{v-q}\right)^{\frac{q}{v}}c_1^{\frac{1}{v}}. \tag{15}$$

That is,

$$|I| \le \left[\frac{C\left(\int_M|\widetilde{u}|^q\right)^{\frac{1}{q}}}{\lambda}\right]^{\frac{qv}{v-q}}.$$

Recall that $I = \left\{x\in M : \left|(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}(x)\right|\ge\lambda\right\}$. So, actually, we arrive at

$$\left|\left\{x\in M : \left|(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}(x)\right|\ge\lambda\right\}\right| \le \left[\frac{C\left(\int_M|\widetilde{u}|^q\right)^{\frac{1}{q}}}{\lambda}\right]^{\frac{qv}{v-q}}. \tag{16}$$

Note that the last inequality holds for $1<q<v$. Since $v>2$, we can choose q_1 and q_2 such that $1<q_1<2<q_2<v$ such that

$$\left|\left\{x\in M : \left|(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}(x)\right|\ge\lambda\right\}\right| \le \left[\frac{C_1\left(\int_M|\widetilde{u}|^{q_1}\right)^{\frac{1}{q_1}}}{\lambda}\right]^{\frac{q_1v}{v-q_1}}$$

and

$$\left|\left\{x\in M : \left|(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}(x)\right|\ge\lambda\right\}\right| \le \left[\frac{C_2\left(\int_M|\widetilde{u}|^{q_2}\right)^{\frac{1}{q_2}}}{\lambda}\right]^{\frac{q_2v}{v-q_2}}$$

with

$$C_i = 2\Gamma\left(\frac{1}{2}\right)^{-1}\left(\frac{v-q_i}{2q_i}\right)^{-\frac{q_i}{v}}c_1^{\frac{1}{v}}$$

for $i=1,2$. Let $\theta = \frac{q_1}{2}\cdot\frac{q_2-2}{q_2-q_1}$, then $0<\theta<1$. Also, for such θ, we have

$$\theta\cdot\frac{1}{q_1}+(1-\theta)\frac{1}{q_2}=\frac{1}{2}$$

and

$$\theta\cdot\frac{v-q_1}{q_1v}+(1-\theta)\frac{v-q_1}{q_2v}=\frac{v-2}{2v}.$$

Thus, by the Marcinkiewicz interpolation theorem or, more precisely, following the proof of the Marcinkiewicz interpolation theorem, there is some constant $K = K(q_1, q_2) > 0$ such that

$$\left(\int_M \left[(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}\right]^{\frac{2v}{v-2}}\right)^{\frac{v-2}{2v}} \le KC_1^\theta C_2^{1-\theta}\left(\int_M \widetilde{u}^2\right)^{\frac{1}{2}}.$$

To see $C_1^\theta C_2^{1-\theta}$, we exactly choose

$$\begin{cases} q_1 = \frac{3}{2} \\ q_2 = \frac{2+v}{2} \end{cases}.$$

Then, for such q_1 and q_2, we have

$$\begin{aligned} \theta &= \frac{q_1}{2}\cdot\frac{q_2-2}{q_2-q_1} \\ &= \frac{3}{4}\cdot\frac{v-2}{v-1} \end{aligned}$$

and thus

$$\begin{aligned} C_1^\theta C_2^{1-\theta} &= 2\Gamma\left(\frac{1}{2}\right)^{-1} c_1^{\frac{1}{v}}\left[\left(\frac{v-q_1}{2q_1}\right)^{-\frac{q_1}{v}}\right]^\theta\left[\left(\frac{v-q_2}{2q_2}\right)^{-\frac{q_2}{v}}\right]^{1-\theta} \\ &= 2\Gamma\left(\frac{1}{2}\right)^{-1} c_1^{\frac{1}{v}}\left[\left(\frac{2v-3}{6}\right)^{-\frac{3}{2v}}\right]^{\frac{3}{4}\cdot\frac{v-2}{v-1}}\left[\left(\frac{v-2}{4+2v}\right)^{-\frac{2+v}{2v}}\right]^{\frac{1}{4}\cdot\frac{v+2}{v-1}} \\ &= 2\Gamma\left(\frac{1}{2}\right)^{-1} c_1^{\frac{1}{v}}\left(\frac{6}{2v-3}\right)^{\frac{9(v-2)}{8v(v-1)}}\left(\frac{4+2v}{v-2}\right)^{\frac{(v+2)^2}{8v(v-1)}} \\ &= Rc_1^{\frac{1}{v}}, \end{aligned}$$

where R is a constant determined only by v. Therefore, the last inequality becomes

$$\left(\int_M \left|(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}\right|^{\frac{2v}{v-2}}\right)^{\frac{v-2}{2v}} \le KRc_1^{\frac{1}{v}}\left(\int_M \widetilde{u}^2\right)^{\frac{1}{2}}.$$

Thus we get

$$\begin{aligned} \left(\int_M \left|(c_2-\Delta)^{-\frac{1}{2}}\widetilde{u}\right|^{\frac{2v}{v-2}}\right)^{1-\frac{2}{v}} &\le K^2R^2c_1^{\frac{2}{v}}\int_M \widetilde{u}^2 \\ &\le K^2R^2\frac{r^2c}{|E|^{\frac{2}{v}}}\int_M \widetilde{u}^2 \text{ (by (1))} \\ &= \frac{C_Sr^2}{|E|^{\frac{2}{v}}}\int_M \widetilde{u}^2, \end{aligned}$$

where $C_S = cK^2R^2$.Since R is a constant dependent only on v, C_S is a constant dependent on both c and v. So we obtain (3) and thus

$$\left(\int_M u^{\frac{2v}{v-2}}\right)^{1-\frac{2}{v}} \le \frac{C_Sr^2}{|E|^{\frac{2}{v}}}\left(\int_M |\nabla u|^2 + c_2\int_M u^2\right)$$

holds by equivalency. Finally, since $c_2 = \frac{1}{r^2}$, the proof is finished. □

Applying Theorem 3.1, 3.2 and 3.3, we get the following result immediately.

Corollary 3.1. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. If M has both VDC and WPI, then SI holds for all functions in $C_0^\infty(E)$ on M. Moreover, the controlling constant in SI is dependent only on d_0 and P_2 in this case, where d_0 and P_2 are the controlling constants in VDC and WPI, respectively.*

Proof. First we note that the controlling constant in VDC can be enlarged without loss of generality such that $d_0 > 4$. Then, since M has both VDC and WPI, by Theorem 3.1, there exists a constant $C_N = C_N(d_0, P_2) > 0$ such that

$$\left(\int_M u^2\right)^{1+\frac{2}{v}} \leq \frac{C_N r^2}{|E|^{\frac{2}{v}}} \left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)\left(\int_M |u|\right)^{\frac{4}{v}}$$

holds for any $u \in C_0^\infty(E)$, where $v = \log_2 d_0 > 2$, d_0 and P_2 are the controlling constant in VDC and WPI, respectively. That is, NI holds for all functions in $C_0^\infty(E)$ on M. Thus, by Theorem 3.2, for the Dirichlet heat kernel H for E, we have

$$H \leq \frac{r^v}{|E|}\left(\frac{vC_N}{t}\right)^{\frac{v}{2}} e^{\frac{t}{r^2}}$$

holds for any $x, y \in M$ and $t > 0$. Let $c = vC_N$, then, by Theorem 3.3, since $v > 2$, there exists a constant $C_S = C_S(v, c) > 0$ such that

$$\left(\int_M u^{\frac{2v}{v-2}}\right)^{1-\frac{2}{v}} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\int_M |\nabla u|^2 + \frac{1}{r^2}\int_M u^2\right)$$

holds for any $u \in C_0^\infty(E)$. That is, SI holds for all functions in $C_0^\infty(E)$ on M. Note that $v = \log_2 d_0$ and $c = vC_N = (\log_2 d_0)\, C_N(d_0, P_2)$, the controlling constant in SI, C_S, is actually dependent only on d_0 and P_2. So we get the proof. □

4 Subsolutions and supersolutions for the heat equation

Definition 4.1. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M and $[a, b] \subseteq \mathbb{R}^+$. Let u be a real-valued function defined on $E\times [a, b]$. u is said to be a subsolution (supersolution, solution) to the Dirichlet heat equation on $E\times [a, b]$ if $u(\cdot, t) \in C_0^\infty(E)$ for any $t \in [a, b]$, $u(x, \cdot) \in C^1(a, b)$ for any $x \in E$ and u satisfies*

$$\Delta u - \partial_t u \geq (\leq, =)\, 0$$

on $\Omega \times [a, b]$.

Remark 4.1. *By Definition 4.1, we know that if u is a solution to the Dirichlet heat equation, then u is both the subsolution and supersolution to the Dirichlet heat equation.*

4.1 Subsolutions

Theorem 4.1. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $t_0 > r^2$ and u be a positive subsolution to the Dirichlet heat equation on $E \times [t_0 - r^2, t_0]$. If SI holds for all functions in $C_0^\infty(E)$ on M, then, for any $0 < \delta' < \delta \le 1$ and $p > 0$, there exists a constant $C_M = C_M(C_S, \delta', \delta, p) > 0$ such that*

$$\sup_{Q_{\delta'}} u \le C_M \left(\frac{1}{|E| r^2} \int_{Q_\delta} u^p \right)^{\frac{1}{p}}, \tag{4.1}$$

where

$$Q_\kappa \equiv \kappa E \times \left[t_0 - \kappa r^2, t_0\right]$$

with $\kappa E = B_{x_0}(\kappa r)$ for all $\kappa \in (0,1]$ and C_S is the controlling constant in SI. The inequality in this theorem is called the mean value inequality (MVI) for subsolutions to the Dirichlet heat equation and C_M is called the controlling contant in MVI.

Proof. For any $0 < \sigma' < \sigma < 1$, let $\tau = \sigma - \sigma'$. Let $\alpha, \beta : [0, \infty) \to \mathbb{R}$ be smooth functions such that

$$\begin{cases} \alpha(l) = 0 & \text{for } l \ge \sigma r \\ \alpha(l) = 1 & \text{for } 0 \le l \le \sigma' r \\ |\alpha'| \le \frac{2}{\tau r} \end{cases} \tag{1}$$

and

$$\begin{cases} \beta(t) = 1 & \text{for } t \ge t_0 - \sigma' r^2 \\ \beta(t) = 0 & \text{for } 0 \le l \le t_0 - \sigma r^2 \\ |\beta'| \le \frac{2}{\tau^2 r^2} \end{cases}, \tag{2}$$

respectively. Also, let $\varphi : M \times [0, \infty) \to \mathbb{R}$ be a test function defined by

$$\varphi(x,t) = \alpha(d(x, x_0))\, \beta(t)$$

for all $x \in M$ and $t \ge 0$. By a direct calculation, since $u(\cdot, t) \in C_0^\infty(E)$, for all $t \in [t_0 - r^2, t_0]$, and u is a subsolution to the Dirichlet heat equation, we have

$$\begin{aligned} \int_E \nabla\left(\varphi^2 u\right) \nabla u &= -\int_E \varphi^2 u \Delta u \text{ (by Green's identity)} \\ &\le -\int_E \varphi^2 u \partial_t u \\ &= -\frac{1}{2} \int_E \varphi^2 \partial_t u^2 \\ &= -\frac{1}{2} \int_E \left[\partial_t\left(\varphi^2 u^2\right) - u^2 \partial_t \varphi^2\right] \\ &= -\frac{1}{2} \partial_t \int_E \varphi^2 u^2 + \int_E u^2 \varphi \partial_t \varphi. \end{aligned} \tag{3}$$

Note that, on the other hand, we have

$$\begin{aligned}\int_E \nabla\left(\varphi^2 u\right)\nabla u &= \int_E \left(\varphi^2\nabla u + u\nabla\varphi^2\right)\nabla u \\ &= \int_E \left(\varphi^2\left|\nabla u\right|^2 + 2\varphi u\nabla\varphi\nabla u\right) \\ &= \int_E \left[\left|\varphi\nabla u\right|^2 + 2\left(\varphi\nabla u\right)\left(u\nabla\varphi\right) + \left|u\nabla\varphi\right|^2 - \left|u\nabla\varphi\right|^2\right] \\ &= \int_E \left|\varphi\nabla u + u\nabla\varphi\right|^2 - \int_E \left|u\nabla\varphi\right|^2 \\ &= \int_E \left|\nabla\left(\varphi u\right)\right|^2 - \int_E \left|u\nabla\varphi\right|^2. \end{aligned} \tag{4}$$

So, combining (3) and (4), we get

$$\int_E \left|\nabla\left(\varphi u\right)\right|^2 - \int_E \left|u\nabla\varphi\right|^2 \leq -\frac{1}{2}\partial_t \int_E \varphi^2 u^2 + \int_E u^2\varphi\partial_t\varphi$$

and thus

$$\begin{aligned}\int_E \left|\nabla\left(\varphi u\right)\right|^2 + \frac{1}{2}\partial_t \int_E \varphi^2 u^2 &\leq \int_E \left|u\nabla\varphi\right|^2 + \int_E u^2\varphi\partial_t\varphi \\ &= \int_E u^2\left(\left|\nabla\varphi\right|^2 + \varphi\partial_t\varphi\right) \\ &\leq \int_{\sigma E} u^2\left(\frac{4}{\tau^2 r^2} + \frac{2}{\tau^2 r^2}\right) \text{ (by (1) and (2))} \\ &= \frac{6}{\tau^2 r^2}\int_{\sigma E} u^2. \end{aligned} \tag{5}$$

The last inequality induces two important informations. First, since $\int_E \left|\nabla\left(\varphi u\right)\right|^2 \geq 0$, we have

$$\begin{aligned}\frac{1}{2}\partial_t \int_E \varphi^2 u^2 &\leq \int_E \left|\nabla\left(\varphi u\right)\right|^2 + \frac{1}{2}\partial_t \int_E \varphi^2 u^2 \\ &\leq \frac{6}{\tau^2 r^2}\int_{\sigma E} u^2. \text{ (by (5))}\end{aligned}$$

That is,

$$\partial_t \int_E \varphi^2 u^2 \leq \frac{12}{\tau^2 r^2}\int_{\sigma E} u^2.$$

Let

$$I_{\sigma'} \equiv \left[t_0 - \sigma' r^2, t_0\right],$$

then, for any $s \in I_{\sigma'}$, by the last inequality, we have

$$\begin{aligned}\int_{t_0-\sigma r^2}^{s} \partial_t \int_E \varphi^2 u^2 &\leq \frac{12}{\tau^2 r^2}\int_{t_0-\sigma r^2}^{s}\int_{\sigma E} u^2 \\ &\leq \frac{12}{\tau^2 r^2}\int_{t_0-\sigma r^2}^{t_0}\int_{\sigma E} u^2 \\ &= \frac{12}{\tau^2 r^2}\int_{Q_\sigma} u^2. \end{aligned} \tag{6}$$

Note that

$$\begin{aligned}\int_{t_0-\sigma r^2}^{s}\partial_t\int_E\varphi^2u^2 &= \int_E \varphi^2u^2\big|_{t=s}-\int_E\varphi^2u^2\big|_{t=t_0-\sigma r^2}\\ &= \int_E\varphi^2u^2\big|_{t=s}.\ \text{(by (2))}\end{aligned}$$

Pluging this into (6), we obtain that

$$\int_E\varphi^2u^2\big|_{t=s}\le\frac{12}{\tau^2r^2}\int_{Q_\sigma}u^2.$$

Since s is arbitrary, we have

$$\sup_{I_{\sigma'}}\int_E\varphi^2u^2\le\frac{12}{\tau^2r^2}\int_{Q_\sigma}u^2. \tag{7}$$

Second, if we integrate (5) directly on $[t_0-\sigma r^2,t_0]$, then we get

$$\begin{aligned}\int_{t_0-\sigma r^2}^{t_0}\int_E|\nabla(\varphi u)|^2+\frac{1}{2}\int_{t_0-\sigma r^2}^{t_0}\partial_t\int_E\varphi^2u^2 &\le \frac{6}{\tau^2r^2}\int_{t_0-\sigma r^2}^{t_0}\int_{B_{x_0}(\sigma r)}u^2\\ &= \frac{6}{\tau^2r^2}\int_{Q_\sigma}u^2.\end{aligned} \tag{8}$$

