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Abstract Semilinear Differential Equations and C-regularized semigroups

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The main concern of this paper is under some suitable conditions on the forcing term and the operator A to find the unique classical solution, strong solution or mild solution for the abstract semilinear initial value problem:

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t,u(t)) & 0\leqslant t_0<t<T \\ u(t_0)=u_0 & u_0\in C(D(A)) \end{cases} \tag{0.1}$$

where A is an infinitesimal generator of a C-semigroup $\{T(t):t\geqslant 0\}$, $f:[t_0,T]\times X\to X$ and X is a Banach space. We also discussed the maximum interval of the existence for the mild solutions and continuous dependence of initial data. The basic technique used in this paper is the fixed point theory for differential equations in Banach space. For this purpose, we prove first that the corresponding inhomogeneous equation

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t) & 0\leqslant t_0<t<T \\ u(t_0)=u_0 & u_0\in C(D(A)) \end{cases} \tag{0.2}$$

has a unique classical solution, strong solution or mild solution. However, the most enjoy here is that we do not need to assume that the C-semigroup is exponentially bounded.

Keywords: C-regularized semigroups exponentially bounded C-regularized semigroups abstract inhomogeneous differential equations abstract semilinear differential equations

Introduction and Preliminaries

The following illustrations explain the historical and mathematical motivation for the studying in this paper. Suppose $\Omega\subseteq R^N$ be an open bounded domain, the partial differential equation

$$\begin{cases} \frac{\partial u(t)}{\partial t}=\triangle u(t,x)+f(t,u(t)) & t>0,\ x\in\Omega \\ u(0,x)=f(x) & x\in\Omega \end{cases}$$

with Dirichlet boundary condition or period condition can be transformed to consider an abstract semilinear initial problem:

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t,u(t)) & 0<t<T \\ u(0)=x & x\in D(A) \end{cases} \tag{1.1}$$

with A is defined as the Drichlet Laplacian operator $\triangle$ on R^N. If we consider the Banach space $X=L^2(\Omega)$, then it can be shown that the semigroup $\{S(t)=e^{\triangle t}:t \geq 0\}$, $S(t):L^2(\Omega) \to L^2(\Omega)$, generated by A is a strongly continuous semigroup (denote by C_0-semigroup) on X. In 1970's many mathematicians were interesting in these type topics. Some of them studied the nonlinear equations and others investigated the semilinear equations. Namely, P. B'enilan [1] studied the nonlinear homogeneous initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) \ni 0 & 0 \leq t \leq T \\ u(0)=u_0 \end{cases} \tag{1.1)'$$

where A is a multivalued, nonlinear (in general), m-accretive operator on the L^p space ($1 \leq p \leq \infty$). L. C. Evans [6] considered the evolution equation in an arbitrary Banach space X of the form

$$\begin{cases} \frac{du(t)}{dt} + A(t)u(t) \ni f(t) & 0 \leq t \leq T \\ u(0)=x_0 \end{cases} \tag{1.1)"$$

where almost every $A(t)$ is an m-accretive operator and, loosely speaking, have an L^1 modulus of continuity in t and $f(t)$ is an integrable function. He used a discrete scheme to approximate the solution of (1.1)" under some restrictions on $A_\lambda(t) \equiv \frac{I-J_\lambda(t)}{\lambda}$ (where $J_\lambda(t) \equiv (I+A(t))^{-1}$) and the range condition on $\overline{D(A(t))}$. M. G. Crandall [3] gave some marvelous examples of quasi linear partial differential equations and nonlinear accretive operators which relative to the abstract problem (1.1)"'.

$$\begin{cases} \frac{du(t)}{dt} + Au(t) \ni f(t) & 0 \leq t \leq T \\ u(0)=x_0 \end{cases} \tag{1.1)"'$$

He also investigated the corresponding homogeneous equation under the tangency conditions. A. Pazy [10] formed some sufficient conditions for the semilinear initial value problem (1.1) has an unique solution under the assumption that the function f is Lipschitz continuous in the space variable. However, we can not expect that all semigroups are strongly continuous (that is, all semigroups are exponentially bounded). For instance, R. Delaubenfel [5] consider A as a linear operator on $C_0(R)$, which is defined by

$$Af(s)=sf(s) \quad \text{for all } f \in C_0(R)$$

then the semigroup generated by A is not exponentially bounded. For an example of partial differential equation, if we consider $-\triangle$ be the Dirichlet Laplacian operator on R^N, then the semigroup $\{e^{i\triangle t}:t \geq 0\}$ generated by $-i\triangle$ is neither exponentially bounded semigroup on $L^p(R^N)$, provide $p \neq 2$ (see L. Hormander [7]). The basic difference of C_0-semigroup and C-semigroup can see the Definition.1.1 in this paper and Theorem 1 in M. G. Grandall's paper [3]. Even the C-regularized semigroups were found in 1950's, but it did not cause many mathematicians pay attention to these type semigroups until the Schrodinger semigroups and integral semigroups were introduced. From 1980's, many mathematicians start to consider the C-regularized semigroups. Recently, B. Simon [15], M. M. Pang [9], Naoki Tanaka and Isao Miyadera [16], Ralph Delaubenfels [5], ⋯, they had some results concerning to the Cauchy problem:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) & 0<t<T \\ u(0)=x & x \in C(D(A)) \end{cases} \tag{1.2}$$

for A to be an infinitesimal generator of a C-regularized semigroup.

In this paper, we generalize the Pazy's results of C_0-semigroup to the C-regularized semigroup circumstances. We consider the semilinear initial value problem:

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t,u(t)) & 0 \leqslant t_0<t<T \\ u(t_0)=u_0 & u_0 \in C(D(A)) \end{cases} \tag{1.3}$$

where T is any given positive real number, A is the infinitesimal generator of C-regularized semigroup and $f:[t_0,T) \times X \to X$ is a function satisfying certain properties. To see this, we proved first that the abstract inhomogeneous initial value problem

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t) & 0<t<T \\ u(0)=x & x \in C(D(A)) \end{cases} \tag{1.4}$$

for A is the infinitesimal generator of a C-regularized semigroups, has a unique solution under some restrictions on the function f.

In the rest of this section, we will introduce some properties of the C-semigroup which was proved in [5], [16]. However, for the convenience, we still list here without proofs. In section 2, we find some sufficient conditions for the existence and uniqueness of the solutions, strong solutions, and mild solutions to the initial value problem (1.4). We find some sufficient conditions for the existence and uniqueness of the solutions, strong solutions, and mild solutions to the initial value problem (1.3) in section 3. We also discussed the maximum interval of the existence for the mild solutions. Through out this paper we denote X to be a Banach space endowed with the norm $|\cdot|$, we always assume that A to be a closed linear operator on X with nonempty resolvent set and which generates a C-regularized semigroup $\{T(t):t \geqslant 0\}$. For the convenience, we will write C-semigroup instead of written C-regularized semigroup. We assume that C be a bounded injective linear operator, $R(C)$ is a closed subspace of X and denotes $\|\cdot\|$ to be the norm for $B(X)$, where $B(X)$ is the space of all bounded linear operators on X. The integrals in this paper are considered as the Bachner integrals on the Banach space X.

Definition 1.1. Suppose C is a bounded, injective operator. The family of bounded linear operators $\{T(t):t \geqslant 0\}$ is a C-semigroup if it satisfies the following properties:

(1) $T(t)$ is strongly continuous, i.e., for each fixed $x \in X$, $t \mapsto T(t)x$ is continuous.

(2) $T(t)T(s)=CT(t+s)$, for all $t,s \geqslant 0$.

(3) $T(0)=C$.

The linear operator A defined by

$$D(A)=\left\{x\in X:\ \lim_{t\downarrow 0}\frac{T(t)x-Cx}{t} \text{ exists and lies in } R(C)\right\}$$

and

$$Ax=C^{-1}\lim_{t\downarrow 0}\frac{T(t)x-Cx}{t} \qquad \text{for } x \in D(A)$$

is the infinitesimal generator of the C-semigroup $T(t)$ where $D(A)$ is the domain of A.

Definition 1.2. A function $u:[t_0,T] \to X$ is said to be a (classical) solution of the initial value problem (1.3) (or (1.4)) on $[t_0,T)$ if it satisfies the following conditions:

(1) $u \in C[t_0,T)$ and $u \in C^1(t_0,T)$.

(2) $u(t) \in D(A)$, for $t_0<t<T$.

(3) (1.3) (or (1.4)) is satisfied on $[t_0,T)$.

Definition 1.3. A continuous function u on $[t_0,T]$ which is differentiable almost everywhere on $[t_0,T]$ such that $u' \in L^1([t_0,T]:X)$ is called a strong solution of the initial value problem (1.3) if $u(\cdot) \in D(A)$ a.e. on

$[t_0,T]$, $u'(t)=Au(t)+f(t,u(t))$ a.e. on $[t_0,T]$ and $u(t_0)=u_0$.

Definition 1.4. A function $f:[t_0,T]\times X\rightarrow X$ is said satisfying the Lipschitz condition on X if there exists a constant $L\geqslant 0$ such that

$$|f(t,x)-f(t,y)|\leqslant L|x-y| \quad \text{for all } x,y\in X \text{ and } t\in[t_0,T] \tag{1.5}$$

Definition 1.5. A function $f:[t_0,\infty)\times X\rightarrow X$ is said satisfying a locally Lipschitz condition in x, uniformly on every bounded interval of t, if for every constant $\alpha>0$, $t'>t_0$, there exists a Lipschitz constant $L(\alpha,t')\geqslant 0$ such that

$$|f(t,x)-f(t,y)|\leqslant L(\alpha,t')|x-y| \tag{1.6}$$

for all $x,y\in X$, with $|x|\leqslant\alpha$, $|y|\leqslant\alpha$ and $t\in[t_0,t']$.

Definition 1.6. Let A be the infinitesimal generator of a C-semigroup $T(t)$. For $x\in R(C)$ and $f\in L^1([0,T]:R(C))$. Then the function $u\in C([0,T]:X)$ defined by

$$u(t)=C^{-1}T(t)x+\int_0^t C^{-1}T(t-s)f(s)ds \quad 0\leqslant t\leqslant T,$$

is called the mild solution of the initial value problem (1.3) on $[0,T]$.

Definition 1.7. A continuous function u on $[0,T]$ which is differentiable almost everywhere on $[0,T]$ such that $u'\in L^1([0,T]:X)$ is called a strong solution of the initial value problem (1.4) if $u(\cdot)\in D(A)$ a.e. on $[0,T)$, $u(0)=x$ and $u'(t)=Au(t)+f(t)$ a.e. on $[0,T]$.

Definition 1.8. A function $f:[t_0,T]\times X\rightarrow X$ is said satisfying Lipschitz condition in both variables if there exists a constant $L\geqslant 0$ such that

$$|f(t_1,x_1)-f(t_2,x_2)|\leqslant L(|t_1-t_2|+|x_1-x_2|) \tag{1.7}$$

for all $x_1,x_2\in X, t_1, t_2\in[t_0,T]$.

Definition 1.9. Let A be an infinitesimal generator of a C-semigroup $\{T(t):t\geqslant 0\}$ on Banach space X, $u_0\in R(C)$ and $f:[t_0,\infty)\times X\rightarrow R(C)$ satisfying $f(s,u(s))$ is integrable, where $u(s)\in C([t_0,\infty),X)$. A continuous solution u of the integral equation

$$u(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^t C^{-1}T(t-s)f(s,u(s))ds \tag{1.8}$$

will be called the mild solution of the initial value problem (1.3).

Lemma 1.1. [5, R. Delaubenfels, Corollary 4.2] Suppose A is the infinitesimal generator of a C-semigroup. Then the equation (1.2) has a unique solution for every initial value $x\in C(D(A))$.

Lemma 1.2. [16, N. Tanaka and I. Miyadera, Corollary 2.2] Let A be a closed linear operator with nonempty resolvent set. Then the following conditions are equivalent:

(1) A is the infinitesimal generator of a C-semigroup.

(2) A satisfies the conditions

(a) for every $x\in D(A)$, $Cx\in D(A)$ and $ACx=CAx$,

(b) the equation (1.2) has a unique solution for every initial value $x\in C(D(A))$.

Lemma 1.3. [16, N. Tanaka and I. Miyadera, Proposition 1.1] Let A be the infinitesimal generator of the C-semigroup $T(t)$. Then we have

$$C^{-1}AC=A.$$

Lemma 1.4. [5, R. Delaubenfels, Lemma 2.6] Suppose x is in $D(A)$. Then $T(t)x$ is in $D(A)$, for all $t\geqslant 0$, $T(t)Ax=AT(t)x$, and $CT(t)x$ is a differentiable function of t, with

$$\frac{d}{dt}CT(t)x = CT(t)Ax,\ t>0.$$

Lemma 1.5. [5, R. Delaubenfels, Lemma 2.7] For any x in X, $t \geqslant 0$, $\int_0^t T(r)xdr$ is in $D(A)$, with

$$A[\int_0^t T(r)xdr] = T(t)x - Cx.$$

Inhomogeneous equations

Lemma 2.1. sup $\{\|T(t)\| : t \in [0,T]\}$ is finite for any given $T>0$.

Proof. Suppose that sup $\{\|T(t)\| : t \in [0,T]\} = \infty$, according to the Banach-Steinhaus theorem (e.g. see[12, P103]) we may have that

$$\sup\{|T(t)x| : t \in [0,T]\} = \infty$$

for all x belonging to some dense subset G_δ of X. Thus, for each $n \in N$, there exists $t_n \in [0,T]$ such that $|T(t_n)x| \geqslant n|x|$, where x is a fixed element belonging to some dense subset G_δ of X. Since $\{t_n : n \in N\}$ is an infinite sequence in a compact set $[0,T]$, there exists a subsequence $\{t_{n_j} : j \in N\}$ of $\{t_n : n \in N\}$ and $t_0 \in [0,T]$ such that $\lim_{j\to\infty} t_{n_j} = t_0$. Since the function $t \mapsto T(t)x$ is continuous, this implies that $|T(t_0)x| \geqslant n|x|$ for each $n \in N$. Thus, $T(t_0)$ is unbounded which contradict to the definition of $T(t_0)$. The proof of this Lemma is complete now.

From now on, we denote $M_T = \sup\{\|T(t)\| : t \in [0,T]\}$ (*)
(or more simply M, if no ambiguous arise).

Lemma 2.2. If $x \in R(C)$, then $T(t)C^{-1}x = C^{-1}T(t)x$, for all $t \geqslant 0$.

Proof. For $x \in R(C)$, there exists $y \in X$ such that $x = Cy$. This implies $y \in C^{-1}x$ and $CT(t)y = T(t)T(0)y = T(t)Cy = T(t)x$. Since C is an injective operator, we have that $T(t)C^{-1}x = T(t)y = C^{-1}T(t)x$.

Lemma 2.3. If A is the infinitesimal generator of a C-semigroup, then $C(D(A)) \subseteq D(A)$.

Proof. If $x \in C(D(A))$, then there exists $y \in D(A)$ such that $x = Cy$. From Lemma 1.4, $T(t)y \in D(A)$, for all $y \in D(A)$ and $t \geqslant 0$. This implies that

$$x = Cy = T(0)y \in D(A).$$

Lemma 2.4. If $x \in C(D(A))$ then $T(t)AC^{-1}x = AC^{-1}T(t)x$.

Proof. Since $x \in C(D(A)) \subset R(C)$, $T(t)C^{-1}x = C^{-1}T(t)x$ and $C^{-1}x \in D(A)$. Thus, from Lemma 1.4 and Lemma 2.2,

$$\begin{aligned} T(t)AC^{-1}x &= AT(t)C^{-1}x \\ &= AC^{-1}T(t)x. \end{aligned}$$

Lemma 2.5. If $g \in C([0,T]:X)$, then the function $t \mapsto \int_0^t T(t-s)g(s)ds$ is continuous on $(0,T)$.

Proof. For each fixed $t \in (0,T)$ and any $h>0$,

$$\begin{aligned} &\left|\int_0^{t+h} T(t+h-s)g(s)ds - \int_0^t T(t-s)g(s)ds\right| \\ &= \left|\int_0^{t+h} T(s)g(t+h-s)ds - \int_0^t T(s)g(t-s)ds\right| \\ &\leqslant \left|\int_0^t T(s)[g(t+h-s)-g(t-s)]ds\right| + \left|\int_t^{t+h} T(s)g(t+h-s)ds\right| \end{aligned}$$

From $s \mapsto T(s)g(t+h-s)$ is a continuous function, there exists K > 0 such that

$$\left|\int_0^{t+h} T(s)g(t+h-s)ds\right| \leqslant Kh.$$

This implies that

$$\lim_{h \to 0} \left| \int_t^{t+h} T(s)g(t+h-s)ds \right| = 0.$$

Moreover, from (*),

$$\left| \int_0^t T(s)[g(t+h-s)-g(t-s)]ds \right| \leq M \int_0^t |g(t+h-s)-g(t-s)|ds.$$

From the assumption of the function g, g is uniformly continuous on $[0,T]$ and hence, for each $\varepsilon>0$, there exists $h_0>0$ such that $|g(t+h-s)-g(t-s)| < \varepsilon$, whenever $0 \leq |h| < h_0$. This implies that

$$\left| \int_0^t T(s)[g(t+h-s)-g(t-s)]ds \right| \leq M \varepsilon t,$$

whenever $0 \leq |h| < h_0$. Thus,

$$\lim_{h \to 0} \left| \int_0^t T(s)[g(t+h-s)-g(t-s)]ds \right| = 0$$

and this Lemma is completely proved now.

Lemma 2.6. If $f(s) \in C(D(A))$, for all $0<s<T$, then $T(t-s)f(s) \in C(D(A))$ for $0<s<t<T$.

Proof. From the assumption of the function f, for each fixed $s \in (0,T)$, there exists $x_s \in D(A)$ such that $f(s)=Cx_s$. Thus, for all $t \geq s$, $T(t-s)f(s)=T(t-s)Cx_s=CT(t-s)x_s$. Since $x_s \in D(A)$, from Lemma 1.4, $T(t-s)x_s \in D(A)$ and hence

$$T(t-s)f(s) \in C(D(A)).$$

Lemma 2.7. Suppose that $f \in L^1([0,T]:R(C))$ and $v(t)=\int_0^t T(t-s)f(s)ds$, for all $0 \leq t \leq T$. If $AC^{-1}v \in L^1([0,T]:X)$, then $C^{-1}v(t)$ is absolutely continuous on $[0,T]$.