Note that

$$\begin{aligned}\int_{t_0-\sigma r^2}^{t_0}\partial_t\int_E\varphi^2u^2 &= \int_E\varphi^2u^2\big|_{t=t_0}-\int_E\varphi^2u^2\big|_{t=t_0-\sigma r^2}\\ &= \int_E\varphi^2u^2\big|_{t=t_0}\ \text{(by (2))}\\ &\ge 0,\end{aligned}$$

so

$$\begin{aligned}\int_{t_0-\sigma r^2}^{t_0}\int_E|\nabla(\varphi u)|^2 &\le \int_{t_0-\sigma r^2}^{t_0}\int_E|\nabla(\varphi u)|^2+\frac{1}{2}\int_{t_0-\sigma r^2}^{t_0}\partial_t\int_E\varphi^2u^2\\ &\le \frac{6}{\tau^2r^2}\int_{Q_\sigma}u^2.\ \text{(by (8))}\end{aligned} \tag{9}$$

Now, let $v>2$ be fixed, then we have

$$\begin{aligned}\int_E(\varphi u)^{2\left(1+\frac{2}{v}\right)} &= \int_E(\varphi u)^2(\varphi u)^{\frac{4}{v}}\\ &\le \left(\int_E(\varphi u)^{\frac{2v}{v-2}}\right)^{\frac{v-2}{v}}\left(\int_E(\varphi u)^2\right)^{\frac{2}{v}}\ \text{(by Hölder's inequality)}\\ &\le \left(\frac{C_Sr^2}{|E|^{\frac{2}{v}}}\int_E\left[|\nabla(\varphi u)|^2+r^{-2}(\varphi u)^2\right]\right)\left(\int_E(\varphi u)^2\right)^{\frac{2}{v}}\ \text{(by (3.2))}\end{aligned}$$

and thus, by (7), we get

$$\begin{aligned}
\int_{t_0-\sigma' r^2}^{t_0}\int_E (\varphi u)^{2\left(1+\frac{2}{v}\right)} &\le \int_{t_0-\sigma' r^2}^{t_0}\left(\frac{C_S r^2}{|E|^{\frac{2}{v}}}\int_E \left[|\nabla(\varphi u)|^2 + r^{-2}(\varphi u)^2\right]\right)\left(\int_E (\varphi u)^2\right)^{\frac{2}{v}} \\
&\le \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\sup_{I_{\sigma'}}\int_E \varphi^2 u^2\right)^{\frac{2}{v}}\int_{t_0-\sigma' r^2}^{t_0}\int_E \left[|\nabla(\varphi u)|^2 + r^{-2}(\varphi u)^2\right] \\
&\le \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{12}{\tau^2 r^2}\int_{Q_\sigma} u^2\right)^{\frac{2}{v}}\int_{t_0-\sigma' r^2}^{t_0}\int_E \left[|\nabla(\varphi u)|^2 + r^{-2}(\varphi u)^2\right] \\
&= \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{12}{\tau^2 r^2}\int_{Q_\sigma} u^2\right)^{\frac{2}{v}}\cdot A, \qquad (10)
\end{aligned}$$

where

$$A = \int_{t_0-\sigma' r^2}^{t_0}\int_E \left[|\nabla(\varphi u)|^2 + r^{-2}(\varphi u)^2\right].$$

Since

$$\begin{aligned}
A &= \int_{t_0-\sigma' r^2}^{t_0}\int_E |\nabla(\varphi u)|^2 + r^{-2}\int_{t_0-\sigma' r^2}^{t_0}\int_E (\varphi u)^2 \\
&\le \int_{t_0-\sigma r^2}^{t_0}\int_E |\nabla(\varphi u)|^2 + r^{-2}\left(\sup_{I_{\sigma'}}\int_E \varphi^2 u^2\right)\int_{t_0-\sigma' r^2}^{t_0} 1 \\
&\le \frac{6}{\tau^2 r^2}\int_{Q_\sigma} u^2 + r^{-2}\frac{12}{\tau^2 r^2}\left(\int_{Q_\sigma} u^2\right)\sigma' r^2 \text{ (by (7) and (9))} \\
&= \frac{6}{\tau^2 r^2}\int_{Q_\sigma} u^2 + \frac{12\sigma'}{\tau^2 r^2}\int_{Q_\sigma} u^2 \\
&= \frac{6(1+2\sigma')}{\tau^2 r^2}\int_{Q_\sigma} u^2 \\
&\le \frac{6(1+2)}{\tau^2 r^2}\int_{Q_\sigma} u^2 \text{ (since } 0<\sigma'<1) \\
&= \frac{18}{\tau^2 r^2}\int_{Q_\sigma} u^2,
\end{aligned}$$

we have

$$\begin{aligned}
\int_{t_0-\sigma' r^2}^{t_0}\int_E (\varphi u)^{2\left(1+\frac{2}{v}\right)} &\le \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{12}{\tau^2 r^2}\int_{Q_\sigma} u^2\right)^{\frac{2}{v}}\cdot A \text{ (by (10))} \\
&\le \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{12}{\tau^2 r^2}\int_{Q_\sigma} u^2\right)^{\frac{2}{v}}\frac{18}{\tau^2 r^2}\int_{Q_\sigma} u^2 \\
&= \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{c_1}{\tau^2 r^2}\int_{Q_\sigma} u^2\right)^{1+\frac{2}{v}}, \qquad (11)
\end{aligned}$$

where

$$c_1 = \left(12^2 18^v\right)^{\frac{1}{v+2}}.$$

Note that

$$
\begin{aligned}
\int_{t_0-\sigma' r^2}^{t_0}\int_E (\varphi u)^{2\left(1+\frac{2}{v}\right)} &\geq \int_{t_0-\sigma' r^2}^{t_0}\int_{\sigma' E} (\varphi u)^{2\left(1+\frac{2}{v}\right)} \\
&= \int_{t_0-\sigma' r^2}^{t_0}\int_{\sigma' E} u^{2\left(1+\frac{2}{v}\right)} \text{ (by (1))} \\
&= \int_{Q_{\sigma'}} u^{2\left(1+\frac{2}{v}\right)},
\end{aligned}
$$

so (11) becomes

$$
\int_{Q_{\sigma'}} u^{2\left(1+\frac{2}{v}\right)} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{c_1}{\tau^2 r^2}\int_{Q_\sigma} u^2\right)^{1+\frac{2}{v}}. \tag{12}
$$

Recall that (12) holds for all positive subsolution to the Dirichlet heat equation. For any $q>1$, since

$$
\begin{aligned}
\Delta u^q - \partial_t u^q &= \operatorname{div}(\nabla u^q) - \partial_t u^q \\
&= \operatorname{div}\left(q u^{q-1}\nabla u\right) - q u^{q-1}\partial_t u \\
&= q\left[\nabla u^{q-1}\nabla u + u^{q-1}\operatorname{div}(\nabla u)\right] - q u^{q-1}\partial_t u \\
&= q(q-1)u^{q-2}|\nabla u|^2 + q u^{q-1}\Delta u - q u^{q-1}\partial_t u \\
&\geq q u^{q-1}\Delta u - q u^{q-1}\partial_t u \text{ (since } q>1) \\
&= q u^{q-1}(\Delta u - \partial_t u) \\
&\geq 0,
\end{aligned}
$$

u^q is also a positive subsolution to the Dirichlet heat equation and thus u^q also satisfies (12). That is,

$$
\int_{Q_{\sigma'}} u^{2q\theta} \leq \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{c_1}{\tau^2 r^2}\int_{Q_\sigma} u^{2q}\right)^{\theta},
$$

where $\theta = 1+\frac{2}{v}$. Now, for the given $0<\delta'<\delta<1$, we let $\eta=\delta-\delta'$ and set $\sigma_0=\delta$, $\sigma_i = \delta - \sum_{j=1}^{i}\frac{\eta}{2^j}$ and $q_i=\theta^i$ for all $i\in\mathbb{N}$. Note that, by this setting, we have

$$
\begin{aligned}
\tau_{i+1} &= \sigma_{i+1}-\sigma_i \\
&= \frac{\eta}{2^{i+1}}.
\end{aligned}
$$

Thus, applying Moser's iteration on the last inequality with such setting, we obtain that

$$
\begin{aligned}
\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}} &\leq \frac{C_S r^2}{|E|^{\frac{2}{v}}} \left(\frac{c_1}{\tau_{i+1}^2 r^2} \int_{Q_{\sigma_i}} u^{2\theta^i} \right)^{\theta} \\
&= \frac{C_S r^2}{|E|^{\frac{2}{v}}} \left(\frac{4^{i+1} c_1}{\eta^2 r^2} \int_{Q_{\sigma_i}} u^{2\theta^i} \right)^{\theta} \\
&= \frac{C_S r^2}{|E|^{\frac{2}{v}}} \left(\frac{c_1}{\eta^2 r^2} \right)^{\theta} 4^{(i+1)\theta} \left(\int_{Q_{\sigma_i}} u^{2\theta^i} \right)^{\theta} \\
&= K \cdot L^{\theta} \cdot 4^{(i+1)\theta} \left(\int_{Q_{\sigma_i}} u^{2\theta^i} \right)^{\theta} \\
&\leq K \cdot L^{\theta} \cdot 4^{(i+1)\theta} \left[K \cdot L^{\theta} \cdot 4^{i\theta} \left(\int_{Q_{\sigma_{i-1}}} u^{2\theta^{i-1}} \right)^{\theta} \right]^{\theta} \\
&= K^{1+\theta} \cdot L^{\theta+\theta^2} \cdot 4^{(i+1)\theta + i\theta^2} \left(\int_{Q_{\sigma_{i-1}}} u^{2\theta^{i-1}} \right)^{\theta^2} \\
&\;\;\vdots \\
&\leq K^{1+\theta+\cdots+\theta^i} \cdot L^{\theta+\theta^2+\cdots+\theta^{i+1}} \cdot 4^{(i+1)\theta + i\theta^2 + \cdots + \theta^{i+1}} \left(\int_{Q_{\sigma_0}} u^{2\theta^0} \right)^{\theta^{i+1}},
\end{aligned}
$$

where

$$
\begin{cases} K = \frac{C_S r^2}{|E|^{\frac{2}{v}}} \\ L = \frac{c_1}{\eta^2 r^2} \end{cases}.
$$

So,

$$
\begin{aligned}
\left(\int_{Q_{\sigma_{i+1}}} \left|u^2\right|^{\theta^{i+1}} \right)^{\frac{1}{\theta^{i+1}}} &\leq K^{\frac{1}{\theta^{i+1}} + \frac{1}{\theta^i} + \cdots + \frac{1}{\theta}} \cdot L^{\frac{1}{\theta^i} + \frac{1}{\theta^{i-1}} + \cdots + 1} \cdot 4^{\frac{(i+1)}{\theta^i} + \frac{i}{\theta^{i-1}} + \cdots + 1} \int_{Q_{\sigma_0}} u^{2\theta^0} \\
&= K^{\sum_{j=1}^{i+1} \frac{1}{\theta^j}} \cdot L^{\sum_{j=0}^{i} \frac{1}{\theta^j}} \cdot 4^{\sum_{j=0}^{i} \frac{j+1}{\theta^j}} \int_{Q_\delta} u^2.
\end{aligned}
$$

letting $i \to \infty$ in the last inequality, we get

$$\begin{aligned}
\sup_{Q_{\delta'}} u^2 &\le K^{\sum_{j=1}^{\infty} \frac{1}{\theta^j}} \cdot L^{\sum_{j=0}^{\infty} \frac{1}{\theta^j}} \cdot 4^{\sum_{j=0}^{\infty} \frac{j+1}{\theta^j}} \int_{Q_\delta} u^2 \\
&= K^{\frac{1}{\theta-1}} \cdot L^{\frac{\theta}{\theta-1}} \cdot 4^{\left(\frac{\theta}{\theta-1}\right)^2} \int_{Q_\delta} u^2 \\
&= K^{\frac{v}{2}} \cdot L^{1+\frac{v}{2}} \cdot 4^{\left(1+\frac{v}{2}\right)^2} \int_{Q_\delta} u^2 \text{ (since } \theta = 1 + \frac{2}{v}) \\
&= \left(\frac{C_S r^2}{|E|^{\frac{2}{v}}} \right)^{\frac{v}{2}} \cdot \left(\frac{c_1}{\eta^2 r^2} \right)^{1+\frac{v}{2}} \cdot 4^{\left(1+\frac{v}{2}\right)^2} \int_{Q_\delta} u^2 \\
&= \frac{C_S^{\frac{v}{2}} c_1^{1+\frac{v}{2}} 4^{\left(1+\frac{v}{2}\right)^2}}{\eta^{2+v} r^2 |E|} \int_{Q_\delta} u^2 \\
&= \frac{c_2}{\eta^{2+v} r^2 |E|} \int_{Q_\delta} u^2, \qquad (13)
\end{aligned}$$

where

$$c_2 = C_S^{\frac{v}{2}} c_1^{1+\frac{v}{2}} 4^{\left(1+\frac{v}{2}\right)^2}.$$

Now, for the given $p > 0$, if $p \ge 2$, then

$$\begin{aligned}
\int_{Q_\delta} u^2 &\le \left(\int_{Q_\delta} u^p \right)^{\frac{2}{p}} \left(\int_{Q_\delta} 1 \right)^{1-\frac{2}{p}} \text{ (by Hölder's inequality)} \\
&\le \left(\int_{Q_\delta} u^p \right)^{\frac{2}{p}} \left(\int_{Q_1} 1 \right)^{1-\frac{2}{p}} \\
&= \left(\int_{Q_\delta} u^p \right)^{\frac{2}{p}} \left(r^2 |E| \right)^{1-\frac{2}{p}} \qquad (14)
\end{aligned}$$

and thus

$$\begin{aligned}
\sup_{Q_{\delta'}} u &= \left(\sup_{Q_{\delta'}} u^2 \right)^{\frac{1}{2}} \\
&\le \frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}} r |E|^{\frac{1}{2}}} \left(\int_{Q_\delta} u^2 \right)^{\frac{1}{2}} \text{ (by (13))} \\
&\le \frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}} r |E|^{\frac{1}{2}}} \left(\int_{Q_\delta} u^p \right)^{\frac{1}{p}} \left(r^2 |E| \right)^{\frac{1}{2}-\frac{1}{p}} \text{ (by (14))} \\
&= \frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}}} \left(\frac{1}{r^2 |E|} \int_{Q_\delta} u^p \right)^{\frac{1}{p}}. \qquad (15)
\end{aligned}$$

On the other hand, in the case of $0 < p < 2$, we note that

$$\sup_{Q_{\delta'}} u \le \frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}} r |E|^{\frac{1}{2}}} \left(\int_{Q_\delta} u^2 \right)^{\frac{1}{2}} \text{ (by (13))}$$

holds for any $0<\delta'<\delta\leq 1$. So, for any $0<\sigma<\sigma+\varepsilon\leq\delta$, we have

$$\sup_{Q_\sigma} u \leq \frac{c_2^{\frac{1}{2}}}{\varepsilon^{1+\frac{v}{2}} r\,|E|^{\frac{1}{2}}}\left(\int_{Q_{\sigma+\varepsilon}} u^2\right)^{\frac{1}{2}}.$$

Since

$$\begin{aligned}
\left(\int_{Q_{\sigma+\varepsilon}} u^2\right)^{\frac{1}{2}} &= \left(\int_{Q_{\sigma+\varepsilon}} u^p u^{2-p}\right)^{\frac{1}{2}}\\
&\leq \left(\sup_{Q_{\sigma+\varepsilon}} u^{2-p}\right)^{\frac{1}{2}}\left(\int_{Q_{\sigma+\varepsilon}} u^p\right)^{\frac{1}{2}}\\
&\leq \left(\sup_{Q_{\sigma+\varepsilon}} u\right)^{1-\frac{p}{2}}\left(\int_{Q_\delta} u^p\right)^{\frac{1}{2}},
\end{aligned}$$

the previous inequality becomes

$$\sup_{Q_\sigma} u \leq \frac{c_2^{\frac{1}{2}}}{\varepsilon^{1+\frac{v}{2}} r\,|E|^{\frac{1}{2}}}\left(\sup_{Q_{\sigma+\varepsilon}} u\right)^{1-\frac{p}{2}}\left(\int_{Q_\delta} u^p\right)^{\frac{1}{2}}.$$