Proof. Since $f \in L^1([0,T]:R(C))$ and $AC^{-1}v \in L^1([0,T]:X)$, there exist two constants M_1 and $M_2>0$ such that

$$|f(s)| \leq M_1 \qquad \text{a.e. on } [0,T]$$

and

$$|AC^{-1}v(t)| \leq M_2 \qquad \text{a.e. on } [0,T].$$

For each $\varepsilon>0$, if we take $\delta = \dfrac{\varepsilon}{\|C^{-1}\|M(M_1+M_2)}$, where M is defined as in (*), then for each $t_i>s_i$ in $[0,T]$ $i=1,2,\cdots\cdots,n$, we have

$$\begin{aligned}
&\sum_{i=1}^n |C^{-1}v(t_i)-C^{-1}v(s_i)| \\
&= \sum_{i=1}^n \left|C^{-1} \int_0^{t_i} T(t_i-s)f(s)ds - C^{-1}v(s_i)\right| \\
&\leq \sum_{i=1}^n \left|C^{-1} \int_0^{s_i} C^{-1}T(t_i-s_i)T(s_i-s)\, f(s)ds - C^{-1}CC^{-1}v(s_i)\right| + \sum_{i=1}^n \left|C^{-1} \int_{s_i}^{t_i} T(t_i-s)f(s)ds\right| \\
&= \sum_{i=1}^n |C^{-1}[T(t_i-s_i)-C]C^{-1}v(s_i)| + \sum_{i=1}^n \left| \int_{s_i}^{t_i} C^{-1}T(t_i-s)f(s)ds \right|.
\end{aligned}$$

If $\sum_{i=1}^n |t_i-s_i| < \delta$, then

$$\begin{aligned}
&\sum_{i=1}^n \left| \int_{s_i}^{t_i} C^{-1}T(t_i-s)f(s)ds \right| \\
&\leq \sum_{i=1}^n \int_{s_i}^{t_i} \|C^{-1}\|\, \|T(t_i-s)\|\, |f(s)|\, ds \\
&\leq \|C^{-1}\| M M_1 \sum_{i=1}^n (t_i-s_i) \\
&\leq \|C^{-1}\| M M_1 \delta,
\end{aligned}$$

and there exists some $\theta \in (0,1)$ such that

$$\sum_{i=1}^{n} |C^{-1}[T(t_i-s_i)-C]C^{-1}v(s_i)|$$
$$=\sum_{i=1}^{n} |C^{-1}(t_i-s_i)T(\theta(t_i-s_i))AC^{-1}v(s_i)|$$
$$\leqslant \sum_{i=1}^{n} \|C^{-1}\|M M_2(t_i-s_i)$$
$$\leqslant \|C^{-1}\|M M_2 \delta .$$

This implies

$$\sum_{i=1}^{n} |C^{-1}v(t_i)-C^{-1}v(s_i)| < \|C^{-1}\| M(M_1+M_2)\delta < \varepsilon$$

and hence, $C^{-1}v(t)$ is absolutely continuous on $[0,T]$.

Proposition 2.1. If $f \in L^1([0,T]:R(C))$, then for each $x \in C(D(A))$ (1.4) has at most one solution. Moreover, if (1.4) has a solution, then it is given by

$$u(t)=C^{-1}T(t)x+\int_0^t C^{-1}T(t-s)f(s)ds.$$

Proof. Suppose u is a solution of (1.4), let $g(s) = T(t-s)u(s)$, $0<s<t<T$. Since $u \in C^1(0,T)$, $g(s)$ is differentiable for all $0<s<t<T$ and

$$\begin{aligned}\frac{d}{ds}g(s)&=-AT(t-s)u(s)+T(t-s)u'(s)\\&=-AT(t-s)u(s)+T(t-s)Au(s)+T(t-s)f(s)\\&=T(t-s)f(s).\end{aligned}$$

From the assumption of the function f and (*)

$$\begin{aligned}\left|\int_0^t T(t-s)f(s)ds\right| &\leqslant \int_0^t \|T(t-s)\|\,|f(s)|ds\\&\leqslant M\int_0^t |f(s)|ds\\&< \infty .\end{aligned}$$

This implies $T(t-s)f(s)$ is integrable, and

$$\int_0^t \frac{d}{ds}g(s)ds=\int_0^t T(t-s)f(s)\,ds .$$

According to the fundamental theorem of the calculus for Bochner integrals,

$$g(t)-g(0)=\int_0^t T(t-s)f(s)ds.$$

Since $g(t)=Cu(t)$ and $g(0)=T(t)u(0)$,

$$\begin{aligned}Cu(t)&=T(t)u(0)+\int_0^t T(t-s)f(s)\,ds\\&=T(t)x+\int_0^t T(t-s)f(s)\,ds.\end{aligned}$$

Furthermore, since $x \in C(D(A))$ and $f(s) \in R(C)$, from Lemma 2.2, $T(t)x$, $\int_0^t T(t-s)f(s)\,ds$ are in $R(C)$ and hence,

$$\begin{aligned}u(t)&=C^{-1}T(t)x+C^{-1}\int_0^t T(t-s)f(s)\,ds\\&=C^{-1}T(t)x+\int_0^t C^{-1}T(t-s)f(s)ds.\end{aligned}$$

This proposition is completely proved now.

Theorem 2.2. Let A be the infinitesimal generator of a C-semigroup $T(t)$, f is continuous on $[0,T]$,

$f \in L^1([0,T]:R(C))$ and $v(t)= \int_0^t T(t-s)f(s)\,ds$, for all $0 \leq t \leq T$. Then
(a)the initial value problem (1.4) has a solution u on $[0,T)$ for every $x \in C(D(A))$ if one of the following conditions is satisfied:

(1) $v(t)$ is continuously differentiable on $(0,T)$ and $v'(t) \in R(C)$,

(2) $v(t) \in C(D(A))$ for $0<t<T$ and $A\,C^{-1}v(t)$ is continuous on $(0,T)$;

(b)if (1.4) has a solution u on $[0,T)$ for some $x \in C(D(A))$, then v satisfies both conditions (1) and (2) in (a).

Proof. (a) For each $h>0$

$$\begin{aligned}
&\frac{1}{h}[v(t+h)-v(t)]\\
&=\frac{1}{h}[\int_0^{t+h} T(t+h-s)f(s)ds-v(t)]\\
&=\frac{1}{h}[\int_0^t C^{-1}T(h)T(t-s)f(s)ds-C^{-1}Cv(t)]+\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds\\
&=C^{-1}\frac{1}{h}[T(h)-C]v(t)+\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds
\end{aligned} \tag{2.1}$$

From the hypothesis of function f, $v(t) \in R(C)$ and hence, by Lemma 2.2 and (2.1), we have that

$$\begin{aligned}
&\frac{1}{h}[T(h)-C]C^{-1}v(t)\\
&=C^{-1}\frac{1}{h}[T(h)-C]v(t)\\
&=\frac{1}{h}[v(t+h)-v(t)]-\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds
\end{aligned} \tag{2.2}$$

Since $f \in C([0,T]:R(C))$ there exists a $0 \leq \theta \leq 1$ such that

$$\begin{aligned}
\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds &= T(t+h-(t+\theta h))f(t+\theta h)\\
&=T((1-\theta)h)f(t+\theta h),
\end{aligned}$$

which converges to $T(0)f(t)$, as $h \to 0$. i.e.,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds=Cf(t). \tag{2.3}$$

Under the condition (1), from (2.2) and (2.3) one immediately has

$$\lim_{h\downarrow 0}\frac{1}{h}[T(h)-C]C^{-1}v(t)=v(t)-Cf(t) \in R(C).$$

This implies $C^{-1}v(t) \in D(A)$ and $AC^{-1}v(t)=C^{-1}v(t)-f(t)$, for $0<t<T$. Thus, $u(t)=C^{-1}T(t)x+C^{-1}v(t)$ is the solution of (1.4), for all $x \in C(D(A))$. Under the condition (2), $C^{-1}v(t) \in D(A)$, for $0<t<T$, this implies that $C^{-1}v(t)$ is differentiable from the right at any $t \in (0,T)$ and

$$\begin{aligned}
&D^+C^{-1}v(t)\\
&=\lim_{h\downarrow 0}\frac{1}{h}[C^{-1}v(t+h)-C^{-1}v(t)]\\
&=C^{-1}\lim_{h\downarrow 0}\frac{1}{h}[v(t+h)-v(t)]\\
&=C^{-1}\lim_{h\downarrow 0}\{\frac{1}{h}[T(h)-C]C^{-1}v(t)+\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds\}\\
&=AC^{-1}v(t)+C^{-1}Cf(t)\\
&=AC^{-1}v(t)+f(t).
\end{aligned}$$

Moreover, $AC^{-1}v$ and f both are continuous on $(0,T)$ and hence $D^+C^{-1}v(t)$ is continuous on $(0,T)$. Thus, $C^{-1}v(t)$ is differentiable and,

$$\frac{d}{dt}C^{-1}v(t)=AC^{-1}v(t)+f(t).$$

This shows that $u(t)=C^{-1}T(t)x+C^{-1}v(t)$ is the solution of (1.4) for all $x\in C(D(A))$.

(b) If there exists $x\in C(D(A))$ such that u is the solution of (1.4) then

$$\begin{aligned}u(t)&=C^{-1}T(t)x+\int_0^t C^{-1}T(t-s)f(s)ds\\&=C^{-1}T(t)x+C^{-1}\int_0^t T(t-s)f(s)ds\\&=C^{-1}T(t)x+C^{-1}v(t).\end{aligned}$$

This implies $v(t)=Cu(t)-T(t)x$. Since $u(t)$ is the solution of (1.4), A is the infinitesimal generator of a C-semigroup $T(t)$ and $x\in C(D(A))\subset D(A)$ (by Lemma 2.3), this implies that $u(t)$ and $T(t)x$ both are differentiable. Hence, $v(t)$ is differentiable and

$$v'(t)=Cu(t)-T(t)Ax. \tag{2.4}$$

Because $x\in C(D(A))$, there is a $y\in D(A)$ such that $x=Cy$. From Lemma 1.2,

$$\begin{aligned}T(t)Ax&=T(t)ACy\\&=T(t)CAy\\&=CT(t)Ay\\&\in R(C).\end{aligned} \tag{2.5}$$

According to (2.4) and (2.5), $v(t)\in R(C)$. Furthermore, since $u(t)\in C(0,T)$ and the function $t\mapsto T(t)Ax$ is continuous on $(0,T)$, $v(t)\in C(0,T)$ and hence $v(t)$ belongs to the set $C^1(0,T)$. Condition (1) is clearly satisfied now.

Since $u(t)$ is the solution of (1.4), $u(t)\in D(A)$, for $0<t<T$. This implies that $\lim_{h\downarrow 0}\frac{1}{h}[T(h)u(t)-Cu(t)]$ exists and $\lim_{h\downarrow 0}\frac{1}{h}[T(h)u(t)-Cu(t)]\in R(C)$. Since $y\in D(A)$, this implies $T(t)y\in D(A)$ and thus from Lemma 1.4,

$$\begin{aligned}&\lim_{h\downarrow 0}\frac{1}{h}[T(h)-C](u(t)-T(t)y)\\&=\lim_{h\downarrow 0}\frac{1}{h}[T(h)-C]u(t)-\lim_{h\downarrow 0}\frac{1}{h}[T(h)-C]T(t)y\end{aligned}$$

exists and $\lim_{h\downarrow 0}[T(h)-C](u(t)-T(t)y)\in R(C)$. This shows that $u(t)-T(t)y\in D(A)$ and hence

$$\begin{aligned}v(t)&=Cu(t)-T(t)x\\&=C(u(t)-T(t)y)\in C(D(A)),\end{aligned}$$

and $C^{-1}v(t)=u(t)-C^{-1}T(t)x\in D(A)$. Moreover, from Lemma 2.4,

$$\begin{aligned}AC^{-1}v(t)&=Au(t)-AC^{-1}T(t)x\\&=u'(t)-f(t)-T(t)AC^{-1}x.\end{aligned}$$

Since $u'(t)$, $f(t)\in C(0,T)$ and the function $t\mapsto T(t)AC^{-1}x$ is continuous, this implies $AC^{-1}v(t)\in C(0,T)$. Thus, condition (2) is satisfied and Theorem 2.2 is completely proved now.

Corollary 2.3. Suppose A generates a C-semigroup $T(t)$. If f is continuously differentiable from $[0,T]$ into $R(C)$, then (1.4) has a unique solution u on $[0,T)$, for every $x\in C(D(A))$.

Proof. From the assumption of function f and the fact that

$$\begin{aligned}v(t)&=\int_0^t T(t-s)f(s)\,ds\\&=\int_0^t T(s)f(t-s)\,ds,\end{aligned}$$

$v(t)$ is differentiable for $t>0$ and

$$v'(t)=T(t)f(0)+\int_0^t T(t-s)f'(s)ds.$$

Since both $f(0)$ and $f(s)$ are in the range of C, $T(t)f(0)$ and $\int_0^t T(t-s)f'(s)\,ds$ are in the range of C. Because the range of C is a closed subspace of X, so

$$v'(t)=T(t)f(0)+\int_0^t T(t-s)f'(s)\,ds \in R(C).$$

From the fact that $t\rightarrow T(t)f(0)$ and $t\rightarrow \int_0^t T(t-s)f'(s)ds$ are continuous maps from $(0,T)$ into $R(C)$, this shows that $v'(t)\in C((0,T):R(C))$ (by Lemma 2.5). Thus, the conclusion of this corollary is followed from Theorem 2.2, (a)-(1) and Proposition 2.1.

Corollary 2.4. Suppose A generates a C-semigroup $T(t)$, $f\in L^1([0,T]:R(C))$ and f is continuous on $[0,T]$. If

$$f(s)\in C(D(A)),\quad \text{for}\quad 0<s<T$$

and

$$AC^{-1}f(s)\in L^1([0,T]:X),$$

then for each $x\in C(D(A))$, (1.4) has a unique solution on $[0,T)$.

Proof. From the assumption of the function f and Lemma 2.6, we obtain the fact $T(t-s)f(s)\in C(D(A))$. This implies $C^{-1}T(t-s)f(s)\in D(A)$ and hence,

$$AC^{-1}T(t-s)f(s)=T(t-s)AC^{-1}f(s).$$

(by Lemma 2.4). Since $AC^{-1}f(s)\in L^1([0,T]:X)$,

$$\int_0^T |AC^{-1}f(s)|\,ds<\infty.$$

From Lemma 2.1, we have

$$\int_0^T |T(t-s)AC^{-1}f(s)|\,ds\leq \int_0^T \|T(t-s)\|\,|AC^{-1}f(s)|\,ds$$
$$\leq M\int_0^T |AC^{-1}f(s)|\,ds<\infty.$$

Thus, $AC^{-1}T(t-s)f(s)=T(t-s)AC^{-1}f(s)$ is integrable. Since A is a closed linear operator and

$$\int_0^t AC^{-1}T(t-s)f(s)ds=\int_0^t T(t-s)AC^{-1}f(s)ds$$

exists, which implies

$$A\int_0^t C^{-1}T(t-s)f(s)ds=\int_0^t AC^{-1}T(t-s)f(s)ds$$

and $\int_0^t C^{-1}T(t-s)f(s)ds\in D(A)$. Furthermore, since C^{-1} is a bounded linear operator,

$$C^{-1}\int_0^t T(t-s)f(s)ds=\int_0^t C^{-1}T(t-s)f(s)ds\in D(A).$$

This implies that $v(t)=\int_0^t T(t-s)f(s)ds\in C(D(A))$ and

$$AC^{-1}v(t)=\int_0^t AC^{-1}T(t-s)f(s)ds=\int_0^t T(t-s)AC^{-1}f(s)ds.$$

The proof of this corollary will be accomplished as long as we prove that $AC^{-1}v(t)$ is continuous on $(0,T)$. Indeed, since $AC^{-1}f(s)\in L^1([0,T]:X)$, for each $\varepsilon>0$, there is a function $g(s)\in C[0,T]$ such that $|g(s)-AC^{-1}f(s)|<\varepsilon$, a.e. on $[0,T]$. This implies

$$|AC^{-1}v(t+h)-AC^{-1}v(t)|$$
$$=\left|\int_0^{t+h} T(t+h-s)AC^{-1}f(s)ds-\int_0^t T(t-s)AC^{-1}f(s)ds\right|$$
$$=\left|\int_0^{t+h} T(s)AC^{-1}f(t+h-s)ds-\int_0^t T(s)AC^{-1}f(t-s)ds\right|$$

$$\leq \left| \int_0^{t+h} T(s)[AC^{-1}f(t+h-s)-g(t+h-s)]ds \right| + \left| \int_0^{t+h} T(s)g(t+h-s))ds- \int_0^t T(s)g(t-s)ds \right|$$
$$+ \left| \int_0^t T(s)[AC^{-1}f(t-s)-g(t-s)]ds \right|$$
$$\leq \int_0^{t+h} \|T(s)\| \varepsilon ds- \int_0^t \|T(s)\| \varepsilon ds + \left| \int_0^{t+h} T(s)g(t+h-s)ds- \int_0^t T(s)g(t-s)ds \right|.$$

From Lemma 2.5, the function $t \to \int_0^t T(t-s)g(s)ds$ is continuous on $(0,T)$ and hence there is a $h_0>0$ such that $\left| \int_0^{t+h} T(t+h-s)g(s)ds- \int_0^t T(t-s)g(s)ds \right| < \varepsilon$ whenever $|h| < h_0$. This shows that

$$|AC^{-1}v(t+h)-AC^{-1}v(t)|$$
$$\leq M(t+h) \varepsilon +Mt \varepsilon + \varepsilon$$
$$\leq (2MT+Mh_0+1) \varepsilon .$$

for $|h| < h_0$. Hence, $AC^{-1}v(t)$ is continuous on $(0,T)$ and the assertion of the corollary is followed from Theorem 2.2 (a)-(2) and Proposition 2.1.

Theorem 2.5. Suppose A generates a C-semigroup $T(t)$, $C(D(A))$ is dense in X and $f \in L^1([0,T];R(C))$. If u is the mild solution of

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t) & 0<t<T'<T \\ u(0)=x & x \in X \end{cases} \tag{1.4'}$$

then for each $T'<T$, u is the uniform limit on $[0,T]$ of solutions of (1.4).

Proof. Since the closure of $C(D(A))$ is X and the closure of $C^1([0,T];R(C))$ is $L^1([0,T];R(C))$, there are $x_n \in C(D(A))$ and $f_n \in C^1([0,T];R(C))$ such that $x_n \to x$ in X and $f_n \to f$ in $L^1([0,T];R(C))$ respectively. From the hypothesis of this Theorem and Corollary 2.3, it follows that for each $n \geq 1$ the initial value problem

$$\begin{cases} \frac{d}{dt}u_n(t)=Au_n(t)+f_n(t) & 0<t<T \\ u_n(0)=x_n & x_n \in C(D(A)) \end{cases}$$

has a unique solution $u_n(t)$ on $[0,T]$ given by

$$u_n(t)=C^{-1}T(t)x_n+ \int_0^t C^{-1}T(t-s)f_n(s)ds.$$

Since u is the mild solution of (1.4)' on $[0,T]$,

$$u(t)=C^{-1}T(t)x+ \int_0^t C^{-1}T(t-s)f(s)ds.$$

Thus,

$$|u_n(t)-u(t)|$$
$$\leq \|C^{-1}T(t)\| \, |x_n-x| + \int_0^t \|C^{-1}T(t-s)\| \, |f_n(s)-f(s)|ds$$
$$\leq M \|C^{-1}\| \, |x_n-x| + \int_0^t M \|C^{-1}\| \, |f_n(s)-f(s)|ds$$
$$\leq M \|C^{-1}\|(|x_n-x| + \int_0^t |f_n(s)-f(s)| ds). \tag{2.6}$$

Therefore, the result of this Theorem follows readily from (2.6).

Proposition 2.6. If $f \in L^1([0,T]):R(C))$, then (1.4) has at most one strong solution for each $x \in C(D(A))$. Moreover, if (1.4) has a strong solution, then it is given by

$$u(t)=C^{-1}T(t)x+ \int_0^t C^{-1}T(t-s)f(s)ds.$$

Proof. Suppose u be a strong solution of (1.4), let $g(s)=T(t-s)u(s)$, $0<s<t$. Since u is differentiable a.e. on $[0,T]$, $g(s)$ is differentiable a.e. on $[0,T]$ and

$$\frac{d}{ds}g(s)$$
$$=-AT(t-s)u(s)+T(t-s)u'(s)$$
$$=-AT(t-s)u(s)+T(t-s)Au(s)+T(t-s)f(s)$$
$$=T(t-s)f(s)$$

a.e. on $[0,T]$. From the assumption of the function $f\in L^1([0,T]:R(C))$, with the same technique used in the proof of Proposition 2.1, we have $T(t-s)f(s)$ is integrable and

$$\int_0^t \frac{d}{ds}g(s)ds=\int_0^t T(t-s)f(s)ds.$$

This implies $g(t)-g(0)=\int_0^t T(t-s)f(s)ds$. Since $g(t)=Cu(t)$ and $g(0)=T(t)u(0)$,

$$Cu(t)=T(t)u(0)+\int_0^t T(t-s)f(s)ds$$
$$=T(t)x+\int_0^t T(t-s)f(s)ds.$$

Furthermore, since $x\in C(D(A))$ and $f(s)\in R(C)$,

$$u(t)=C^{-1}T(t)x+C^{-1}\int_0^t T(t-s)f(s)ds$$
$$=C^{-1}T(t)x+\int_0^t C^{-1}T(t-s)f(s)ds.$$

This proposition is proved now.