Set $\sigma_0=\delta'$, $\sigma_{i+1}=\sigma_i+\varepsilon_{i+1}$, $\varepsilon_i=\frac{\eta}{2^i}$ and $\vartheta=1-\frac{p}{2}$. Iterating on the last inequality by such setting, we obtain that

$$\begin{aligned}
\sup_{Q_{\delta'}} u &= \sup_{Q_{\sigma_0}} u\\
&\leq \frac{c_2^{\frac{1}{2}}}{\left(\frac{\eta}{2^1}\right)^{1+\frac{v}{2}} r\,|E|^{\frac{1}{2}}}\left(\sup_{Q_{\sigma_1}} u\right)^{\vartheta}\left(\int_{Q_\delta} u^p\right)^{\frac{1}{2}}\\
&= \frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}} r\,|E|^{\frac{1}{2}}}\left(\int_{Q_\delta} u^p\right)^{\frac{1}{2}}\left(2^{1+\frac{v}{2}}\right)\left(\sup_{Q_{\sigma_1}} u\right)^{\vartheta}\\
&= M\cdot N^1\cdot\left(\sup_{Q_{\sigma_1}} u\right)^{\vartheta}\\
&\leq M\cdot N^1\cdot\left[M\cdot N^2\cdot\left(\sup_{Q_{\sigma_2}} u\right)^{\vartheta}\right]^{\vartheta}\\
&= M^{1+\vartheta}\cdot N^{1+2\vartheta}\cdot\left(\sup_{Q_{\sigma_2}} u\right)^{\vartheta^2}\\
&\;\;\vdots\\
&\leq M^{1+\vartheta+\cdots+\vartheta^i}\cdot N^{1+2\vartheta+\cdots+(i+1)\vartheta^i}\cdot\left(\sup_{Q_{\sigma_{i+1}}} u\right)^{\vartheta^{i+1}}\\
&= M^{\sum_{j=0}^{i}\vartheta^j}\cdot N^{\sum_{j=0}^{i}(j+1)\vartheta^j}\cdot\left(\sup_{Q_{\sigma_{i+1}}} u\right)^{\vartheta^{i+1}},
\end{aligned}$$

where

$$\begin{cases} M=\frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}} r|E|^{\frac{1}{2}}}\left(\int_{Q_\delta} u^p\right)^{\frac{1}{2}}\\ N=2^{1+\frac{v}{2}}\end{cases}.$$

In the last inequality, letting $i \to \infty$, we obtain that

$$\begin{aligned}
\sup_{Q_{\delta'}} u &\leq M^{\sum_{j=0}^{\infty}\vartheta^j} \cdot N^{\sum_{j=0}^{\infty}(j+1)\vartheta^j} \cdot \lim_{i\to\infty} \left(\sup_{Q_{\sigma_{i+1}}} u \right)^{\vartheta^{i+1}} \\
&= M^{\frac{1}{1-\vartheta}} \cdot N^{\frac{1}{(1-\vartheta)^2}} \cdot 1 \\
&= M^{\frac{2}{p}} \cdot N^{\frac{4}{p^2}} \\
&= \left[\frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}} r \left|E\right|^{\frac{1}{2}}} \left(\int_{Q_\delta} u^p \right)^{\frac{1}{2}} \right]^{\frac{2}{p}} \left(2^{1+\frac{v}{2}}\right)^{\frac{4}{p^2}} \\
&= \frac{c_2^{\frac{1}{p}} \left(2^{1+\frac{v}{2}}\right)^{\frac{4}{p^2}}}{\eta^{\frac{2+v}{p}}} \left(\frac{1}{r^2\left|E\right|} \int_{Q_\delta} u^p \right)^{\frac{1}{p}}. \qquad (16)
\end{aligned}$$

By (15) and (16), we know that, for any $p > 0$,

$$\sup_{Q_{\delta'}} u \leq C_M \left(\frac{1}{r^2\left|E\right|} \int_{Q_\delta} u^p \right)^{\frac{1}{p}},$$

where

$$C_M = \max\left\{ \frac{c_2^{\frac{1}{2}}}{\eta^{1+\frac{v}{2}}}, \frac{c_2^{\frac{1}{p}} \left(2^{1+\frac{v}{2}}\right)^{\frac{4}{p^2}}}{\eta^{\frac{2+v}{p}}} \right\}.$$

Recall that

$$\begin{aligned}
c_2 &= C_S^{\frac{v}{2}} c_1^{1+\frac{v}{2}} 4^{\left(1+\frac{v}{2}\right)^2} \\
&= C_S^{\frac{v}{2}} \left[\left(12^2 18^v\right)^{\frac{1}{v+2}} \right]^{1+\frac{v}{2}} 4^{\left(1+\frac{v}{2}\right)^2} \\
&= 12 \cdot 18^{\frac{v}{2}} \cdot 4^{\left(1+\frac{v}{2}\right)^2} C_S^{\frac{v}{2}},
\end{aligned}$$

and

$$\eta = \delta - \delta'.$$

So C_M is just a constant dependent on C_S, δ', δ and p. Thus we get the proof. □

4.2 Supersolutions

Theorem 4.2. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $t_0 > r^2$ and u be a positive supersolution to the Dirichlet heat equation on $E \times [t_0 - r^2, t_0]$. Fix $v > 2$ and let $\theta = 1 + \frac{v}{2}$. If SI holds for all functions in $C_0^\infty(E)$ on M, then, for any $0 < \delta' < \delta \leq 1$ and $0 < p \leq \frac{p_0}{\theta} < p_0 \leq \infty$, there exists a constant $C_{RH} = C_{RH}(C_S, \delta', \delta, v, p, p_0, r) > 0$ such that*

$$\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0} \right)^{\frac{1}{p_0}} \leq C_{RH} \left(\frac{1}{\left|E\right| r^2} \right)^{\frac{1}{p}-\frac{1}{p_0}} \left(\int_{\widetilde{Q}_\delta} u^p \right)^{\frac{1}{p}} \qquad (4.2)$$

where

$$\widetilde{Q}_\kappa \equiv \kappa E \times \left[t_0 - r^2, t_0 - (1-\kappa) r^2\right]$$

with $\kappa E = B_{x_0}(\kappa r)$ for all $\kappa \in (0,1]$ and C_S is the controlling constant in SI. The inequality in this theorem is called the reverse Hölder inequality (RHI) and C_{RH} is called the controlling constant in RHI.

Proof. For any $0 < \sigma' < \sigma < 1$, let $\tau = \sigma - \sigma'$. Let $\alpha, \beta : [0, \infty) \to \mathbb{R}$ be smooth functions such that

$$\begin{cases} \alpha(l) = 0 & \text{for } l \geq \sigma r \\ \alpha(l) = 1 & \text{for } 0 \leq l \leq \sigma' r \\ |\alpha'| \leq \frac{2}{\tau r} \end{cases} \tag{1}$$

and

$$\begin{cases} \beta(t) = 0 & \text{for } t \geq t_0 - (1-\sigma) r^2 \\ \beta(t) = 1 & \text{for } 0 \leq l \leq t_0 - (1-\sigma') r^2 \\ |\beta'| \leq \frac{2}{\tau^2 r^2} \end{cases}, \tag{2}$$

respectively. Also, let $\varphi : M \times [0, \infty) \to \mathbb{R}$ be the test function defined by

$$\varphi(x,t) = \alpha(d(x, x_0)) \beta(t)$$

for all $x \in M$ and $t \geq 0$. Then, for any $q \in \left(0, \frac{p_0}{\theta}\right]$, since $u(\cdot, t) \in C_0^\infty(E)$, for all $t \in [t_0 - r^2, t_0]$, and u is a supersolution to the Dirichlet heat equation, we have

$$\begin{aligned} \int_E \nabla\left(\varphi^2 u^{q-1}\right) \nabla u &= -\int_E \varphi^2 u^{q-1} \Delta u \text{ (by Green's identity)} \\ &\geq -\int_E \varphi^2 u^{q-1} \partial_t u \\ &= -\frac{1}{q} \int_E \varphi^2 q u^{q-1} \partial_t u \\ &= -\frac{1}{q} \int_E \varphi^2 \partial_t u^q \\ &= -\frac{1}{q} \int_E \left[\partial_t\left(\varphi^2 u^q\right) - u^q \partial_t \varphi^2\right] \\ &= -\frac{1}{q} \partial_t \int_E \varphi^2 u^q + \frac{2}{q} \int_E u^q \varphi \partial_t \varphi. \end{aligned} \tag{1}$$

Note that, on the other hand, we have

$$\begin{aligned} \int_E \nabla\left(\varphi^2 u^{q-1}\right) \nabla u &= \int_E \left(\varphi^2 \nabla u^{q-1} + u^{q-1} \nabla \varphi^2\right) \nabla u \\ &= \int_E \left[(q-1) \varphi^2 u^{q-2} |\nabla u|^2 + 2\varphi u^{q-1} \nabla \varphi \nabla u\right] \\ &= \int_E \left[(q-1) \varphi^2 \frac{4}{q^2} \cdot \frac{q^2}{4} \left(u^{\frac{q}{2}-1}\right)^2 |\nabla u|^2 + 2\varphi u^{\frac{q}{2}} \frac{2}{q} \cdot \frac{q}{2} u^{\frac{q}{2}-1} \nabla \varphi \nabla u\right] \\ &= \int_E \left[\frac{4(q-1)}{q^2} \varphi^2 \left|\nabla u^{\frac{q}{2}}\right|^2 + \frac{4}{q} \varphi u^{\frac{q}{2}} \nabla \varphi \nabla u^{\frac{q}{2}}\right] \\ &= \frac{4(q-1)}{q^2} \int_E \left[\left|\varphi \nabla u^{\frac{q}{2}}\right|^2 + \frac{q}{q-1} \left(\varphi \nabla u^{\frac{q}{2}}\right) \left(u^{\frac{q}{2}} \nabla \varphi\right)\right]. \end{aligned} \tag{2}$$

Since

$$\begin{aligned} \left|\varphi \nabla u^{\frac{q}{2}}\right|^2 &= \left|\nabla\left(\varphi u^{\frac{q}{2}}\right) - u^{\frac{q}{2}} \nabla \varphi\right|^2 \\ &= \left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2 + u^q |\nabla \varphi|^2 - 2u^{\frac{q}{2}} \nabla \varphi \nabla\left(\varphi u^{\frac{q}{2}}\right) \end{aligned}$$

and

$$\begin{aligned}\frac{q}{q-1}\left(\varphi\nabla u^{\frac{q}{2}}\right)\left(u^{\frac{q}{2}}\nabla\varphi\right) &= \frac{q}{q-1}\left[\nabla\left(\varphi u^{\frac{q}{2}}\right)-u^{\frac{q}{2}}\nabla\varphi\right]\left(u^{\frac{q}{2}}\nabla\varphi\right)\\ &= \frac{q}{q-1}u^{\frac{q}{2}}\nabla\varphi\nabla\left(\varphi u^{\frac{q}{2}}\right)-\frac{q}{q-1}u^{q}\left|\nabla\varphi\right|^{2},\end{aligned}$$

we get

$$(2)=\frac{4(q-1)}{q^{2}}\int_{E}\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}+\frac{2-q}{q-1}u^{\frac{q}{2}}\nabla\varphi\nabla\left(\varphi u^{\frac{q}{2}}\right)-\frac{1}{q-1}u^{q}\left|\nabla\varphi\right|^{2}\right]. \qquad (3)$$

Since

$$\begin{aligned}0 &\leq \left|\left(\frac{2-q}{q-1}\right)u^{\frac{q}{2}}\nabla\varphi+\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}\\ &= \left(\frac{2-q}{q-1}\right)^{2}u^{q}\left|\nabla\varphi\right|^{2}+\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}+2\left(\frac{2-q}{q-1}\right)u^{\frac{q}{2}}\nabla\varphi\nabla\left(\varphi u^{\frac{q}{2}}\right),\end{aligned}$$

we have

$$\frac{2-q}{q-1}u^{\frac{q}{2}}\nabla\varphi\nabla\left(\varphi u^{\frac{q}{2}}\right)\geq-\frac{1}{2}\left(\frac{2-q}{q-1}\right)^{2}u^{q}\left|\nabla\varphi\right|^{2}-\frac{1}{2}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}.$$

Note that, since $q\leq\frac{p_0}{\theta}<1$,

$$\frac{4(q-1)}{q^{2}}<0,$$

so

$$\begin{aligned}(3) &\leq \frac{4(q-1)}{q^{2}}\int_{E}\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}-\frac{1}{2}\left(\frac{2-q}{q-1}\right)^{2}u^{q}\left|\nabla\varphi\right|^{2}-\frac{1}{2}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}-\frac{1}{q-1}u^{q}\left|\nabla\varphi\right|^{2}\right]\\ &= \frac{4(q-1)}{q^{2}}\int_{E}\left[\frac{1}{2}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}-\frac{q^{2}-2q+2}{2(q-1)^{2}}u^{q}\left|\nabla\varphi\right|^{2}\right].\end{aligned}$$

That is,

$$\int_{E}\nabla\left(\varphi^{2}u^{q-1}\right)\nabla u\leq\frac{4(q-1)}{q^{2}}\int_{E}\left[\frac{1}{2}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}-\frac{q^{2}-2q+2}{2(q-1)^{2}}u^{q}\left|\nabla\varphi\right|^{2}\right].$$

Combining the last inequality and (1), we obtain that

$$\frac{4(q-1)}{q^{2}}\int_{E}\left[\frac{1}{2}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}-\frac{q^{2}-2q+2}{2(q-1)^{2}}u^{q}\left|\nabla\varphi\right|^{2}\right]\geq-\frac{1}{q}\partial_{t}\int_{E}\varphi^{2}u^{q}+\frac{2}{q}\int_{E}u^{q}\varphi\partial_{t}\varphi.$$

It implies that

$$\int_{E}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}-\frac{q^{2}-2q+2}{(q-1)^{2}}\int_{E}u^{q}\left|\nabla\varphi\right|^{2}\leq-\frac{q}{2(q-1)}\partial_{t}\int_{E}\varphi^{2}u^{q}+\frac{q}{(q-1)}\int_{E}u^{q}\varphi\partial_{t}\varphi.$$

So, if we set

$$I=\int_{E}\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^{2}+\frac{q}{2(q-1)}\partial_{t}\int_{E}\varphi^{2}u^{q},$$

then

$$
\begin{aligned}
I &\leq \frac{q}{(q-1)}\int_E u^q\varphi\partial_t\varphi+\frac{q^2-2q+2}{(q-1)^2}\int_E u^q\left|\nabla\varphi\right|^2\\
&= \frac{-q}{(q-1)}\int_E u^q\varphi\left(-\partial_t\varphi\right)+\frac{q^2-2q+2}{(q-1)^2}\int_E u^q\left|\nabla\varphi\right|^2\\
&= \frac{-q}{(q-1)}\int_E u^q\varphi\left|\partial_t\varphi\right|+\frac{q^2-2q+2}{(q-1)^2}\int_E u^q\left|\nabla\varphi\right|^2\\
&\leq \left[\frac{-q}{(q-1)}+\frac{q^2-2q+2}{(q-1)^2}\right]\left(\int_E u^q\varphi\partial_t\varphi+\int_E u^q\left|\nabla\varphi\right|^2\right)\\
&= \frac{2-q}{(q-1)^2}\int_E u^q\left(\varphi\partial_t\varphi+\left|\nabla\varphi\right|^2\right),
\end{aligned}
$$

where $-\partial_t\varphi=|\partial_t\varphi|$ since $\partial_t\varphi$ is always non-positive by its definition and the second inequality holds since $\frac{-q}{(q-1)}$, $\int_E u^q\varphi\left|\partial_t\varphi\right|$, $\frac{q^2-2q+2}{(q-1)^2}$ and $\int_E u^q\left|\nabla\varphi\right|^2$ are all non-negative with $q\in\left(0,\frac{p_0}{\theta}\right]\subseteq(0,1)$. That is,

$$
\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+\frac{q}{2(q-1)}\partial_t\int_E\varphi^2u^q\leq\frac{2-q}{(q-1)^2}\int_E u^q\left(\varphi\partial_t\varphi+\left|\nabla\varphi\right|^2\right). \tag{4}
$$