Theorem 2.7. Let A be the infinitesimal generator of a C-semigroup $T(t)$, $f\in L^1([0,T]:R(C))$ and

$$v(t)=\int_0^t T(t-s)f(s)\,ds,\quad 0\leqslant t\leqslant T.$$

Then

(a) for every $x\in C(D(A))$, the initial value problem (1.4) has a strong solution u on $[0,T)$, if one of the following conditions is satisfied

(1) $v(t)$ is differentiable a.e. on $[0,T]$ and $v'(t)\in L^1([0,T]:R(C))$,

(2) $v(t)\in C(D(A))$ a.e. on $[0,T]$ and $AC^{-1}v(t)\in L^1([0,T]:X)$;

(b)if (1.4) has a strong solution u on $[0,T)$ for some $x\in C(D(A))$, then v satisfies both conditions (1) and (2) in (a).

Proof. (a) For each $h>0$, as in the proof of Theorem 2.4. we have

$$\frac{1}{h}[T(h)-C]C^{-1}v(t)$$
$$=\frac{1}{h}[v(t+h)-v(t)]-\frac{1}{h}\int_t^{t+h}T(t+h-s)f(s)ds \tag{2.7}$$

Since $f\in L^1([0,T]:R(C))$, $C([0,T]:R(C))$ is dense in $L^1([0,T]:R(C))$ and from mean value theorem,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h}T(t+h-s)f(s)ds=Cf(t). \tag{2.8}$$

a.e. on $[0,T]$.

If condition (1) is satisfied, then $v(t)$ is differentiable a.e. on $[0,T]$ and hence

$$\lim_{h\downarrow 0}\frac{1}{h}[v(t+h)-v(t)]=v'(t), \tag{2.9}$$

a.e. on $[0,T]$. From (2.7), (2.8) and (2.9) one immediately has

$$\lim_{h\downarrow 0}\frac{1}{h}[T(h)-C]C^{-1}v(t)=v'(t)-Cf(t)\in R(C),$$

a.e. on $[0,T]$. This implies $C^{-1}v(t) \in D(A)$ a.e. on $[0,T]$ and

$$AC^{-1}v(t) = C^{-1}\lim_{h\downarrow 0}\frac{1}{h}[T(h)-C]C^{-1}v(t)$$
$$= C^{-1}v'(t)-f(t),$$

a.e. on $[0,T]$. Moreover, from the fact that $v(0)=0$ we have, for all $x \in C(D(A))$, $u(t)=C^{-1}T(t)x+C^{-1}v(t)$ is the strong solution of (1.4) .
If condition (2) is satisfied, then $v(t) \in C(D(A))$ a.e. on $[0,T]$ and hence $C^{-1}v(t) \in D(A)$ a.e. on $[0,T]$. With the same technique used in the proof of Theorem 2.4, we have $D^{+}C^{-1}v(t)$ exists a.e. on $[0,T]$ and

$$D^{+}C^{-1}v(t) = AC^{-1}v(t)+f(t) \quad \text{a.e. on } [0,T].$$

From Lemma 2.7, $C^{-1}v(t)$ is absolutely continuous on $[0,T]$, this implies $\frac{d}{dt}C^{-1}v(t)$ exists a.e. on $[0,T]$ and

$$\frac{d}{dt}C^{-1}v(t) = D^{+}C^{-1}v(t)$$
$$= AC^{-1}v(t)+f(t)$$

a.e. on $[0,T]$ (See [6, Royden, P 109]). Since $AC^{-1}v(t), f(t) \in L^{1}([0,T]:X)$, $v(0)=0$, which implies $C^{-1}v'(t) \in L^{1}([0,T]:X)$ and for all $x \in C(D(A))$, $u(t)=C^{-1}T(t)x+C^{-1}v(t)$ is the strong solution of (1.4).

(b) Let u be the strong solution of (1.4) for some $x \in C(D(A))$, then

$$u(t) = C^{-1}T(t)x+ \int_0^t C^{-1}T(t-s)f(s)ds$$
$$= C^{-1}T(t)x+C^{-1}\int_0^t T(t-s)f(s)ds$$
$$= C^{-1}T(t)x+C^{-1}v(t) .$$

This implies $v(t)=Cu(t)-T(t)x$. Since A generates a C-semigroup $T(t)$ and $x \in C(D(A)) \subset D(A)$ (by Lemma 2.3), $T(t)x$ is differentiable for every t in $(0,T)$. Since $u(t)$ is differentiable a.e. on $[0,T]$, $v(t)$ is differentiable a.e. on $[0,T]$ and

$$v'(t) = Cu'(t)-T(t)Ax.$$

Because of $Cu'(t) \in L^{1}([0,T]:X)$ and $T(t)Ax \in C([0,T]:R(C))$, we have

$$v'(t) \in L^{1}([0,T]:R(C)).$$

Thus, condition (1) is satisfied.
From the assumption, $x \in C(D(A))$ and $u(t) \in D(A)$ a.e. on $[0,T]$, so

$$T(x) = T(t)Cy = CT(t)y \in C(D(A)),$$

for $x=Cy$, $y \in D(A)$ and

$$Cu(t) \in C(D(A)) \text{ a.e. on } [0,T].$$

Hence, $v(t) \in C(D(A))$ a.e. on $[0,T]$ and $C^{-1}v(t) \in D(A)$ a.e. on $[0,T]$. Moreover,

$$AC^{-1}v(t) = Au(t)-AC^{-1}T(t)x$$
$$= u'(t)-f(t)-T(t)AC^{-1}x$$
$$\in L^{1}([0,T]:X).$$

Thus condition (2) is satisfied and this theorem is completely proved now.

Corollary 2.8. Suppose A generates a C-semigroup $T(t)$. If f is differentiable a.e. on $[0,T]$ and $f' \in L^{1}([0,T]:R(C))$, then (1.4) has a unique strong solution u on $[0,T]$, for every $x \in C(D(A))$.
Proof. Since f is differentiable a.e. on $[0,T]$ and the fact that

$$v(t)=\int_0^t T(t-s)f(s)ds = \int_0^t T(t-s)f(s)ds,$$

$v(t)$ is differentiable a.e. on $[0,T]$ and

$$v'(t)=T(t)f(0)+\int_0^t T(t-s)f'(s)ds.$$

for those t where $v(t)$ is differentiable. From the assumption of $f'\in L^1([0,T]:R(C))$, we infer that the function $t\mapsto \int_0^t T(t-s)f'(s)ds$ is continuous. Since both $f(0)$ and $f'(s)$ are in the range of C for all $0\leqslant s\leqslant T$, $T(t)f(0)$ and $\int_0^t T(t-s)f'(s)\,ds$ are in the range of C. Because the range of C is a closed subspace of X and so,

$$v'(t)=T(t)f(0)+\int_0^t T(t-s)f'(s)ds\in R(C).$$

From the fact that $t\mapsto T(t)f(0)$ and $t\mapsto \int_0^t T(t-s)f'(s)ds$ are continuous on $(0,T)$, we have that $v'(t)\in L^1([0,T]:R(C))$. Thus, the conclusion of this corollary is followed from Theorem 2.7, (a)-(1) and Proposition 2.6.

Corollary 2.9. Suppose A generates a C-semigroup $T(t)$, $f\in L^1([0,T]:R(C))$. If $f(s)\in C(D(A))$ a.e. on $[0,T]$ and $AC^{-1}f(s)\in L^1([0,T]:X)$, then the equation (1.4) has a unique strong solution u on $[0,T]$, for each $x\in C(D(A))$.

Proof. From our assumption of f, we have that $T(t-s)f(s)\in C(D(A))$ a.e. on $[0,T]$ followed from Lemma 2.6. This implies $C^{-1}T(t-s)f(s)\in D(A)$ a.e. on $[0,T]$ and from Lemma 2.4, we have that

$$AC^{-1}T(t-s)f(s)=T(t-s)AC^{-1}f(s)$$

a.e. on $[0,T]$. As in the proof of Corollary 2.4,

$$v(t)=\int_0^t T(t-s)f(s)ds\in C(D(A))$$

a.e. on $[0,T]$ and

$$AC^{-1}v(t)=\int_0^t T(t-s)AC^{-1}f(s)ds$$

is continuous a.e. on $[0,T]$ and hence $AC^{-1}v(t)\in L^1([0,T]:X)$. According to Theorem 2.7, (a)-(2) and Proposition 2.6, the assertion of the corollary is desired.

Proposition 2.10. Suppose that X is a reflexive Banach space. If f is Lipschitz continuous on $[0,T]$ and $f(s)\in R(C)$, for $0\leqslant s\leqslant T$, then f is differentiable a.e. on $[0,T]$ and $f'\in L^1([0,T]:R(C))$.

Proof. From the assumption of the function f and the Banach space X, it is well known that (see, e.g. [5, P.109]) f is differentiable a.e. on $[0,T]$ and $f'\in L^1([0,T]:R(C))$. Since

$$f(s)\in R(C),\ \text{for all } 0\leqslant s\leqslant T,$$

and

$$f'(s)=\lim_{h\downarrow 0}\frac{1}{h}[f(s+h)-f(s)]\in\overline{R(C)}=R(C),\ \text{for all } 0\leqslant s\leqslant T.$$

Thus, $f'\in L^1([0,T]:R(C)$ and Proposition 2.10 is completely proved now.

From Proposition 2.10 and Corollary 2.8, we immediately have following result :

Corollary 2.11. Suppose that X is a reflexive Banach space, and let A be the infinitesimal generator of C-semigroup $T(t)$. If f is Lipschitz continuous on $[0,T]$, and $f(s)\in R(C)$, for all $0\leqslant s\leqslant T$, then for every $x\in C(D(A))$ the initial value problem (1.4) has a unique strong solution u on $[0,T]$ given by

$$u(t)=C^{-1}T(x)x+\int_0^t C^{-1}T(t-s)f(s)ds.$$

Semilinear equations

Proposition 3.1. Let $f:[t_0,T]\times X\rightarrow R(C)$ be continuous in t and satisfies Lipschitz condition in x. If u is a classical or strong solution of the initial value problem (1.3), then the solution u satisfies the integral equation (1.8),

$$u(t)=C^{-1}T(t-t_0)u_0+\int_0^t C^{-1}T(t-s)f(s,u(s))ds$$

i.e., u is the mild solution of the initial value problem (1.3).

Proof. (a) Suppose u is a classical solution of (1.3), let $g(s)=T(t-s)u(s)$ for all $t_0\leqslant s\leqslant t$. Since $u\in C^1(t_0,T)$, we infer that $g(s)$ is differentiable for $t_0<s<t$ and

$$\frac{d}{ds}g(s)=T(t-s)f(s,u(s)).$$

According to the assumption of f and u

$$\begin{aligned}|f(t,u(t)-f(s,u(s))| &\leqslant |f(t,u(t))-f(t,u(s))|+|f(t,u(s))-f(s,u(s))| \\ &\leqslant L|u(t)-u(s)|+|f(t,u(s))-f(s,u(s))| ,\end{aligned} \tag{3.1}$$

where the constant L is given by (1.5). This implies that $\lim_{t\rightarrow s}|f(t,u(t))-f(s,u(s))|=0$. Hence, the function $t\mapsto f(t,u(t))$ is continuous on $[t_0,T)$ and $\int_0^T|f(s,u(u))|ds<\infty$. Set $M=sup\{\|T(s)\|:s\in[0,T]\}$, we have

$$\begin{aligned}|\int_{t_0}^t T(t-s)f(s,u(s))ds| &\leqslant \int_{t_0}^t\|T(t-s)\|\ |f(s,u(s))|ds \\ &\leqslant M\int_{t_0}^T|f(s,u(s)|ds \\ &<\infty .\end{aligned}$$

This implies $T(t-s)f(s,u(s))$ is integrable and

$$\int_{t_0}^t\frac{d}{ds}g(s)ds=\int_{t_0}^t T(t-s)f(s,u(s))ds.$$

According to the fundamental theorem of the calculus for Bochner integrals, we have

$$g(t)-g(t_0)=\int_{t_0}^t T(t-s)f(s,u(s))ds.$$

From the facts that $g(t)=Cu(t)$ and $g(t_0)=T(t-t_0)u(t_0)$, we have that

$$\begin{aligned}Cu(t)&=T(t-t_0)u(t_0)+\int_{t_0}^t T(t-s)f(s,u(s))ds \\ &=T(t-t_0)u_0+\int_{t_0}^t T(t-s)f(s,u(s))ds.\end{aligned}$$

Since $u_0\in C(D(A))$ and $f(s,u(s))\in R(C)$, thus from Lemma 2.2 and the assumption of C, we have that

$$u(t)=C^{-1}T(t-t_0)u(t_0)+\int_{t_0}^t C^{-1}T(t-s)f(s,u(s))ds.$$

(b) Suppose u is a strong solution of (1.3), let $g(s)=T(t-s)u(s)$ for all $t_0\leqslant s\leqslant t$. Since u is differentiable a.e. on $[t_0,T]$, this implies that $g(s)$ is differentiable a.e. on $[t_0,T]$ and

$$\frac{d}{ds}g(s)=T(t-s)f(s,u(s))$$

a.e. on $[t_0,T]$. From the assumption of the functions f and u, with the same technique as used in the above, we have $T(t-s)f(s,u(s))$ is integrable and

$$\int_{t_0}^t\frac{d}{ds}g(s)ds=\int_{t_0}^t T(t-s)f(s,u(s))ds.$$

This implies $g(t)-g(t_0)=\int_{t_0}^{t} T(t-s)f(s,u(s))ds$ and hence

$$u(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds.$$

This proposition is proved now.

Theorem 3.2. Let A be an infinitesimal generator of a C-semigroup $\{T(t):t \geq 0\}$ on a Banach space X. If $f:[t_0,T]\times X \to R(C)$ is continuous in t and satisfies Lipschitz condition (with Lipschitz constant L) on X, then for every $u_0 \in R(C)$ the initial value problem (1.3) has a unique mild solution $u \in C([t_0,T]:X)$. Moreover, the mapping $u_0 \mapsto u$ satisfies Lipschitz condition from $R(C)$ into $C([t_0,T]:X)$.

Proof. For any given $u_0 \in R(C)$, we defined a mapping

$$F:C([t_0,T]:X) \to C([t_0,T]:X)$$

by

$$F(u)(t)=C^{-1}T(t-t_0)u(t_0)+\int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds \tag{3.2}$$

for $t_0 \leqslant t \leqslant T$. At first, we show that the mapping F is well-defined.

Since $u_0 \in R(C)$ and $f(s,u(s)) \in R(C)$, from Lemma 2.2 $C^{-1}T(t-t_0)u_0$ and $C^{-1}T(t-s)f(s,u(s))$ both exist. From the assumption of f and $u \in C[t_0,T]$, with the same technique used in the proof of Proposition 3.1, the function $t \mapsto f(t,u(t))$ is continuous on $[t_0,T)$. Thus, $\int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds$ exists for $t \in [t_0,T)$ and hence F is well-defined. Denoting $\|u\|_\infty$ for the norm of u as an element of $C([t_0,T]:X)$, it follows readily from the definition of F that

$$\begin{aligned}|F(u)(t)-F(v)(t)| &= |\int_{t_0}^{t} C^{-1}T(t-s)[f(s,u(s))-f(s,v(s))]ds| \\ &= \|C^{-1}\| M \int_{t_0}^{t} L|u(s)-v(s)|ds \\ &= \|C^{-1}\| ML \|u-v\|_\infty (t-t_0),\end{aligned} \tag{3.3}$$

for $t_0 \leqslant t \leqslant T$, where $M=sup\{\|T(s)\|:s \in [0,T]\}$. We denote F^n to be the n times' compositions of the function F for each $n \in N$. Using (3.3) and induction on n, one may have

$$F^n(u)(t)-F^n(v)(t) \leqslant \frac{[\|C^{-1}\|ML(t-t_0)]^n}{n!} \|u-v\|_\infty$$

for $t_0 \leqslant t \leqslant T$. Thus,

$$|F^n(u)(t)-F^n(v)(t)| \leqslant \frac{[\|C^{-1}\|MLT]^n}{n!} \|u-v\|_\infty$$

for $t_0 \leqslant t \leqslant T$ and hence,

$$\|F^n(u)-F^n(v)\|_\infty \leqslant \frac{[\|C^{-1}\|MLT]^n}{n!} \|u-v\|_\infty.$$

For n large enough, $\frac{[\|C^{-1}\|MLT]^n}{n!}<1$ (i.e., F^n is a contraction mapping from the Banach space $C([t_0,T]:X)$ into itself) and by a well known extension of the contraction principle, F has a unique fixed point u in $C([t_0,T]:X)$ (see [8, E. Kreysig, P323]). This fixed point is the desired solution of the integral equation (1.8) and which is the mild solution of (1.3).

The uniqueness of u and the Lipschitz continuity of the map $u_0 \mapsto u$ are consequences of the following argument. Let v be a mild solution of (1.1) on $[t_0,T]$ with the initial value v_0. Then, for $t_0 \leqslant t \leqslant T$

$$\begin{aligned}&|u(t)-v(t)| \\ &\leqslant |C^{-1}T(t-t_0)u_0-C^{-1}T(t-t_0)v_0| + \int_{t_0}^{t} |C^{-1}T(t-s)[f(s,u(s))-f(s,v(s))]|ds \\ &\leqslant \|C^{-1}\| \|T(t-t_0)\| |u_0-v_0| + \int_{t_0}^{t} \|C^{-1}\| \|T(t-s)\| |f(s,u(s))-f(s,v(s))|ds\end{aligned}$$

$$\leq \|C^{-1}\|M\ |u_0-v_0| + \|C^{-1}\|ML \int_{t_0}^{t} |u(s)-v(s)|\,ds \tag{3.4}$$

From Gronwall's inequality (see [14, P. 96]), the inequality (3.4) implies that

$$\begin{aligned}|u(t)-v(t)| &\leq \|C^{-1}\|M|u_0-v_0| \ \exp(\int_{t_0}^{t} M\|C^{-1}\|Lds)\\ &= \|C^{-1}\|M|u_0-v_0| \exp(M\|C^{-1}\|L(t-t_0))\\ &\leq (\|C^{-1}\|M \exp(M\|C^{-1}\|LT)|u_0-v_0|\end{aligned}$$

and therefore,

$$\|u-v\|_\infty \leq [\|C^{-1}\|M\ exp(M\ \|C^{-1}\|LT)]\ |u_0-v_0|$$

which yields the uniqueness of u and the Lipschitz continuity of the map $u_0 \mapsto u$. This theorem is completely proved now.

Corollary 3.3. If A and f satisfy the conditions of Theorem 3.2, then for every $g \in C([t_0,T]:X)$ the integral equation

$$w(t) = g(t) + \int_{t_0}^{t} C^{-1}T(t-s)f(s,w(s))ds \tag{3.5}$$

has a unique solution $w \in C([t_0,T]:X)$.