Note that $2-q>0$ for any $q\in\left(0,\frac{p_0}{\theta}\right]\subseteq(0,1)$, so we have

$$
\begin{aligned}
\frac{2-q}{(q-1)^2}\int_E u^q\left(\varphi\left|\partial_t\varphi\right|+\left|\nabla\varphi\right|^2\right) &\leq \frac{2-q}{(q-1)^2}\int_{\sigma E}u^q\left(\frac{2}{\tau^2r^2}+\frac{4}{\tau^2r^2}\right)\ \text{(by (1) and (2))}\\
&= \frac{2-q}{(q-1)^2}\cdot\frac{6}{\tau^2r^2}\int_{\sigma E}u^q\\
&\leq \frac{6c_1(q)}{\tau^2r^2}\int_{\sigma E}u^q,
\end{aligned}
$$

where

$$
c_1(q)=\frac{2-q}{(q-1)^2}.
$$

Pluging this into (4), we get

$$
\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+\frac{q}{2(q-1)}\partial_t\int_E\varphi^2u^q\leq\frac{6c_1(q)}{\tau^2r^2}\int_{\sigma E}u^q. \tag{5}
$$

Being similar to the proof of Theorem 4.1, the last inequality also induces two important informations. First, since $\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2\geq 0$, we have

$$
\begin{aligned}
\frac{q}{2(q-1)}\partial_t\int_E\varphi^2u^q &\leq \int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+\frac{q}{2(q-1)}\partial_t\int_E\varphi^2u^q\\
&\leq \frac{6c_1(q)}{\tau^2r^2}\int_{\sigma E}u^q.\ \text{(by (5))}
\end{aligned}
$$

Let

$$
\widetilde{I}_{\sigma'}\equiv\left[t_0-r^2,t_0-(1-\sigma')\,r^2\right],
$$

then, for any $s \in \widetilde{I}_{\sigma'}$, integrating the last inequality from s to $t_0 - (1-\sigma)r^2$, we get

$$\begin{aligned} \frac{q}{2(q-1)} \int_s^{t_0-(1-\sigma)r^2} \partial_t \int_E \varphi^2 u^q &\leq \frac{6c_1(q)}{\tau^2 r^2} \int_s^{t_0-(1-\sigma)r^2} \int_{\sigma E} u^q \\ &\leq \frac{6c_1(q)}{\tau^2 r^2} \int_{t_0-r^2}^{t_0-(1-\sigma)r^2} \int_{\sigma E} u^q \\ &= \frac{6c_1(q)}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q. \end{aligned}$$

Since

$$\begin{aligned} \int_s^{t_0-(1-\sigma)r^2} \partial_t \int_E \varphi^2 u^q &= \int_E \varphi^2 u^q\Big|_{t=t_0-(1-\sigma)r^2} - \int_E \varphi^2 u^q\Big|_{t=s} \\ &= -\int_E \varphi^2 u^q\Big|_{t=s}, \text{ (by (2))} \end{aligned}$$

we arrive at

$$-\frac{q}{2(q-1)} \int_E \varphi^2 u^q\Big|_{t=s} \leq \frac{6c_1(q)}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q.$$

Note that $q < 1$ so $-\frac{2(q-1)}{q} > 0$ and thus the last inequality becomes

$$\begin{aligned} \int_E \varphi^2 u^q\Big|_{t=s} &\leq -\frac{2(q-1)}{q} \cdot \frac{6c_1(q)}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q \\ &= c_1(q) \cdot \frac{(1-q)}{q} \cdot \frac{12}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q \\ &= \frac{2-q}{(q-1)^2} \cdot \frac{(1-q)}{q} \cdot \frac{12}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q \\ &= \frac{q-2}{q(q-1)} \cdot \frac{12}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q \\ &= \frac{12c_2(q)}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q, \end{aligned}$$

where

$$c_2(q) = \frac{q-2}{q(q-1)}.$$

Since s is arbitrary, we get

$$\sup_{\widetilde{I}_{\sigma'}} \int_E \varphi^2 u^q \leq \frac{12c_2(q)}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q. \tag{6}$$

Second, integrating (5) directly on $[t_0 - r^2, t_0 - (1-\sigma)r^2]$, we obatain that

$$\begin{aligned} \int_{t_0-r^2}^{t_0-(1-\sigma)r^2} \int_E \left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2 + \frac{q}{2(q-1)} \int_{t_0-r^2}^{t_0-(1-\sigma)r^2} \partial_t \int_E \varphi^2 u^q &\leq \frac{6c_1(q)}{\tau^2 r^2} \int_{t_0-r^2}^{t_0-(1-\sigma)r^2} \int_{\sigma E} u^q \\ &= \frac{6c_1(q)}{\tau^2 r^2} \int_{\widetilde{Q}_\sigma} u^q. \end{aligned} \tag{7}$$

Since

$$\begin{aligned}\frac{q}{2(q-1)}\int_{t_0-r^2}^{t_0-(1-\sigma)r^2}\partial_t\int_E\varphi^2u^q &= \frac{q}{2(q-1)}\left(\int_E\varphi^2u^q\Big|_{t=t_0-(1-\sigma)r^2}-\int_E\varphi^2u^q\Big|_{t=t_0-r^2}\right)\\ &= -\frac{q}{2(q-1)}\int_E\varphi^2u^q\Big|_{t=t_0-r^2}\ \text{(by (2))}\\ &\geq 0\ (\text{since } 0<q<1)\end{aligned}$$

and

$$\int_{t_0-r^2}^{t_0-(1-\sigma)r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2\geq\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2,$$

we get

$$\begin{aligned}\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2 &\leq \int_{t_0-r^2}^{t_0-(1-\sigma)r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+\frac{q}{2(q-1)}\int_{t_0-r^2}^{t_0-(1-\sigma)r^2}\partial_t\int_E\varphi^2u^q\\ &\leq \frac{6c_1(q)}{\tau^2r^2}\int_{\widetilde{Q}_\sigma}u^q.\ \text{(by (7))}\end{aligned}$$

That is,

$$\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2\leq\frac{6c_1(q)}{\tau^2r^2}\int_{\widetilde{Q}_\sigma}u^q. \tag{8}$$

Now, note that, for the given $v>2$, we have

$$\begin{aligned}\int_E\left(\varphi u^{\frac{q}{2}}\right)^{2\left(1+\frac{2}{v}\right)} &= \int_E\left(\varphi u^{\frac{q}{2}}\right)^2\left(\varphi u^{\frac{q}{2}}\right)^{\frac{4}{v}}\\ &\leq \left(\int_E\left(\varphi u^{\frac{q}{2}}\right)^{\frac{2v}{v-2}}\right)^{\frac{v-2}{v}}\left(\int_E\left(\varphi u^{\frac{q}{2}}\right)^2\right)^{\frac{2}{v}}\ \text{(by Hölder's inequality)}\\ &\leq \left(\frac{C_Sr^2}{|E|^{\frac{2}{v}}}\int_E\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\left(\varphi u^{\frac{q}{2}}\right)^2\right]\right)\left(\int_E\left(\varphi u^{\frac{q}{2}}\right)^2\right)^{\frac{2}{v}}\ \text{(by (3.2))}\\ &= \left(\frac{C_Sr^2}{|E|^{\frac{2}{v}}}\int_E\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\varphi^2u^q\right]\right)\left(\int_E\varphi^2u^q\right)^{\frac{2}{v}}.\end{aligned}$$

So, if we let

$$A=\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left(\varphi u^{\frac{q}{2}}\right)^{2\left(1+\frac{2}{v}\right)},$$

then

$$\begin{aligned}A &\leq \int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\left(\frac{C_Sr^2}{|E|^{\frac{2}{v}}}\int_E\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\varphi^2u^q\right]\right)\left(\int_E\varphi^2u^q\right)^{\frac{2}{v}}\\ &\leq \frac{C_Sr^2}{|E|^{\frac{2}{v}}}\left(\sup_{\widetilde{I}_{\sigma'}}\int_E\varphi^2u^q\right)^{\frac{2}{v}}\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\varphi^2u^q\right]\\ &\leq \frac{C_Sr^2}{|E|^{\frac{2}{v}}}\left(\frac{12c_2(q)}{\tau^2r^2}\int_{\widetilde{Q}_\sigma}u^q\right)^{\frac{2}{v}}\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\varphi^2u^q\right]\ \text{(by (6))}\\ &= \frac{C_Sr^2}{|E|^{\frac{2}{v}}}\left(\frac{12c_2(q)}{\tau^2r^2}\int_{\widetilde{Q}_\sigma}u^q\right)^{\frac{2}{v}}\cdot B,\end{aligned}$$

where

$$B=\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left[\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\varphi^2u^q\right].$$

Note that

$$\begin{aligned}
B &= \int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\varphi^2u^q \\
&\le \int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left|\nabla\left(\varphi u^{\frac{q}{2}}\right)\right|^2+r^{-2}\left(\sup_{\tilde{I}_{\sigma'}}\int_E\varphi^2u^q\right)\int_{t_0-r^2}^{t_0-(1-\sigma')r^2}1 \\
&\le \frac{6c_1(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q+r^{-2}\left(\frac{12c_2(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q\right)\sigma'r^2 \text{ (by (6) and (8))} \\
&= \frac{6c_1(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q+\frac{12\sigma'c_2(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q \\
&\le \frac{12c_1(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q+\frac{12c_2(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q \text{ (since } 0<\sigma'<1) \\
&= \frac{12\left[c_1(q)+c_2(q)\right]}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q \\
&= \frac{12}{\tau^2r^2}\left[\frac{2-q}{(q-1)^2}+\frac{q-2}{q(q-1)}\right]\int_{\tilde{Q}_\sigma}u^q \\
&= \frac{12}{\tau^2r^2}\left[\frac{2-q}{q(q-1)^2}\right]\int_{\tilde{Q}_\sigma}u^q \\
&= \frac{12c_3(q)}{\tau^2r^2}\int_{\tilde{Q}_\sigma}u^q,
\end{aligned}$$

where

$$c_3(q)=\frac{2-q}{q(q-1)^2},$$

and

$$\begin{aligned}
A &= \int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_E\left(\varphi u^{\frac{q}{2}}\right)^{2\left(1+\frac{2}{v}\right)} \\
&= \int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_{\sigma E}\left(u^{\frac{q}{2}}\right)^{2\left(1+\frac{2}{v}\right)} \text{ (by (1))} \\
&\ge \int_{t_0-r^2}^{t_0-(1-\sigma')r^2}\int_{\sigma'E}\left(u^q\right)^{1+\frac{2}{v}} \\
&= \int_{\tilde{Q}_{\sigma'}}u^{q\left(1+\frac{2}{v}\right)} \\
&= \int_{\tilde{Q}_{\sigma'}}u^{q\theta},
\end{aligned}$$

so the previous inequality becomes

$$\begin{aligned}\int_{\widetilde{Q}_{\sigma'}} u^{q\theta} &\leq \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{12c_2(q)}{\tau^2r^2}\right)^{\frac{2}{v}}\cdot\left(\frac{12c_3(q)}{\tau^2r^2}\int_{\widetilde{Q}_{\sigma}} u^q\right)\\ &= \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{c_4(q)}{\tau^2r^2}\int_{\widetilde{Q}_{\sigma}} u^q\right)^{1+\frac{2}{v}}\\ &= \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{c_4(q)}{\tau^2r^2}\int_{\widetilde{Q}_{\sigma}} u^q\right)^{\theta},\end{aligned}$$

where

$$c_4(q) = 12\left[c_2(q)\right]^{\frac{2}{2+v}}\left[c_3(q)\right]^{\frac{v}{2+v}}.$$

Now, for the given $0<\delta'<\delta<1$, we let $\eta=\delta-\delta'$ and set $\sigma_0=\delta'$, $\sigma_i=\delta'+\sum_{j=1}^{i}\frac{\eta}{2^j}$ and $q_i=\frac{p_0}{\theta^i}$ for all $i\in\mathbb{N}$. Note that, by this setting, we have

$$\begin{aligned}\tau_{i+1} &= \sigma_{i+1}-\sigma_i\\ &= \frac{\eta}{2^{i+1}}.\end{aligned}$$

Thus, applying Moser's iteration on the last inequality with such setting, we obtain that

$$\begin{aligned}\int_{\widetilde{Q}_{\sigma'}} u^{p_0} &= \int_{\widetilde{Q}_{\sigma_0}}\left(u^{\frac{p_0}{\theta}}\right)^{\theta}\\ &\leq \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{c_4(q_1)}{\tau_1^2r^2}\int_{\widetilde{Q}_{\sigma_1}} u^{\frac{p_0}{\theta}}\right)^{\theta}\\ &= \frac{C_S r^2}{|E|^{\frac{2}{v}}}\left(\frac{1}{\eta^2r^2}\right)^{\theta}\cdot 4^{1\theta}\cdot\left[c_4(q_1)\right]^{\theta}\left(\int_{\widetilde{Q}_{\sigma_1}} u^{\frac{p_0}{\theta}}\right)^{\theta}\\ &= K\cdot L^{\theta}\cdot 4^{1\theta}\cdot\left[c_4(q_1)\right]^{\theta}\left(\int_{\widetilde{Q}_{\sigma_1}} u^{\frac{p_0}{\theta}}\right)^{\theta}\\ &\leq K\cdot L^{\theta}\cdot 4^{1\theta}\cdot\left[c_4(q_1)\right]^{\theta}\left\{K\cdot L^{\theta}\cdot 4^{2\theta}\cdot\left[c_4(q_2)\right]^{\theta}\left(\int_{\widetilde{Q}_{\sigma_2}} u^{\frac{p_0}{\theta^2}}\right)^{\theta}\right\}^{\theta}\\ &= K^{1+\theta}\cdot L^{\theta+\theta^2}\cdot 4^{1\theta+2\theta^2}\cdot\left[c_4(q_1)\right]^{\theta}\left[c_4(q_2)\right]^{\theta^2}\left(\int_{\widetilde{Q}_{\sigma_2}} u^{\frac{p_0}{\theta^2}}\right)^{\theta^2}\\ &\;\;\vdots\\ &\leq K^{1+\theta+\cdots+\theta^{i-1}}\cdot L^{\theta+\theta^2+\cdots+\theta^i}\cdot 4^{1\theta+2\theta^2+\cdots+i\theta^i}\cdot d_1(i)\left(\int_{\widetilde{Q}_{\sigma_i}} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i}\\ &\leq K^{\frac{1-\theta^i}{1-\theta}}\cdot L^{\frac{\theta\left(1-\theta^i\right)}{1-\theta}}\cdot 4^{\sum_{j=1}^{i} j\theta^j}\cdot d_1(i)\left(\int_{\widetilde{Q}_{\delta}} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i},\end{aligned}\tag{9}$$

where

$$\begin{cases} K=\frac{C_S r^2}{|E|^{\frac{2}{v}}}\\ L=\frac{1}{\eta^2r^2}\\ d_1(i)=\left[c_4(q_1)\right]^{\theta}\left[c_4(q_2)\right]^{\theta^2}\cdots\left[c_4(q_i)\right]^{\theta^i}\end{cases}.$$

Note that

$$\theta = 1 + \frac{2}{v} > 1$$

which implies that

$$1 - \theta^i < 0.$$

Thus

$$\begin{aligned}\sum_{j=1}^{i} j\theta^j &= \frac{\theta}{(1-\theta)^2}\left(1-\theta^i\right) + \frac{i\theta^{i+1}}{\theta-1} \\ &< \frac{i\theta^{i+1}}{\theta-1} \\ &= \frac{v}{2}\cdot i\theta^{i+1}.\end{aligned}$$