Proof. For any given $g \in C([t_0,T]:X)$, we define a mapping

$$F:C([t_0,T):X) \to C([t_0,T]:X)$$

by

$$F(u)(t) = g(t) + \int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds$$

for every $t_0 \leq t \leq T$. Denoting $\|u\|_\infty$ for the norm of u as an element of $C([t_0,T]:X)$, it follows readily from the definition of F that

$$\begin{aligned}|F(u)(t)-F(v)(t)| &= |\int_{t_0}^{t} C^{-1}T(t-s)[f(s,u(s))-f(s,v(s))]ds|\\ &\leq \|C^{-1}\|ML\|u-v\|_\infty (t-t_0).\end{aligned}$$

Using the same technique used in proof of Theorem 2.2, we have that

$$\|F^n(u)-F^n(v)\|_\infty \leq \frac{[\|C^{-1}\|\ MLT]^n}{n!}\ \|u-v\|_\infty.$$

For n large enough, $\dfrac{[\|C^{-1}\|MLT]^n}{n!} < 1$ and hence, F has a unique fixed point u in $C([t_0,T:]X)$. This fixed point is the desired solution of the integral equation (3.5).

Theorem 3.4. Let A be the infinitesimal generator of a C-semigroup $\{T(t):t \geq 0\}$ on X. If $f:[t_0,\infty) \times X \to R(C)$ is continuous in t for $t \geq 0$ and satisfies locally Lipschitz condition in x, uniformly in t on bounded intervals, then for every $u_0 \in R(C)$ the initial value problem (1.3) has a unique mild solution u on $[t_0,t_0+\delta(u_0)]$, where δ is a function from $R(C)$ into R^+.

Proof. Denote that

$$\begin{aligned}M(t_0) &= \sup\{\ \|T(t)\|:0 \leq t \leq t_0+1\},\\ N(t_0) &= \sup\{\ \|f(t,0)\|:0 \leq t \leq t_0+1\}\end{aligned}$$

and $L(\alpha,t')$ is the locally Lipschitz constant of f (as defined in (1.6)).
In the case of $u_0 \neq 0$, we set $\alpha(t_0) = 2|u_0|M(t_0)\|C^{-1}\|$ and

$$\delta(u_0)=\min\left\{1,\frac{|u_0|}{\alpha(t_0)L(\alpha(t_0),t_0+1)+N(t_0)}\right\}$$

Define a mapping

$$F:C([t_0,t_0+\delta(u_0)]:X)\to C([t_0,t_0+\delta(u_0)]:X)$$

by

$$F(u)(t)=C^{-1}T(t-t_0)u(t_0)+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds$$

for every $t_0\leq t\leq t_0+\delta(u_0)$, then for every $t\in[t_0,t_0+\delta(u_0)]$ and u belongs to the closed ball $B(0;\alpha(t_0))$ in $C([t_0,t_0+\delta(u_0)]:X)$ with radius $\alpha(t_0)$ centered at 0, we have the following estimate:

$$\begin{aligned}|F(u)(t)| &= |C^{-1}T(t-t_0)u_0+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds|\\
&\leq\|C^{-1}\|M(t_0)|u_0|+\int_{t_0}^{t}\|C^{-1}\|M(t_0)|f(s,u(s))-f(s,0)|ds+\int_{t_0}^{t}\|C^{-1}\|M(t_0)|f(s,0)|ds\\
&\leq\|C^{-1}\|M(t_0)|u_0|+\|C^{-1}\|M(t_0)L(\alpha(t_0),t_0+1)\|u\|_\infty(t-t_0)+\|C^{-1}\|M(t_0)N(t_0)(t-t_0)\\
&\leq\|C^{-1}\|M(t_0)\{|u_0|+[L(\alpha(t_0),t_0+1)\alpha(t_0)+N(t_0)](t-t_0)\}\\
&\leq\|C^{-1}\|M(t_0)\{|u_0|+[L(\alpha(t_0),t_0+1)\alpha(t_0)+N(t_0)]\delta(u_0)\}\\
&\leq 2\|C^{-1}\|M(t_0)|u_0|\\
&=\alpha(t_0).\end{aligned}$$

This implies $\|F(u)\|_\infty\leq\alpha(t_0)$ for all $u\in B(0;\alpha(t_0))$ and hence the mapping F maps the closed ball $B(0;\alpha(t_0))$ of $C([t_0,t_0+\delta(u_0)]:X)$ into itself. From the assumption of f, f satisfy a uniformly Lipschitz condition with Lipschitz constant $L=L(\alpha(t_0),t_0+1)$ and thus as in the proof of Theorem 3.2, it possesses a unique fixed point u in the closed ball $B(0;\alpha(t_0))$, [see 8, Kreysig P.303]. This fixed point is the desired solution of (1.3) on the interval $[t_0,t_0+\delta(u_0)]$. In the case of $u_0=0$, we take

$$\delta(u_0)=\delta(0)=\min\left\{1,\frac{1}{\|C^{-1}\|M(t_0)[L(1,t_0+1)+N(t_0)]}\right\}$$

and define a mapping

$$F:C([t_0,t_0+\delta(u_0)]:X)\to C([t_0,t_0+\delta(u_0)]:X)$$

by

$$F(u)(t)=C^{-1}T(t-t_0)u(t_0)+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds$$

for every $t_0\leq t\leq t_0+\delta(u_0)$, then for every $t\in[t_0,t_0+\delta(u_0)]$ and u belongs to the closed ball $B(0;1)$ of $C([t_0,t_0+\delta(u_0)]:X)$, we have the following estimate:

$$\begin{aligned}|F(u)(t)| &= |\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s)ds|\\
&\leq\|C^{-1}\|M(t_0)\int_{t_0}^{t}[|f(s,u(s))-f(s,0)|+|f(s,0)|]ds\\
&\leq\|C^{-1}\|M(t_0)\int_{t_0}^{t}[L(1,t_0+1)|u(s)|+N(t_0)]ds\\
&\leq\|C^{-1}\|M(t_0)[L(1,t_0+1)\|u\|_\infty+N(t_0)](t-t_0)\\
&\leq\|C^{-1}\|M(t_0)[L(1,t_0+1)+N(t_0)]\ \delta(u_0)\\
&\leq 1.\end{aligned}$$

This implies that $\|F(u)\|_\infty\leq 1$ for all $u\in B(0;1)$ and hence the mapping F maps the closed ball $B(0;1)$ of $C([t_0,t_0+\delta(u_0)]:X)$ into itself. From the assumption of f, f satisfy a uniformly Lipschitz condition with Lipschitz constant $L=L(1,t_0+1)$ and hence as in the proof of Theorem 3.2, it possesses a unique fixed point u in the closed ball $B(0;1)$. This fixed point is the desired solution of (1.3) on the interval $[t_0,t_0+\delta(u_0)]=[t_0,t_0+\delta(0)]$ and this theorem is completely proved now.

Corollary 3.5. Let A be the infinitesimal generator of a C-semigroup $\{T(t):t\geq 0\}$ on X and $R(C)=X$. If $f:[0,\infty)\times X\rightarrow R(C)$ is continuous in t and satisfies locally Lipschitz condition in x, uniformly in t on bounded intervals, then for each $u_0\in R(C)$, there exists $t_{max}\leqslant\infty$ such that

$$\begin{cases}\frac{d}{dt}u(t)=Au(t)+f(t,u(t))\\ u(0)=u_0\end{cases}\tag{3.6}$$

has a unique mild solution u on $[0,t_{max})$.
Moreover, if $t_{max}<\infty$, then $\lim_{t\uparrow t_{max}}\|u(t)\|=\infty$.

Proof. From Theorem 3.4 and the assumption $R(C)=X$, it follows that if u is a mild solution of (3.6) on the interval $[0,\tau]$, then u can be extended to the interval $[0,\tau+\delta]$ for some $\delta>0$, by defining $u(t)=w(t)$ on $[\tau,\tau+\delta]$ where $w(t)$ is the solution of the integral equation

$$\begin{cases}w(t)=C^{-1}T(t-\tau)u(\tau)+\int_\tau^t C^{-1}T(t-s)f(s,w(s))ds\\ w(\tau)=u(\tau)\end{cases}$$

for all $\tau\leqslant t\leqslant\tau+\delta$. Moreover, this δ depends only on the constants $\|u(\tau)\|$, $\alpha(\tau),N(\tau)$ which were defined in the proof of Theorem 3.4.

Let $[0,t_{max})$ be the maximal interval of existence of the mild solution u for the initial value problem (3.6). We will prove that the uniqueness of the local mild solution u for the initial value problem (3.6) as following. If v is a mild solution of the initial value problem (3.6), then on every closed interval $[0,t_0]$ both u and v exist, they coincide by the uniqueness argument given at the end of the proof of Theorem 3.2 with Lipschitz constant $L=L(\alpha,t_0+1)$ while $\|u(t)\|,\|v(t)\|\leqslant\alpha$, for $t\in[0,t_0]$. Therefore, both u and v have the same t_{max} and $u\equiv v$ on $[0,t_{max})$. Suppose that $t_{max}<\infty$ and $\lim_{t\uparrow t_{max}}\|u(t)\|<\infty$ then there exists a sequence $\{t_n:n\in N\}$, such that $t_n\uparrow t_{max}$ as $n\rightarrow\infty$, and $\|u(t_n)\|\leqslant K$ for all $n\in N$. Since $t_{max}<\infty$, we may set $M(t_{max})=\sup\{\|T(t)\|:0\leqslant t\leqslant t_{max}+1\}$, $N(t_{max})=\sup\{\|f(t,0)\|:0\leqslant t\leqslant t_{max}+1\}$ and denoted them by $\overline{M}$ and $\overline{N}$ respectively. We note that both $\overline{M}$and $\overline{N}$ are finite. If the sequence $\{t_n:n\in N\}$ has infinitely many $n\in N$ such that $\|u(t_n)\|=0$, then without of loss generality, we can assume that $\|u(t_n)\|=0$ for all $n\in N$ and hence as in the proof of Theorem 3.4, we may take

$$\delta(u(t_n))=\min\left\{1,\frac{1}{\|C^{-1}\|M(t_n)[L(1,t_n+1)+N(t_n)]}\right\}$$

for all $n\in N$. Since $t_n\uparrow t_{max}$ *as* $n\rightarrow\infty$, for each ε, $0<\varepsilon\leqslant\min\left\{1,\frac{1}{2\|C^{-1}\|\overline{M}[L(1,t_{max}+1)+\overline{N}]}\right\}$ there exists a $n_1\in N$ such that $t_{max}-t_n<\varepsilon$ for all $n\geqslant n_1$.