Pluging this result into (9), we obtain that

$$\begin{aligned}\int_{\widetilde{Q}_{\sigma'}} u^{p_0} &\le K^{\frac{1-\theta^i}{1-\theta}} \cdot L^{\frac{\theta\left(1-\theta^i\right)}{1-\theta}} \cdot 4^{\frac{v}{2}\cdot i\theta^{i+1}} \cdot d_1(i)\left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i} \\ &= K^{\frac{1-\theta^i}{1-\theta}} \cdot L^{\frac{\theta\left(1-\theta^i\right)}{1-\theta}} \cdot (2^v)^{i\theta^{i+1}} \cdot d_1(i)\left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i} \\ &= \left[K\cdot L^\theta\right]^{\frac{1-\theta^i}{1-\theta}} \cdot d_2(i)\left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i} \\ &= \left[K\cdot L^\theta\right]^{\frac{\theta^i-1}{\theta-1}} \cdot d_2(i)\left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i} \\ &= \left[\frac{C_S r^2}{|E|^{\frac{2}{v}}}\cdot\left(\frac{1}{\eta^2 r^2}\right)^{1+\frac{2}{v}}\right]^{\frac{\theta^i-1}{\frac{2}{v}}} \cdot d_2(i)\left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i} \\ &= \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}|E|\,r^2}\right)^{\theta^i-1} \cdot d_2(i)\left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\theta^i},\end{aligned}$$

where

$$d_2(i) = (2^v)^{i\theta^{i+1}} \cdot d_1(i).$$

Thus

$$\begin{aligned}\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} &\le \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}|E|\,r^2}\right)^{\frac{\theta^i}{p_0}-\frac{1}{p_0}} \cdot \left[d_2(i)\right]^{\frac{1}{p_0}} \cdot \left(\int_{\widetilde{Q}_\delta} u^{\frac{p_0}{\theta^i}}\right)^{\frac{\theta^i}{p_0}} \\ &= c^{\frac{1}{p_0\theta^{-i}}-\frac{1}{p_0}} \cdot \left[d_2(i)\right]^{\frac{1}{p_0}} \cdot \left(\int_{\widetilde{Q}_\delta} u^{p_0\theta^{-i}}\right)^{p_0\theta^{-i}},\end{aligned}$$

where

$$c = \frac{C_S^{\frac{v}{2}}}{\eta^{2+v}|E|\,r^2}.$$

Note that the last inequality holds for $0 < \delta' < \delta < 1$. So, in particular, for the given $0 < \delta' < \delta < 1$, since

$$0 < \delta' < \frac{\delta+\delta'}{2} < \delta < 1$$

we have

$$\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} \leq c^{\frac{1}{p_0\theta^{-i}}-\frac{1}{p_0}} \cdot [d_2(i)]^{\frac{1}{p_0}} \cdot \left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{p_0\theta^{-i}}\right)^{\frac{1}{p_0\theta^{-i}}} \tag{10}$$

and

$$\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{p_0}\right)^{\frac{1}{p_0}} \leq c^{\frac{1}{p_0\theta^{-i}}-\frac{1}{p_0}} \cdot [d_2(i)]^{\frac{1}{p_0}} \cdot \left(\int_{\widetilde{Q}_{\delta}} u^{p_0\theta^{-i}}\right)^{\frac{1}{p_0\theta^{-i}}}. \tag{11}$$

Now, for any $p \in \left(0, \frac{p_0}{\theta}\right]$, there exists some $k = k(p) \geq 2$ such that

$$\frac{p_0}{\theta^k} \leq p \leq \frac{p_0}{\theta^{k-1}}$$

and thus we can write

$$p = \lambda \cdot \frac{p_0}{\theta^{k-1}} = \lambda p_0 \theta^{1-k}$$

for some $\frac{1}{\theta} \leq \lambda \leq 1$. Note that, for such $0 < \lambda \leq 1$, since u is a supersolution to the Dirichlet heat equation, we have

$$\begin{aligned}
\Delta u^\lambda - \partial_t u^\lambda &= \operatorname{div}\left(\nabla u^\lambda\right) - \partial_t u^\lambda \\
&= \operatorname{div}\left(\lambda u^{\lambda-1}\nabla u\right) - \lambda u^{\lambda-1}\partial_t u \\
&= \lambda\left(\nabla u^{\lambda-1}\nabla u + u^{\lambda-1}\operatorname{div}(\nabla u)\right) - \lambda u^{\lambda-1}\partial_t u \\
&= \lambda(\lambda-1)u^{\lambda-2}|\nabla u|^2 + \lambda u^{\lambda-1}(\Delta u - \partial_t u) \\
&\leq 0.
\end{aligned}$$

So u^λ is also a supersolution to the heat equation and thus it also satisfies (11). That is,

$$\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} \left(u^\lambda\right)^{p_0}\right)^{\frac{1}{p_0}} \leq c^{\frac{1}{p_0\theta^{-i}}-\frac{1}{p_0}} \cdot [d_2(i)]^{\frac{1}{p_0}} \cdot \left(\int_{\widetilde{Q}_{\delta}} \left(u^\lambda\right)^{p_0\theta^{-i}}\right)^{\frac{1}{p_0\theta^{-i}}}.$$

Thus, with letting $i = k-1$, we get

$$\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} \left(u^\lambda\right)^{p_0}\right)^{\frac{1}{p_0}} \leq c^{\frac{1}{p_0\theta^{1-k}}-\frac{1}{p_0}} \cdot [d_2(k-1)]^{\frac{1}{p_0}} \cdot \left(\int_{\widetilde{Q}_{\delta}} \left(u^\lambda\right)^{p_0\theta^{1-k}}\right)^{\frac{1}{p_0\theta^{1-k}}}$$

and thus

$$\begin{aligned}
\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\lambda p_0}\right)^{\frac{1}{\lambda p_0}} &= \left[\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} \left(u^\lambda\right)^{p_0}\right)^{\frac{1}{p_0}}\right]^{\frac{1}{\lambda}} \\
&\leq c^{\frac{1}{\lambda p_0\theta^{1-k}}-\frac{1}{\lambda p_0}} \cdot [d_2(k-1)]^{\frac{1}{\lambda p_0}} \cdot \left(\int_{\widetilde{Q}_{\delta}} \left(u^\lambda\right)^{p_0\theta^{1-k}}\right)^{\frac{1}{\lambda p_0\theta^{1-k}}} \\
&= c^{\frac{1}{p}-\frac{1}{\lambda p_0}} \cdot [d_2(k-1)]^{\frac{1}{\lambda p_0}} \cdot \left(\int_{\widetilde{Q}_{\delta}} u^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{12}$$

Also, in (10), letting $i=1$, we get

$$\begin{aligned}\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} &\le c^{\frac{1}{p_0\theta^{-1}}-\frac{1}{p_0}}\cdot\left[d_2(1)\right]^{\frac{1}{p_0}}\cdot\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{p_0\theta^{-1}}\right)^{\frac{1}{p_0\theta^{-1}}}\\ &= c^{\frac{\theta}{p_0}-\frac{1}{p_0}}\cdot\left[d_2(1)\right]^{\frac{1}{p_0}}\cdot\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\frac{p_0}{\theta}}\right)^{\frac{\theta}{p_0}}.\end{aligned}$$

Note that $\lambda\ge\frac{1}{\theta}$ so $\lambda\theta\ge 1$. Applying Hölder's inequality we get

$$\begin{aligned}\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\frac{p_0}{\theta}} &\le \left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}}\left(u^{\frac{p_0}{\theta}}\right)^{\lambda\theta}\right)^{\frac{1}{\lambda\theta}}\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} 1\right)^{1-\frac{1}{\lambda\theta}}\\ &= \left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\lambda p_0}\right)^{\frac{1}{\lambda\theta}}\left|\widetilde{Q}_{\frac{\delta+\delta'}{2}}\right|^{1-\frac{1}{\lambda\theta}}\\ &\le \left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\lambda p_0}\right)^{\frac{1}{\lambda\theta}}\left|\widetilde{Q}_1\right|^{1-\frac{1}{\lambda\theta}}\\ &= \left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\lambda p_0}\right)^{\frac{1}{\lambda\theta}}\left(|E|\,r^2\right)^{1-\frac{1}{\lambda\theta}}.\end{aligned}$$

Thus the previous inequality becomes

$$\begin{aligned}\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} &\le c^{\frac{\theta}{p_0}-\frac{1}{p_0}}\left[d_2(1)\right]^{\frac{1}{p_0}}\left[\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\lambda p_0}\right)^{\frac{1}{\lambda\theta}}\left(|E|\,r^2\right)^{1-\frac{1}{\lambda\theta}}\right]^{\frac{\theta}{p_0}}\\ &= c^{\frac{\theta}{p_0}-\frac{1}{p_0}}\left[d_2(1)\right]^{\frac{1}{p_0}}\left(\int_{\widetilde{Q}_{\frac{\delta+\delta'}{2}}} u^{\lambda p_0}\right)^{\frac{1}{\lambda p_0}}\left(|E|\,r^2\right)^{\frac{\theta}{p_0}-\frac{1}{\lambda p_0}}\\ &\le c^{\frac{\theta}{p_0}-\frac{1}{p_0}}\left[d_2(1)\right]^{\frac{1}{p_0}}c^{\frac{1}{p}-\frac{1}{\lambda p_0}}\left[d_2(k-1)\right]^{\frac{1}{\lambda p_0}}\left(\int_{\widetilde{Q}_{\delta}} u^{p}\right)^{\frac{1}{p}}\left(|E|\,r^2\right)^{\frac{\theta}{p_0}-\frac{1}{\lambda p_0}}\\ &= \left(|E|\,r^2\right)^{\frac{\theta}{p_0}-\frac{1}{\lambda p_0}}c^{\frac{\theta}{p_0}-\frac{1}{p_0}+\frac{1}{p}-\frac{1}{\lambda p_0}}\left[d_2(1)\right]^{\frac{1}{p_0}}\left[d_2(k-1)\right]^{\frac{1}{\lambda p_0}}\left(\int_{\widetilde{Q}_{\delta}} u^{p}\right)^{\frac{1}{p}},\quad(13)\end{aligned}$$

where the second inequality holds by (12). Note that $p=\lambda p_0\theta^{1-k}$, we have $\lambda p_0=\frac{p}{\theta^{1-k}}$. Pluging this into (13), we get

$$\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}}\le\left(|E|\,r^2\right)^{\frac{\theta}{p_0}-\frac{\theta^{1-k}}{p}}c^{\frac{\theta}{p_0}-\frac{1}{p_0}+\frac{1}{p}-\frac{\theta^{1-k}}{p}}\left[d_2(1)\right]^{\frac{1}{p_0}}\left[d_2(k-1)\right]^{\frac{\theta^{1-k}}{p}}\left(\int_{\widetilde{Q}_{\delta}} u^{p}\right)^{\frac{1}{p}}.\quad(14)$$

Since

$$c=\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}\,|E|\,r^2},$$

we have

$$
\begin{aligned}
\left(|E|\,r^2\right)^{\frac{\theta}{p_0}-\frac{\theta^{1-k}}{p}} c^{\frac{\theta}{p_0}-\frac{1}{p_0}+\frac{1}{p}-\frac{\theta^{1-k}}{p}} &= \left(|E|\,r^2\right)^{\frac{\theta}{p_0}-\frac{\theta^{1-k}}{p}} \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}\,|E|\,r^2}\right)^{\frac{\theta}{p_0}-\frac{1}{p_0}+\frac{1}{p}-\frac{\theta^{1-k}}{p}} \\
&= \left(\frac{1}{|E|\,r^2}\right)^{\frac{1}{p}-\frac{1}{p_0}} \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}}\right)^{\frac{\theta}{p_0}-\frac{1}{p_0}+\frac{1}{p}-\frac{\theta^{1-k}}{p}} \\
&= \left(\frac{1}{|E|\,r^2}\right)^{\frac{1}{p}-\frac{1}{p_0}} \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}}\right)^{\frac{\theta-1}{p_0}+\frac{1-\theta^{1-k}}{p}}.
\end{aligned}
$$

Pluging this into (14), we arrive at

$$
\begin{aligned}
\left(\int_{\widetilde{Q}_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} &= \left(\frac{1}{|E|\,r^2}\right)^{\frac{1}{p}-\frac{1}{p_0}} \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}}\right)^{\frac{\theta-1}{p_0}+\frac{1-\theta^{1-k}}{p}} [d_2(1)]^{\frac{1}{p_0}} [d_2(k-1)]^{\frac{\theta^{1-k}}{p}} \left(\int_{\widetilde{Q}_\delta} u^p\right)^{\frac{1}{p}} \\
&= C_{RH} \left(\frac{1}{|E|\,r^2}\right)^{\frac{1}{p}-\frac{1}{p_0}} \left(\int_{\widetilde{Q}_\delta} u^p\right)^{\frac{1}{p}},
\end{aligned}
$$

where

$$
C_{RH} = \left(\frac{C_S^{\frac{v}{2}}}{\eta^{2+v}}\right)^{\frac{\theta-1}{p_0}+\frac{1-\theta^{1-k}}{p}} [d_2(1)]^{\frac{1}{p_0}} [d_2(k-1)]^{\frac{\theta^{1-k}}{p}}.
$$

Recall that

$$
d_2(i) = (2^v)^{i\theta^{i+1}} \cdot d_1(i),
$$

where

$$
\begin{cases}
d_1(i) = [c_4(q_1)]^{\theta} [c_4(q_2)]^{\theta^2} \cdots [c_4(q_i)]^{\theta^i} \\
c_4(q) = 12\,[c_2(q)]^{\frac{2}{2+v}} [c_3(q)]^{\frac{v}{2+v}} \\
c_3(q) = \frac{2-q}{q(q-1)^2} \\
c_2(q) = \frac{q-2}{q(q-1)} \\
q_j = \frac{p_0}{\theta^j}
\end{cases}.
$$

Since $\eta = \delta - \delta'$, $\theta = 1 + \frac{2}{v}$ and $k = k(p)$, we can conclude that C_{RH} depends only on C_S, δ', δ, v, p and p_0. This finishes the proof. □

Theorem 4.3. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $t_0 > r^2$ and u be the positive supersolution to the Dirichlet heat equation on $E \times [t_0 - r^2, t_0]$. If M has both VDC and WPI, then, for any $0 < \delta < 1$ and $0 < \rho < 1$, there exists two constant $a = a(u, \rho) > 0$ and $C = C(\delta, d_0, P_2) > 0$ such that*

$$
\begin{cases}
|\{(x,t) \in Q_+ : \ln u^{-1} > a + \lambda\}| \le C\,|E|\,r^2\lambda^{-1} \\
|\{(x,t) \in Q_- : \ln u > -a + \lambda\}| \le C\,|E|\,r^2\lambda^{-1}
\end{cases}
$$

with

$$
\begin{cases}
Q_+ = \delta E \times [t_0 - \rho r^2, t_0] \\
Q_- = \delta E \times [t_0 - r^2, t_0 - \rho r^2]
\end{cases}
$$

holds for all $\lambda > 0$, where $\delta E \equiv B_{x_0}(\delta r)$. d_0 and P_2 are the controlling constants in VDC and WPI, respectively.

Proof. Let

$$w = -\ln u, \tag{1}$$

then, since u is a supersolution, we have

$$\begin{aligned}
\Delta w - \partial_t w - |\nabla w|^2 &= \operatorname{div}\left[\nabla\left(-\ln u\right)\right] - \partial_t\left(-\ln u\right) - \left|\nabla\left(-\ln u\right)\right|^2 \\
&= -\operatorname{div}\left(\frac{\nabla u}{u}\right) + \frac{\partial_t u}{u} - \left|-\frac{\nabla u}{u}\right|^2 \\
&= -\left[\nabla\left(\frac{1}{u}\right)\nabla u + \frac{1}{u}\operatorname{div}\left(\nabla u\right)\right] + \frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} \\
&= -\left(\frac{-|\nabla u|^2}{u^2} + \frac{\Delta u}{u}\right) + \frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} \\
&= -\frac{1}{u}\left(\Delta u - \partial_t u\right) \\
&\geq 0.
\end{aligned}$$

That is,

$$\partial_t w \leq \Delta w - |\nabla w|^2. \tag{2}$$

Now, for a given $\delta \in (0,1)$, let $f : [0,\infty) \to \mathbb{R}$ be defined by

$$f(l) = \begin{cases} 1 & , l \in [0,\delta] \\ \frac{1-l}{1-\delta} & , l \in [\delta, 1] \\ 0 & , l \in [1,\infty) \end{cases}$$

and $\varphi : M \to \mathbb{R}$ be defined by

$$\varphi(x) = f\left(\frac{d(x,x_0)}{r}\right)$$

for all $x \in M$. Using φ as a test function, since $\varphi \in C_0(E)$, we obtain that

$$\begin{aligned}
\partial_t \int_M \varphi^2 w &= \int_E \varphi^2 \partial_t w \\
&\leq \int_E \varphi^2\left(\Delta w - |\nabla w|^2\right) \text{ (by (2))} \\
&= \int_E \varphi^2 \Delta w - \int_E \varphi^2 |\nabla w|^2 \\
&= -\int_E \left(\nabla \varphi^2\right)\nabla w - \int_E \varphi^2 |\nabla w|^2 \text{ (by Green's identity)} \\
&= -2\int_E \varphi \nabla\varphi \nabla w - \int_E \varphi^2 |\nabla w|^2.
\end{aligned} \tag{3}$$

Note that

$$\begin{aligned}
0 &\leq \left|\sqrt{2}\nabla\varphi + \frac{1}{\sqrt{2}}\varphi\nabla w\right|^2 \\
&= 2|\nabla\varphi|^2 + \frac{1}{2}\varphi^2|\nabla w|^2 + 2\varphi\nabla\varphi\nabla w
\end{aligned}$$

which is equivalent to

$$-2\varphi\nabla\varphi\nabla w \leq 2|\nabla\varphi|^2 + \frac{1}{2}\varphi^2|\nabla w|^2.$$

Pluging this into (3), we get

$$
\begin{aligned}
(3) &\le \left(2\int_E |\nabla\varphi|^2 + \frac{1}{2}\int_E \varphi^2 |\nabla w|^2\right) - \int_E |\varphi\nabla w|^2 \\
&= 2\int_E |\nabla\varphi|^2 - \frac{1}{2}\int_E \varphi^2 |\nabla w|^2 .
\end{aligned}
$$

That is,

$$\partial_t \int_M \varphi^2 w \le 2\int_E |\nabla\varphi|^2 - \frac{1}{2}\int_E \varphi^2 |\nabla w|^2$$

which is equivalent to

$$2\int_E |\nabla\varphi|^2 \ge \partial_t \int_M \varphi^2 w + \frac{1}{2}\int_E \varphi^2 |\nabla w|^2 . \tag{4}$$

Note that, since $\varphi \in C_0(E)$, we have

$$
\begin{aligned}
\int_E \varphi^2 |\nabla w|^2 &= \int_M |\nabla w|^2 \varphi^2 \\
&\ge P_w^{-1} r^{-2} \int_M |w - w_\varphi|^2 \varphi^2, \text{ (by (2.4))}
\end{aligned}
$$

where $P_w = P_w(d_0, P_2) > 0$ and

$$w_\varphi \equiv \frac{1}{\int_M \varphi^2}\int_M w\varphi^2. \tag{5}$$

Pluging this into (4), we get

$$(4) \ge \partial_t \int_M \varphi^2 w + \frac{1}{2}P_w^{-1} r^{-2}\int_M |w - w_\varphi|^2 \varphi^2.$$

That is,

$$\partial_t \int_M \varphi^2 w + \frac{1}{2}P_w^{-1} r^{-2}\int_M |w - w_\varphi|^2 \varphi^2 \le 2\int_E |\nabla\varphi|^2 . \tag{6}$$

By the definition of φ, we have

$$
\begin{aligned}
\int_E |\nabla\varphi|^2 &\le \int_E \frac{1}{[(1-\delta) r]^2} \\
&\le \frac{1}{[(1-\delta) r]^2}\int_E 1 \\
&= \frac{|E|}{[(1-\delta) r]^2}.
\end{aligned}
$$

Pluging this into (6), we obtain that

$$(6) \le \frac{2|E|}{[(1-\delta) r]^2}.$$

That is,

$$\partial_t \int_M \varphi^2 w + \frac{1}{2}P_w^{-1} r^{-2}\int_M |w - w_\varphi|^2 \varphi^2 \le \frac{2|E|}{[(1-\delta) r]^2}.$$

It implies that, since $\varphi \in C_0(E)$,

$$\begin{aligned}
\frac{2}{[(1-\delta)r]^2} &\geq \frac{1}{2}P_w^{-1}|E|^{-1}r^{-2}\int_M |w-w_\varphi|^2\varphi^2 + |E|^{-1}\partial_t\int_M \varphi^2 w \\
&= \frac{P_w^{-1}}{2|E|r^2}\int_M |w-w_\varphi|^2\varphi^2 + \frac{1}{|E|}\partial_t\left(w_\varphi\int_M\varphi^2\right) \text{ (by (5))} \\
&= \frac{P_w^{-1}}{2|E|r^2}\int_M |w-w_\varphi|^2\varphi^2 + \frac{\int_E\varphi^2}{|E|}\partial_t w_\varphi.
\end{aligned}$$

Note that, by the definition of φ, we have

$$\begin{aligned}
\frac{\int_E\varphi^2}{|E|} &\geq \frac{\int_{\delta E}\varphi^2}{|E|} \\
&= \frac{\int_{\delta E}1}{|E|} \\
&= \frac{|\delta E|}{|E|} \\
&= \frac{|B_{x_0}(\delta r)|}{|B_{x_0}(r)|} \\
&= \left(\frac{|B_{x_0}(r)|}{|B_{x_0}(\delta r)|}\right)^{-1} \\
&\geq \left(d_0^2\left(\frac{\delta r}{r}\right)^{\log_2 d_0}\right)^{-1} \text{ (by (1.2))} \\
&= d_0^{-2}\delta^{-\log_2 d_0} \\
&= d_0^{-2}\left(\frac{1}{\delta}\right)^{\log_2 d_0} \\
&> d_0^{-2}. \text{ (since } 0<\delta<1 \text{ and } d_0>1)
\end{aligned}$$

So the last inequality becomes

$$\frac{2}{[(1-\delta)r]^2} \geq \frac{P_w^{-1}}{2|E|r^2}\int_M |w-w_\varphi|^2\varphi^2 + d_0^{-2}\partial_t w_\varphi$$

which is equivalent to

$$\partial_t w_\varphi + \frac{d_0^2 P_w^{-1}}{2|E|r^2}\int_M |w-w_\varphi|^2\varphi^2 \leq \frac{2d_0^2}{[(1-\delta)r]^2}.$$

To simplify the last inequality, we let

$$\begin{cases} C_1 = \frac{d_0^2P_w^{-1}}{2|E|r^2} \\ C_2 = \frac{2d_0^2}{[(1-\delta)r]^2} \end{cases}$$

and get

$$\partial_t w_\varphi + C_1\int_M |w-w_\varphi|^2\varphi^2 \leq C_2. \tag{7}$$

Now, for convenience, we let $t_1 = t_0 - \rho r^2$, $t_2 = t_0 - r^2$ and

$$\begin{cases} w_1(x,t) = w(x,t) - C_2(t - t_1) \\ w_{\varphi,1}(t) = w_\varphi(t) - C_2(t - t_1) \end{cases}, \tag{8}$$

then

$$\begin{aligned} \partial_t w_{\varphi,1} + C_1 \int_M |w_1 - w_{\varphi,1}|^2 \varphi^2 &= w_\varphi - C_2 + C_1 \int_M |w - w_\varphi|^2 \varphi^2 \\ &\leq C_2 - C_2 \text{ (by (7))} \\ &= 0. \end{aligned} \tag{9}$$

In this case, we let

$$a = w_{\varphi,1}(t_1) \tag{10}$$

and, for any $\xi > 0$ and any $t \in [t_0 - r^2, t_0]$,

$$\begin{cases} E_t^+(\xi) = \{x \in \delta E : w_1(x,t) > a + \xi\} \\ E_t^-(\xi) = \{x \in \delta E : w_1(x,t) < a - \xi\} \end{cases}.$$

We observe both on $E_t^+(\xi)$ and $E_t^-(\xi)$.

First, we focus on $E_t^+(\xi)$. For any $x \in E_t^+(\xi)$ and $t \in [t_1, t_0]$, since

$$\begin{aligned} \partial_t w_{\varphi,1} &\leq -C_1 \int_M |w_1 - w_{\varphi,1}|^2 \varphi^2 \text{ (by (9))} \\ &\leq 0 \end{aligned}$$

which indicates that $w_{\varphi,1}$ is decreasing on its domain, $[t_2, t_0]$, we have

$$\begin{aligned} w_1(x,t) - w_{\varphi,1}(t) &> a + \xi - w_{\varphi,1}(t) \\ &= w_{\varphi,1}(t_1) + \xi - w_{\varphi,1}(t) \text{ (by (10))} \\ &= \xi + [w_{\varphi,1}(t_1) - w_{\varphi,1}(t)] \\ &\geq \xi \\ &> 0. \end{aligned} \tag{11}$$

Thus

$$|w_1 - w_{\varphi,1}|^2 \geq \left| a + \widetilde{\lambda} - w_{\varphi,1} \right|^2. \tag{12}$$

Note that, by the definition of φ, we have

$$\begin{aligned} 0 &\geq \partial_t w_{\varphi,1} + C_1 \int_M |w_1 - w_{\varphi,1}|^2 \varphi^2 \text{ (by (9))} \\ &\geq \partial_t w_{\varphi,1} + C_1 \int_{\delta E} |w_1 - w_{\varphi,1}|^2 \varphi^2 \\ &= \partial_t w_{\varphi,1} + C_1 \int_{\delta E} |w_1 - w_{\varphi,1}|^2 \\ &\geq \partial_t w_{\varphi,1} + C_1 \int_{E_t^+(\xi)} |w_1 - w_{\varphi,1}|^2 \text{ (since } E_t^+(\xi) \subseteq \delta E) \\ &\geq \partial_t w_{\varphi,1} + C_1 \int_{E_t^+(\xi)} |a + \xi - w_{\varphi,1}|^2 \text{ (by (12))} \\ &= \partial_t w_{\varphi,1} + C_1 |a + \xi - w_{\varphi,1}|^2 \int_{E_t^+(\xi)} 1 \\ &= \partial_t w_{\varphi,1} + C_1 |a + \xi - w_{\varphi,1}|^2 \left| E_t^+(\xi) \right|. \end{aligned} \tag{13}$$

Let

$$g(t) = w_{\varphi,1}(t) - (a+\xi),$$

then

$$\begin{aligned}\partial_t g + C_1 \left|E_t^+(\xi)\right| g^2 &= \partial_t w_{\varphi,1} + C_1 \left|E_t^+(\xi)\right| \left|w_{\varphi,1}(t) - (a+\xi)\right|^2 \\ &= \partial_t w_{\varphi,1} + C_1 \left|E_t^+(\xi)\right| \left|a+\xi - w_{\varphi,1}\right|^2 \\ &\leq 0. \text{ (by (13))}\end{aligned}$$

Note that, following the deduction in (11), we have $g = w_{\varphi,1} - (a+\xi) \lneqq 0$ and thus $g^2 \gneqq 0$. Dividing the last inequality by $-g^2$, we arrive at

$$-\frac{\partial_t g}{g^2} - C_1 \left|E_t^+(\xi)\right| \geq 0.$$

So

$$\begin{aligned}0 &\leq \int_{t_1}^{t_0} \left(-\frac{\partial_t g}{g^2} - C_1 \left|E_t^+(\xi)\right|\right) \\ &= \int_{t_1}^{t_0} \partial_t \left(\frac{1}{g}\right) - C_1 \int_{t_1}^{t_0} \left|E_t^+(\xi)\right| \\ &= \left[\frac{1}{g(t_0)} - \frac{1}{g(t_1)}\right] - C_1 \int_{t_1}^{t_0} \left|E_t^+(\xi)\right|\end{aligned}$$

which is equivalent to

$$C_1^{-1}\left[\frac{1}{g(t_0)} - \frac{1}{g(t_1)}\right] \geq \int_{t_1}^{t_0} \left|E_t^+(\xi)\right| \tag{14}$$

Recall that

$$\begin{aligned}g(t_1) &= w_{\varphi,1}(t_1) - (a+\xi) \\ &= a - a - \xi \text{ (by (10))} \\ &= -\xi\end{aligned}$$

and $g(t) = w_{\varphi,1}(t) - (a+\xi)$ is decresing on $[t_1, t_0]$ since $w_{\varphi,1}(t)$ is decreasing on $[t_2, t_0]$. So

$$\begin{aligned}g(t_0) &\leq g(t_1) \\ &= -\xi \\ &< 0\end{aligned}$$

and thus

$$\begin{aligned}\frac{1}{g(t_0)} - \frac{1}{g(t_1)} &< -\frac{1}{g(t_1)} \\ &= -\frac{1}{(-\xi)} \\ &= \xi^{-1}.\end{aligned} \tag{15}$$

Also, since $0 < \rho < 1$, we have

$$\begin{aligned}C_2(t - t_1) &\leq C_2(t_0 - t_1) \\ &= \frac{2d_0^2}{\left[(1-\delta)r\right]^2}\rho r^2 \\ &< \frac{2d_0^2}{(1-\delta)^2},\end{aligned}$$

and thus

$$\begin{aligned}
\int_{t_1}^{t_0}\left|E_t^+(\xi)\right| &= \int_{t_1}^{t_0}\left|\{x\in\delta E: w_1(x,t)>a+\xi\}\right| \\
&= \left|\{(x,t)\in\delta E\times[t_1,t_0]: w(x,t)-C_2(t-t_1)>a+\xi\}\right| \\
&= \left|\{(x,t)\in\delta E\times[t_1,t_0]: w(x,t)>a+\xi+C_2(t-t_1)\}\right| \\
&\geq \left|\left\{(x,t)\in\delta E\times[t_1,t_0]: w(x,t)>a+\xi+\frac{2d_0^2}{(1-\delta)^2}\right\}\right| \\
&= \left|\left\{(x,t)\in\delta E\times\left[t_0-\rho r^2,t_0\right]: -\ln u>a+\xi+\frac{2d_0^2}{(1-\delta)^2}\right\}\right| \\
&= \left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\xi+\frac{2d_0^2}{(1-\delta)^2}\right\}\right|. \qquad (16)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\xi+\frac{2d_0^2}{(1-\delta)^2}\right\}\right| &\leq \int_{t_1}^{t_0}\left|E_t^+(\xi)\right| \text{ (by (16))} \\
&\leq C_1^{-1}\left[\frac{1}{g(t_0)}-\frac{1}{g(t_1)}\right] \text{ (by (14))} \\
&< C_1^{-1}\xi^{-1} \text{ (by (15))} \\
&= \frac{2P_w}{d_0^2}\left|E\right|r^2\xi^{-1} \\
&\leq 2P_w\left|E\right|r^2\xi^{-1}. \text{ (since } d_0>1)
\end{aligned}$$

That is,

$$\left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\xi+K\right\}\right|\leq 2P_w\left|E\right|r^2\xi^{-1}, \qquad (17)$$

where $K=\frac{2d_0^2}{(1-\delta)^2}$. Note that the last inequality holds for all $\xi>0$. Now, for a given $\lambda>0$, if $\lambda\geq 2K$, then we have

$$\begin{aligned}
\left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\lambda\right\}\right| &= \left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\frac{\lambda}{2}+\frac{\lambda}{2}\right\}\right| \\
&\leq \left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\frac{\lambda}{2}+K\right\}\right| \\
&\leq 2P_w\left|E\right|r^2\left(\frac{\lambda}{2}\right)^{-1} \text{ (by (17))} \\
&= 4P_w\left|E\right|r^2\lambda^{-1}. \qquad (18)
\end{aligned}$$

Once $\lambda<2K$, since $2K\lambda^{-1}>1$, we get

$$\begin{aligned}
\left|\left\{(x,t)\in Q_+: \ln u^{-1}>a+\lambda\right\}\right| &\leq \left|Q_+\right| \\
&= \left|\delta E\times\left[t_0-\rho r^2,t_0\right]\right| \\
&= \left|\delta E\right|\rho r^2 \\
&< \left|E\right|r^2 \\
&< 2K\lambda^{-1}\left|E\right|r^2 \\
&= 2\left(\frac{2d_0^2}{(1-\delta)^2}\right)\left|E\right|r^2\lambda^{-1} \\
&= \left(\frac{2d_0}{1-\delta}\right)^2\left|E\right|r^2\lambda^{-1}. \qquad (19)
\end{aligned}$$

Therefore, by (18) and (19),

$$\left|\left\{(x,t)\in Q_{+}:\ln u^{-1}>a+\lambda\right\}\right|\leq C_{+}\left|E\right|r^{2}\lambda^{-1}, \tag{20}$$

holds for all $\lambda > 0$, where

$$C_{+}=\max\left\{4P_{w},\left(\frac{2d_{0}}{1-\delta}\right)^{2}\right\}.$$

Next, we focus on $E_t^-(\xi)$ for all $\xi > 0$. For any $x \in E_t^-(\xi)$ and $t \in [t_2, t_1]$, since $w_{\varphi,1}$ is decreasing on $[t_2, t_0]$, we have

$$\begin{aligned} w_{1}(x,t)-w_{\varphi,1}(t) &< a-\xi-w_{\varphi,1}(t)\\ &= w_{\varphi,1}(t_{1})-\xi-w_{\varphi,1}(t)\ \text{(by (10))}\\ &\leq -\widetilde{\lambda}\\ &< 0. \end{aligned} \tag{21}$$

Thus

$$\left|w_{1}-w_{\varphi,1}\right|^{2}\geq\left|a-\xi-w_{\varphi,1}(t)\right|^{2}. \tag{22}$$

Note that, by the definition of φ, we have

$$\begin{aligned} 0 &\geq \partial_{t}w_{\varphi,1}+C_{1}\int_{M}\left|w_{1}-w_{\varphi,1}\right|^{2}\varphi^{2}\ \text{(by (9))}\\ &\geq \partial_{t}w_{\varphi,1}+C_{1}\int_{\delta E}\left|w_{1}-w_{\varphi,1}\right|^{2}\varphi^{2}\\ &= \partial_{t}w_{\varphi,1}+C_{1}\int_{\delta E}\left|w_{1}-w_{\varphi,1}\right|^{2}\\ &\geq \partial_{t}w_{\varphi,1}+C_{1}\int_{E_{t}^{-}(\xi)}\left|w_{1}-w_{\varphi,1}\right|^{2}\ \text{(since } E_{t}^{-}(\xi)\subseteq\delta E)\\ &\geq \partial_{t}w_{\varphi,1}+C_{1}\int_{E_{t}^{-}(\xi)}\left|a-\xi-w_{\varphi,1}\right|^{2}\ \text{(by (22))}\\ &= \partial_{t}w_{\varphi,1}+C_{1}\left|a-\xi-w_{\varphi,1}\right|^{2}\int_{E_{t}^{-}(\xi)}1\\ &= \partial_{t}w_{\varphi,1}+C_{1}\left|a-\xi-w_{\varphi,1}\right|^{2}\left|E_{t}^{-}(\xi)\right|. \end{aligned} \tag{23}$$

Let

$$h(t)=w_{\varphi,1}(t)-(a-\xi),$$

then

$$\begin{aligned} \partial_{t}h+C_{1}\left|E_{t}^{-}(\xi)\right|h^{2} &= \partial_{t}w_{\varphi,1}+C_{1}\left|E_{t}^{-}(\xi)\right|\left|w_{\varphi,1}(t)-(a-\xi)\right|^{2}\\ &= \partial_{t}w_{\varphi,1}+C_{1}\left|E_{t}^{-}(\xi)\right|\left|a-\xi-w_{\varphi,1}\right|^{2}\\ &\leq 0.\ \text{(by (23))} \end{aligned}$$

Note that, following the deduction in (21), we have $h = w_{\varphi,1} - (a - \xi) \gneqq 0$ and thus $h^2 \lneqq 0$. Dividing the last inequality by $-h^2$, we arrive at

$$-\frac{\partial_{t}h}{h^{2}}-C_{1}\left|E_{t}^{-}(\xi)\right|\geq 0.$$

So

$$\begin{aligned} 0 &\leq \int_{t_2}^{t_1} \left(-\frac{\partial_t h}{h^2} - C_1 \left| E_t^- (\xi) \right| \right) \\ &= \int_{t_2}^{t_1} \partial_t \left(\frac{1}{h} \right) - C_1 \int_{t_2}^{t_1} \left| E_t^- (\xi) \right| \\ &= \left[\frac{1}{h(t_1)} - \frac{1}{h(t_2)} \right] - C_1 \int_{t_2}^{t_1} \left| E_t^- (\xi) \right| \end{aligned}$$

which is equivalent to

$$C_1^{-1} \left[\frac{1}{h(t_1)} - \frac{1}{h(t_2)} \right] \geq \int_{t_2}^{t_1} \left| E_t^- (\xi) \right| \tag{24}$$

Recall that

$$\begin{aligned} h(t_1) &= w_{\varphi,1}(t_1) - (a - \xi) \\ &= a - a + \xi \text{ (by (10))} \\ &= \xi \end{aligned}$$

and $h(t) = w_{\varphi,1}(t) - (a - \xi)$ is decresing since $w_{\varphi,1}(t)$ is decreasing everywhere. So

$$\begin{aligned} h(t_2) &\geq h(t_1) \\ &= \xi \\ &> 0 \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{h(t_1)} - \frac{1}{h(t_2)} &< \frac{1}{h(t_1)} \\ &= \frac{1}{\xi} \\ &= \xi^{-1}. \end{aligned} \tag{25}$$

Also, since

$$\begin{aligned} C_2(t - t_1) &\geq C_2(t_2 - t_1) \\ &= \frac{2d_0^2}{[(1-\delta) r]^2} (\rho - 1) r^2 \\ &= \frac{2(\rho-1) d_0^2}{(1-\delta)^2}, \end{aligned}$$

we obtain that

$$\begin{aligned} \int_{t_2}^{t_1} \left| E_t^- (\xi) \right| &= \int_{t_0 - r^2}^{t_1} \left| \{ x \in \delta E : w_1(x,t) < a - \xi \} \right| \\ &= \left| \{ (x,t) \in \delta E \times [t_2, t_1] : w(x,t) - C_2(t - t_1) < a - \xi \} \right| \\ &= \left| \{ (x,t) \in \delta E \times [t_2, t_1] : w(x,t) < a - \xi + C_2(t - t_1) \} \right| \\ &\geq \left| \left\{ (x,t) \in \delta E \times [t_2, t_1] : w(x,t) < a - \xi + \frac{2(\rho-1) d_0^2}{(1-\delta)^2} \right\} \right| \\ &= \left| \left\{ (x,t) \in \delta E \times \left[t_0 - r^2, t_0 - \rho r^2 \right] : -\ln u < a - \xi + \frac{2(\rho-1) d_0^2}{(1-\delta)^2} \right\} \right| \\ &= \left| \left\{ (x,t) \in Q_- : \ln u > -a + \xi + \frac{2(1-\rho) d_0^2}{(1-\delta)^2} \right\} \right|. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}
\left|\left\{(x,t)\in Q_- : \ln u > -a+\xi+\frac{2(1-\rho)d_0^2}{(1-\delta)^2}\right\}\right| &\le \int_{t_2}^{t_1}\left|E_t^-(\xi)\right| \text{ (by (26))}\\
&\le C_1^{-1}\left[\frac{1}{h(t_1)}-\frac{1}{h(t_2)}\right] \text{ (by (24)}\\
&< C_1^{-1}\xi^{-1} \text{ (by (25))}\\
&= \frac{2P_w}{d_0^2}|E|\,r^2\xi^{-1}\\
&\le 2P_w|E|\,r^2\xi^{-1}. \text{ (since } d_0>1)
\end{aligned}$$

That is,

$$|\{(x,t)\in Q_- : \ln u > -a+\xi+M\}| \le 2P_w|E|\,r^2\xi^{-1}, \tag{27}$$

where $M=\frac{2(1-\rho)d_0^2}{(1-\delta)^2}$. Note that the last inequality holds for all $\xi>0$. Now, for a given $\lambda>0$, if $\lambda\ge 2M$, then we have

$$\begin{aligned}
|\{(x,t)\in Q_- : \ln u > -a+\lambda\}| &= \left|\left\{(x,t)\in Q_- : \ln u > a+\frac{\lambda}{2}+\frac{\lambda}{2}\right\}\right|\\
&\le \left|\left\{(x,t)\in Q_- : \ln u > a+\frac{\lambda}{2}+M\right\}\right|\\
&\le 2P_w|E|\,r^2\left(\frac{\lambda}{2}\right)^{-1} \text{ (by (27))}\\
&= 4P_w|E|\,r^2\lambda^{-1}.
\end{aligned} \tag{28}$$

If $\lambda<2M$, since $2M\lambda^{-1}>1$, we get

$$\begin{aligned}
|\{(x,t)\in Q_- : \ln u > -a+\lambda\}| &\le |Q_-|\\
&= \left|\delta E\times\left[t_0-r^2, t_0-\rho r^2\right]\right|\\
&= |\delta E|\,(1-\rho)\,r^2\\
&< (1-\rho)\,|E|\,r^2\\
&< 2M\lambda^{-1}(1-\rho)\,|E|\,r^2\\
&= 2\left[\frac{2(1-\rho)d_0^2}{(1-\delta)^2}\right](1-\rho)\,|E|\,r^2\lambda^{-1}\\
&= \left[\frac{2(1-\rho)d_0}{1-\delta}\right]^2|E|\,r^2\lambda^{-1}.
\end{aligned} \tag{29}$$

Therefore, by (28) and (29),

$$|\{(x,t)\in Q_- : \ln u > -a+\lambda\}| \le C_-|E|\,r^2\lambda^{-1}, \tag{30}$$

holds for all $\lambda>0$, where

$$C_- = \max\left\{4P_w, \left[\frac{2(1-\rho)d_0}{1-\delta}\right]^2\right\}.$$

Finally, if we choose

$$C=\max\{C_+, C_-\},$$

then, by (20) and (30), we have

$$\begin{cases} |\{(x,t)\in Q_+ : \ln u^{-1} > a+\lambda\}| \le C\,|E|\,r^2\lambda^{-1} \\ |\{(x,t)\in Q_- : \ln u > -a+\lambda\}| \le C\,|E|\,r^2\lambda^{-1} \end{cases}.$$

Since $0<\rho<1$,

$$\begin{aligned} C &= \max\left\{4P_w, \left(\frac{2d_0}{1-\delta}\right)^2, \left[\frac{2\,(1-\rho)\,d_0}{1-\delta}\right]^2\right\} \\ &= \max\left\{4P_w, \left(\frac{2d_0}{1-\delta}\right)^2\right\}. \end{aligned}$$

Note that P_w is a constant which depends on d_0 and P_2, we conclude that C is a constant which depends on δ, d_0 and P_2. This finishes the proof. □

5 Harnack inequality for the heat equation

Lemma 5.1. *Let $\{R_\delta\}_{\delta\in(0,1]}$ be collection of measurable subsets of $M\times\mathbb{R}$ such that $R_{\delta'}\subseteq R_\delta$ if $\delta'\le\delta$. Fix $m>0$, $K>0$, $\delta_0\in\left[\frac{1}{2},1\right)$ and $0<p_1<p_0\le\infty$. Let u be positive measurable on R_1. If*

$$\left(\int_{R_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} \le K\left(\frac{1}{(\delta-\delta')^m\,|R_1|}\right)^{\frac{1}{p}-\frac{1}{p_0}}\left(\int_{R_\delta} u^p\right)^{\frac{1}{p}},$$

holds for all δ, δ' and p such that $\frac{1}{2}\le\sigma\le\delta'<\delta\le 1$ and $0<p\le p_1<p_0$, and if, moreover, for any $\delta\in(0,1)$,

$$|\{(x,t)\in R_\delta : \ln u>\lambda\}| \le K\,|R_1|\,\lambda^{-1}$$

holds for all $\lambda>0$. Then there exists a constant $C_0=C_0\,(m,K,\delta_0,p_0,p_1)>0$ such that

$$\left(\int_{R_{\delta_0}} u^{p_0}\right)^{\frac{1}{p_0}} \le C_0\,|R_1|^{\frac{1}{p_0}}.$$

Proof. Define $\psi:[\delta_0,1]\to\mathbb{R}$ by

$$\psi(\delta)=\ln\left(\frac{1}{|R_1|}\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}},$$

for any $\delta\in[\delta_0,1]$. Now, for a given $\delta\in[\delta_0,1]$, we let

$$\begin{cases} R_{\delta,1}=\left\{(x,t)\in R_1 : \ln u>\frac{\psi(\delta)}{2}\right\} \\ R_{\delta,2}=\left\{(x,t)\in R_1 : \ln u\le\frac{\psi(\delta)}{2}\right\} \end{cases},$$

then $R_\delta = R_{\delta,1}\,\dot\cup\,R_{\delta,2}$ and, for any $(x,t)\in R_{\delta,2}$,

$$u\,(x,t)\le e^{\frac{\psi(\delta)}{2}}. \tag{1}$$

So, in this case, for any $0 < p < p_0$, applying Hölder's inequality and by the definition of ψ, we get

$$\begin{aligned}
\int_{R_\delta} u^p &= \int_{R_{\delta,1}} u^p + \int_{R_{\delta,2}} u^p \\
&\le \left(\int_{R_{\delta,1}} (u^p)^{\frac{p_0}{p}}\right)^{\frac{p}{p_0}} \left(\int_{R_{\delta,1}} 1\right)^{1-\frac{p}{p_0}} + \int_{R_{\delta,2}} \left(e^{\frac{\psi(\delta)}{2}}\right)^p \text{ (by (1))} \\
&= \left(\int_{R_{\delta,1}} u^{p_0}\right)^{\frac{p}{p_0}} |R_{\delta,1}|^{1-\frac{p}{p_0}} + \left(e^{\frac{\psi(\delta)}{2}}\right)^p \int_{R_{\delta,2}} 1 \\
&\le \left(\int_{R_\delta} u^{p_0}\right)^{\frac{p}{p_0}} |R_{\delta,1}|^{1-\frac{p}{p_0}} + \left(e^{\frac{\psi(\delta)}{2}}\right)^p \int_{R_1} 1 \\
&= \left[\left(\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} |R_{\delta,1}|^{\frac{1}{p}-\frac{1}{p_0}}\right]^p + \left(e^{\frac{\psi(\delta)}{2}}\right)^p |R_1| . \qquad (2)
\end{aligned}$$

Note that, by second assumption, we have

$$\begin{aligned}
|R_{\delta,1}| &= \left|\left\{(x,t) \in R_1 : \ln u > \frac{\psi(\delta)}{2}\right\}\right| \\
&\le K |R_1| \left(\frac{\psi(\delta)}{2}\right)^{-1} \\
&= \frac{2K |R_1|}{\psi(\delta)}
\end{aligned}$$

and thus

$$\begin{aligned}
(2) &\le \left[\left(\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} \left(\frac{2K |R_1|}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}}\right]^p + \left(e^{\frac{\psi(\delta)}{2}}\right)^p |R_1| \\
&= \left[\left(\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} \left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}} |R_1|^{\frac{1}{p}-\frac{1}{p_0}}\right]^p + \left(e^{\frac{\psi(\delta)}{2}}\right)^p |R_1| \\
&= \left[\left(\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} \left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}} |R_1|^{-\frac{1}{p_0}}\right]^p |R_1| + \left(e^{\frac{\psi(\delta)}{2}}\right)^p |R_1| .
\end{aligned}$$

That is,

$$\int_{R_\delta} u^p \le \left[\left(\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} \left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}} |R_1|^{-\frac{1}{p_0}}\right]^p |R_1| + \left(e^{\frac{\psi(\delta)}{2}}\right)^p |R_1|$$

which is equivalent to

$$\frac{1}{|R_1|}\int_{R_\delta} u^p \le \left[\left(\frac{1}{|R_1|}\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} \left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}}\right]^p + \left(e^{\frac{\psi(\delta)}{2}}\right)^p .$$

Since, by the definition of ψ, $\left(\frac{1}{|R_1|}\int_{R_\delta} u^{p_0}\right)^{\frac{1}{p_0}} = e^{\psi(\delta)}$, the last inequality becomes

$$\frac{1}{|R_1|}\int_{R_\delta} u^p \le \left[e^{\psi(\delta)} \left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}}\right]^p + \left(e^{\frac{\psi(\delta)}{2}}\right)^p . \qquad (3)$$

Solving

$$e^{\psi(\delta)}\left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}}=e^{\frac{\psi(\delta)}{2}}, \tag{4}$$

we get

$$\left(\frac{2K}{\psi(\delta)}\right)^{\frac{1}{p}-\frac{1}{p_0}}=e^{-\frac{\psi(\delta)}{2}}$$

and thus

$$\frac{1}{p}-\frac{1}{p_0}=\frac{\psi(\delta)}{2\ln\left(\frac{\psi(\delta)}{2K}\right)}. \tag{5}$$

Note that, if $\frac{\psi(\delta)}{2K}\leq 1$, then $2K\geq\psi(\delta)=\ln\left(\frac{1}{|R_1|}\int_{R_\delta}u^{p_0}\right)^{\frac{1}{p_0}}$ which implies that $\left(\int_{R_\delta}u^{p_0}\right)^{\frac{1}{p_0}}\leq |R_1|^{\frac{1}{p_0}}e^{2K}$. This satisfies the conclusion of this theorem. Thus, it is enough to analyze the case of $\frac{\psi(\delta)}{2K}>1$. In this case, we have $\ln\left(\frac{\psi(\delta)}{2K}\right)>0$. Since $\psi(\delta)$ is always positive, the solution p to (5) can be choosen so that $p<p_0$. Also, since $\ln\left(\frac{\psi(\delta)}{2K}\right)$ tends to 0^+ as K tends to $\left(\frac{\psi(\delta)}{2}\right)^-$, the solution p to (5) can be choosen so that $p\leq p_1<p_0$ by enlarging K to be large enough. Therefore, (3) also holds for such p. Pluging (4) into (3), we obtain that

$$\begin{aligned}\frac{1}{|R_1|}\int_{R_\delta}u^p &\leq \left(e^{\frac{\psi(\delta)}{2}}\right)^p+\left(e^{\frac{\psi(\delta)}{2}}\right)^p\\ &= 2\left(e^{\frac{\psi(\delta)}{2}}\right)^p\end{aligned}$$

which is equivalent to

$$\int_{R_\delta}u^p\leq 2|R_1|\left(e^{\frac{\psi(\delta)}{2}}\right)^p. \tag{6}$$

Now, for any $\delta_0\leq\delta'<\delta$, if $\psi(\delta')>2K(\delta-\delta')^{-2m}$, then, since

$$\begin{aligned}\psi(\delta') &= \ln\left(\frac{1}{|R_1|}\int_{R_{\delta'}}u^{p_0}\right)^{\frac{1}{p_0}}\\ &\leq \ln\left(\frac{1}{|R_1|}\int_{R_\delta}u^{p_0}\right)^{\frac{1}{p_0}}\\ &= \psi(\delta),\end{aligned}$$

we have

$$\psi(\delta)>2K(\delta-\delta')^{-2m} \tag{7}$$

and thus, by the first assumption,

$$\begin{aligned}
\psi\left(\delta'\right) &= \ln\left(\frac{1}{|R_1|}\int_{R_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} \\
&= \ln|R_1|^{-\frac{1}{p_0}}\left(\int_{R_{\delta'}} u^{p_0}\right)^{\frac{1}{p_0}} \\
&\le \ln|R_1|^{-\frac{1}{p_0}} K\left(\frac{1}{\left(\delta-\delta'\right)^m|R_1|}\right)^{\frac{1}{p}-\frac{1}{p_0}}\left(\int_{R_{\delta'}} u^{p_0}\right)^{\frac{1}{p}} \\
&\le \ln|R_1|^{-\frac{1}{p_0}} K\left(\frac{1}{\left(\delta-\delta'\right)^m|R_1|}\right)^{\frac{1}{p}-\frac{1}{p_0}}\left(2|R_1|\left(e^{\frac{\psi(\delta)}{2}}\right)^p\right)^{\frac{1}{p}} \quad \text{(by (6))} \\
&= -\frac{1}{p_0}\ln|R_1| + \ln K + \left(\frac{1}{p}-\frac{1}{p_0}\right)\ln\left(\delta-\delta'\right)^{-m}|R_1|^{-1} + \frac{1}{p}\ln 2|R_1|\left(e^{\frac{\psi(\delta)}{2}}\right)^p \\
&= \ln K + \left(\frac{1}{p}-\frac{1}{p_0}\right)\ln\left(\delta-\delta'\right)^{-m} + \frac{1}{p}\ln 2 + \frac{\psi(\delta)}{2} \\
&= \ln K + \frac{\psi(\delta)}{2\ln\left(\frac{\psi(\delta)}{2K}\right)}\ln\left(\delta-\delta'\right)^{-m} + \frac{1}{p}\ln 2 + \frac{\psi(\delta)}{2} \quad \text{(by (5))} \\
&\le \ln K + \frac{\psi(\delta)}{2\ln\left(\frac{\psi(\delta)}{2K}\right)}\ln\left(\frac{\psi(\delta)}{2K}\right)^{\frac{1}{2}} + \frac{1}{p}\ln 2 + \frac{\psi(\delta)}{2} \quad \text{(by (7))} \\
&= \ln K + \frac{\psi(\delta)}{4} + \frac{1}{p}\ln 2 + \frac{\psi(\delta)}{2} \\
&= \frac{3}{4}\psi(\delta) + \ln K + \frac{1}{p}\ln 2.
\end{aligned}$$

That is,

$$\psi\left(\delta'\right) \le \frac{3}{4}\psi(\delta) + \ln 2^{\frac{1}{p}} K.$$

Note that the last inequality holds when $\psi\left(\delta'\right) > 2K\left(\delta-\delta'\right)^{-2m}$. Since $\psi\left(\delta'\right)$ may by less or equal to $2K\left(\delta-\delta'\right)^{-2m}$, together with the last inequality, we arrive at

$$\psi\left(\delta'\right) \le \frac{3}{4}\psi(\delta) + \ln K + \frac{1}{p}\ln 2 + 2K\left(\delta-\delta'\right)^{-2m}. \tag{8}$$

Recall that $\frac{1}{p}-\frac{1}{p_0} = \frac{\psi(\delta)}{2\ln\left(\frac{\psi(\delta)}{2K}\right)}$ and $\delta_0 \le \delta < 1$, we have

$$\begin{aligned}
\frac{1}{p} &= \frac{\psi(\delta)}{2\ln\left(\frac{\psi(\delta)}{2K}\right)} + \frac{1}{p_0} \\
&< \frac{\psi(1)}{2\ln\left(\frac{\psi(\delta_0)}{2K}\right)} + \frac{1}{p_0}.
\end{aligned}$$

So, setting

$$L = \frac{\psi(1)}{2\ln\left(\frac{\psi(\delta_0)}{2K}\right)} + \frac{1}{p_0},$$

(8) becomes

$$\begin{aligned}\psi(\delta') &\le \frac{3}{4}\psi(\delta) + \ln K + L\ln 2 + 2K(\delta-\delta')^{-2m} \\ &= \frac{3}{4}\psi(\delta) + \ln 2^L K + 2K(\delta-\delta')^{-2m}. \qquad (9)\end{aligned}$$

Here we should note that L is independent with δ and δ'.

Finally, iterating on (9) by choosing some $\beta > 1$ so that $\frac{3}{4}\beta^{2m} < 1$ and letting $\delta_{i+1} = \delta_i + \beta^{-i-1}(1-\delta_0)$ for all integer $i \ge 0$, we get

$$\begin{aligned}\psi(\delta_0) &\le \frac{3}{4}\psi(\delta_1) + \ln 2^L K + 2K\left[\beta^{-1}(1-\delta_0)\right]^{-2m} \\ &= \frac{3}{4}\psi(\delta_1) + \ln 2^L K + 2K\beta^{2m}(1-\delta_0)^{-2m} \\ &\le \frac{3}{4}\left\{\frac{3}{4}\psi(\delta_2) + \ln 2^L K + 2K\left[\beta^{-2}(1-\delta_0)\right]^{-2m}\right\} + \ln 2^L K + 2K\beta^{2m}(1-\delta_0)^{-2m} \\ &= \frac{3}{4}\left\{\frac{3}{4}\psi(\delta_2) + \ln 2^L K + 2K\beta^{4m}(1-\delta_0)^{-2m}\right\} + \ln 2^L K + 2K\beta^{2m}(1-\delta_0)^{-2m} \\ &= \left(\frac{3}{4}\right)^2\psi(\delta_2) + \left(1+\frac{3}{4}\right)\ln 2^L K + 2K(1-\delta_0)^{-2m}\left(\beta^{2m} + \frac{3}{4}\beta^{4m}\right) \\ &\;\vdots \\ &= \left(\frac{3}{4}\right)^k\psi(\delta_k) + \ln 2^L K\sum_{i=0}^{k-1}\left(\frac{3}{4}\right)^i + 2K(1-\delta_0)^{-2m}\sum_{i=0}^{k-1}\beta^{2m}\left(\frac{3}{4}\beta^{2m}\right)^i.\end{aligned}$$

Letting $k \to \infty$, since $\frac{3}{4}\beta^{2m} < 1$, we obtain that

$$\psi(\delta_0) \le 4\ln 2^L K + 8K(1-\delta_0)^{-2m}.$$

Recall that

$$\psi(\delta_0) = \ln\left(\frac{1}{|R_1|}\int_{R_{\delta_0}} u^{p_0}\right)^{\frac{1}{p_0}},$$

the last inequality becomes

$$\begin{aligned}\ln\left(\frac{1}{|R_1|}\int_{R_{\delta_0}} u^{p_0}\right)^{\frac{1}{p_0}} &\le 4\ln 2^L K + 8K(1-\delta_0)^{-2m} \\ &= \ln 2^{4L}K^4 e^{8K(1-\delta_0)^{-2m}}\end{aligned}$$

which is equivalent to

$$\left(\int_{R_{\delta_0}} u^{p_0}\right)^{\frac{1}{p_0}} \le C_0\,|R_1|^{\frac{1}{p_0}},$$

where $C_0 = 2^{4L}K^4e^{8K(1-\delta_0)^{-2m}}$. Since L depends only on δ_0, p_0 and K, we conclude that C_0 is a constant depends nothing but m, K, δ_0, p_0 and p_1. Therefore we get the proof. □

Theorem 5.1. *Let E be a ball of center $x_0 \in M$ and raius $r > 0$ in M. Let $t_0 > 2r^2$ and u be a positive solution to the Dirichlet heat equation on $E \times [t_0 - 2r^2, t_0]$. If M has both VDC and WPI, then there exists a constant $C_H = C_H(d_0, P_2) > 0$ such that*

$$\sup_{Q_-} u \leq C_H \inf_{Q_+} u,$$

where

$$\begin{cases} Q_+ = \frac{1}{2}E \times \left[t_0 - \frac{1}{2}r^2, t_0\right] \\ Q_- = \frac{1}{2}E \times \left[t_0 - 2r^2, t_0 - \frac{3}{2}r^2\right] \end{cases},$$

d_0 and P_2 are controlling constants in VDC and WPI, respectively. The inequality in this theorem is called the Harnack inequality.

Proof. Let a be the constant in Theorem 4.3 (see page 74). Since u is a positive solution (to the Dirichlet heat equation on $E \times [t_0 - 2r^2, t_0]$), both $e^{-a}u$ and $e^{-a}u^{-1}$ are positive solution. Let $t_1 = t_0 - r^2$, then Q_- can be rewritten as $Q_- = \frac{1}{2}E \times \left[t_1 - r^2, t_1 - \frac{1}{2}r^2\right]$. Also, for any $0 < \varepsilon < 1$, we set

$$\begin{cases} Q_{+,\varepsilon} = \left(\frac{1}{2} + \varepsilon\right) E \times \left[t_0 - \left(\frac{1}{2} + \varepsilon\right) r^2, t_0\right] \\ Q_{-,\varepsilon} = \left(\frac{1}{2} + \varepsilon\right) E \times \left[t_1 - r^2, t_0 - \left(\frac{1}{2} + \varepsilon\right) r^2\right] \end{cases}.$$

Now, since $e^{-a}u$ is a positive solution on $E \times [t_0 - 2r^2, t_0]$, $e^{-a}u$ is a supersolution on $E \times [t_1 - r^2, t_1]$. By Theorem 4.2 and Theorem 4.3, we know that $e^{-a}u$ satiesfies the conditions in Lemma 5.1 and thus

$$\left[\int_{Q_{-,\varepsilon}} \left(e^{-a}u\right)^{p_0}\right]^{\frac{1}{p_0}} \leq C_{0,-} \left(|E| r^2\right)^{\frac{1}{p_0}}$$

or, equivalently,

$$\left(\int_{Q_{-,\varepsilon}} u^{p_0}\right)^{\frac{1}{p_0}} \leq e^a C_{0,-} \left(|E| r^2\right)^{\frac{1}{p_0}}. \tag{1}$$

Thus

$$\begin{aligned} \sup_{Q_-} u &\leq C_{RM} \left(\frac{1}{|E| r^2}\right)^{\frac{1}{p_0}} \left(\int_{Q_{-,\varepsilon}} u^{p_0}\right)^{\frac{1}{p_0}} \quad \text{(by (4.2))} \\ &\leq C_{RM} \left(\frac{1}{|E| r^2}\right)^{\frac{1}{p_0}} e^a C_{0,-} \left(|E| r^2\right)^{\frac{1}{p_0}} \quad \text{(by (1))} \\ &= C_{RM} C_{0,-} e^a. \end{aligned} \tag{2}$$

On the other hand, since $e^{-a}u^{-1}$ is a positive solution on $E \times [t_0 - 2r^2, t_0]$, $e^{-a}u^{-1}$ is a subsolution on $E \times [t_0 - r^2, t_0]$. By Theorem 4.1 and Theorem 4.3, we know that $e^{-a}u^{-1}$ satiesfies the conditions in Lemma 5.1 and thus

$$\left[\int_{Q_{+,\varepsilon}} \left(e^{-a}u^{-1}\right)^{p_0}\right]^{\frac{1}{p_0}} \leq C_{0,+} \left(|E| r^2\right)^{\frac{1}{p_0}}$$

or, equivalently,

$$\left[\int_{Q_{+,\varepsilon}} \left(u^{-1}\right)^{p_0}\right]^{\frac{1}{p_0}} \leq e^a C_{0,+} \left(|E| r^2\right)^{\frac{1}{p_0}}. \tag{3}$$

Note that u^{-1} is a positive solution since u is a positive solution, so u^{-1} is a subsolution and thus, by Theorem 4.1, we have

$$\sup_{Q_+} u^{-1} \leq C_M \left(\frac{1}{|E|\, r^2} \right)^{\frac{1}{p_0}} \left[\int_{Q_{+,\varepsilon}} \left(u^{-1}\right)^{p_0} \right]^{\frac{1}{p_0}}$$

which is equivalent to

$$\left[\int_{Q_{+,\varepsilon}} \left(u^{-1}\right)^{p_0} \right]^{\frac{1}{p_0}} \geq C_M^{-1} \left(|E|\, r^2\right)^{\frac{1}{p_0}} \sup_{Q_+} u^{-1}. \tag{4}$$

So, by (3) and (4), we get

$$C_M^{-1} \left(|E|\, r^2\right)^{\frac{1}{p_0}} \sup_{Q_+} u^{-1} \leq e^a C_{0,+} \left(|E|\, r^2\right)^{\frac{1}{p_0}}$$

or, equivalently,

$$\sup_{Q_+} u^{-1} \leq C_M C_{0,+} e^a.$$

Note that $\sup_{Q_+} u^{-1} = \left(\inf_{Q_+} u\right)^{-1}$, thus the last inequality becomes

$$\inf_{Q_+} u \geq C_M^{-1} C_{0,+}^{-1} e^{-a}$$

which is equivalent to

$$e^a \leq C_M C_{0,+} e^{2a} \inf_{Q_+} u. \tag{5}$$

Finally, plugging (5) into (2), we obtain that

$$\begin{aligned} \sup_{Q_-} u &\leq C_{RM} C_{0,-} C_M C_{0,+} e^{2a} \inf_{Q_+} u \\ &= C_H \inf_{Q_+} u, \end{aligned}$$

where $C_H = C_{RM} C_M C_{0,-} C_{0,+} e^{2a}$. Since C_H is contant depends only on d_0 and P_2, we get the proof. □

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