If there exists $n\geqslant n_1$ such that $\delta(u(t_n))=1$, then we can define a solution u on $[0,t_n+1]$ and $t_n+1>t_{max}$, which is contradicting to the definition of t_{max}. This implies

$$\delta(u(t_n))=\frac{1}{\|C^{-1}\|M(t_n)[L(1,t_n+1)+N(t_n)]}$$

for all $n\geqslant n_1$. Since $L(\alpha,t')$ is the local Lipschitz constant of f as defined by (1.6) for $\|u\|\leqslant\alpha$, $\|v\|\leqslant\alpha$ and $t\in[0,t']$, we can obtain $L(1,t_n+1)<L(1,t_{max}+1)$ for all $n\in N$. From $\overline{M}\geqslant M(t_n)$ and $\overline{N}\geqslant N(t_n)$ for all $n\in N$,

$$\delta(u(t_n))=\frac{1}{\|C^{-1}\|M(t_n)[L(1,t_n+1)+N(t_n)]}$$

$$\geq \frac{1}{\|C^{-1}\|\overline{M}[L(1,t_{max}+1)+\overline{N}]}$$

$$> \varepsilon$$

for all $n \geq n_1$. Thus, we can define a solution u on $[0,t_{n_1}+\delta(u(t_{n_1}))]$ with $t_{n_1}+\delta(u(t_{n_1}))>t_{n_1}+\varepsilon>t_{max}$. This implies u can be extended beyond t_{max} which contradict to the definition of t_{max} again. From above statement, the sequence $\{t_n : n \in N\}$ has at most finite many $n \in N$ such that $|u(t_n)|=0$. Without of loss generality, we can assume that $|u(t_n)| \neq 0$, for all $n \in N$ and hence as in the proof of Theorem 3.4 we may take

$$\delta(u(t_n))=\min\left\{1, \frac{|u(t_n)|}{\alpha(t_n)L(\alpha(t_n),t_n+1)+N(t_n)}\right\}$$

Since $|u(t_n)| \leq K$ for all $n \in N$,

$$\alpha(t_n)=2|u(t_n)|M(t_n)\|C^{-1}\| \leq 2K\overline{M}\|C^{-1}\|,$$

we define $\alpha=2K\overline{M}\|C^{-1}\|$ and $L=L(\alpha,t_{max}+1)$, then $L \geq L(\alpha(t_n),t_n+1)$. Since $t \uparrow t_{max}$ as $n \to \infty$, for each $0<\varepsilon \leq \min\left\{1, \frac{K}{2(\alpha L+\overline{N})}\right\}$ there exist a $n_0 \in N$ such that $|t_n-t_{max}|<\varepsilon$ for all $n \geq n_0$.

If there exists $n \geq n_0$ such that $\delta(u(t_n))=1$, we can define a solution u on $[0,t_n+1]$ and $t_n+1>t_{max}$, which contradict to the definition of t_{max}. This implies

$$\delta(u(t_n))=\frac{|u(t_n)|}{\alpha(t_n)L(\alpha(t_n),t_n+1)+N(t_n)} \geq \frac{|u(t_n)|}{\alpha L+\overline{N}}$$

for all $n \geq n_0$. If $|u(t_n)|$ does not converge to zero as $n \to \infty$, then there exists a subsequence $\{t_m : m \in N\}$ of the sequence $\{t_n : n \in N\}$ such that

$$|u(t_m)| > \varepsilon(\alpha L+\overline{N}).$$

(Because, α, L and $\overline{N}$ are independent for n_0, we can choose ε small enough such that $\varepsilon(\alpha L+\overline{N})$ is a small fixed number). Thus, there exists a positive number $m_0 \geq n_0$ such that

$$\delta(u(t_{m_0})) \geq \frac{|u(t_{m_0})|}{\alpha L+\overline{N}} > \varepsilon,$$

and hence we can be defined a solution u on $[0,t_{m_0}+\delta(u(t_{m_0}))]$ and $t_{m_0}+\delta(u(t_{m_0}))>t_{max}$. This implies u can be extended beyond t_{max} which contradict to the definition of t_{max}. Thus, $\lim_{n\to\infty}|u(t_n)|=0$ and hence $\lim_{n\to\infty}u(t_n)=0$. Now let's consider the initial value problem

$$\begin{cases} \frac{d}{dt}w(t)=Aw(t)+f(t,w(t)) & t \geq t_{max} \\ w(t_{max})=0 \end{cases} \tag{3.8}$$

From Theorem 3.4, we can obtain a mild solution $w(t)$ of the initial value problem (3.8) on $[t_{max},t_{max}+\delta_0]$, where $\delta_0=\min\left\{1, \frac{1}{\|C^{-1}\|\overline{M}(L(1,t_{max}+1)+\overline{N})}\right\}$. Since $\lim_{n\to\infty}u(t_n)=0$ and

$$\begin{aligned}\lim_{n\to\infty}u(t_n)&=\lim_{n\to\infty}[C^{-1}T(t_n)u_0+\int_0^{t_n}C^{-1}T(t_n-s)f(s,u(s))ds]\\ &=C^{-1}T(t_{max})u_0+\int_0^{t_{max}}C^{-1}T(t_{max}-s)f(s,u(s))ds,\end{aligned}$$

we obtain that

$$C^{-1}T(t_{max})u_0 + \int_0^{t_{max}} C^{-1}T(t_{max}-s)f(s,u(s))ds = 0$$

If we define $u(t)$ by

$$u(t)=\begin{cases} u(t) & \text{for } 0 \leqslant t < t_{max} \\ w(t) & \text{for } t_{max} \leqslant t \leqslant t_{max}+\delta_0 \end{cases}$$

then for each $t \in [t_{max}, t_{max}+\delta_0]$, we have the following equality:

$$\begin{aligned} u(t) &= C^{-1}T(t-t_{max})0 + \int_{t_{max}}^{t} C^{-1}T(t-s)f(s,u(s))ds \\ &= C^{-1}T(t-t_{max})[C^{-1}T(t_{max})u_0 + \int_0^{t_{max}} C^{-1}T(t_{max}-s)f(s,u(s))ds] \\ &\quad + \int_{t_{max}}^{t} C^{-1}T(t-s)f(s,u(s))ds \\ &= C^{-1}[C^{-1}T(t-t_{max})T(t_{max})]u_0 + \int_0^{t_{max}} C^{-1}[C^{-1}T(t-t_{max})T(t_{max}-s)]f(s,u(s))ds \\ &\quad + \int_{t_{max}}^{t} C^{-1}T(t-s)f(s,u(s))ds \\ &= C^{-1}T(t)u_0 + \int_0^t C^{-1}T(t-s)f(s,u(s))ds. \end{aligned}$$

This implies that the mild solution u of the initial value problem (3.6) on $[0,t_{max})$ can be extended to the interval $[0,t_{max}+\delta_0]$. It contradicts to the definition of t_{max} again and hence $\lim_{t \uparrow t_{max}} |u(t)| = \infty$.

Theorem 3.6. Suppose A generates a C-semigroup $\{T(t):t \geqslant 0\}$ on X. If $f:[t_0,T] \times X \to R(C)$ is continuously differentiable in the uniformly operator topology from $[t_0,T] \times X$ into $R(C)$, and u is the mild solution of the initial value problem (1.3) with $u_0 \in C(D(A))$ on $[t_0,T']$, where $t_0 < T' \leqslant T$, then u is the classical solution of the initial value problem (1.3) with $u_0 \in C(D(A))$ on $[t_0,T']$.

Proof. Let $B(s)=\frac{\partial}{\partial u}f(s,u)$, for all $s \in [t_0,T]$ and

$$g(t)=C^{-1}T(t-t_0)f(t_0,u(t_0))+C^{-1}AT(t-t_0)u_0 + \int_{t_0}^{t} C^{-1}T(t-s)\frac{\partial}{\partial s}f(s,u(s))ds \tag{3.9}$$

for all $t \in [t_0,T']$. Since f is continuous differentiable, the function $s \mapsto \frac{\partial}{\partial s} f(s,u(s))$ is continuous. Followed from Lemma 2.5, $t \mapsto \int_{t_0}^{t} T(t-s)\frac{\partial}{\partial s} f(s,u(s))ds$ is continuous. Since C^{-1} is linear and bounded, and the facts that the functions $t \mapsto T(t)x$ and $t \mapsto AT(t-t_0)u_0 = T(t-t_0)Au_0$ are continuous, this implies that $g \in C([t_0,T']:X)$. Since f is continuously differentiable, the function $s \mapsto B(s)$ is continuous from $[t_0,T]$ into $B(X)$, where $B(X)$ is the set of all bounded linear operator on X (see [17]). Thus, $h(t,u)=B(t)u$ is a continuous function in t from $[t_0,T]$ into X and $\|B(s)\|$ is bounded on the compact set $[t_0,T]$ and hence

$$\begin{aligned} |h(t,u)-h(t,v)| &= |B(t)u-B(t)v| \\ &\leqslant \|B(t)\| \, |u-v| \\ &\leqslant \alpha \, |u-v| \end{aligned}$$

where $\alpha = \sup\{\|B(s)\|: s \in [t_0,T]\}$ is a finite number. This implies $h(t,u)$ satisfies uniformly Lipschitz condition on X. From Corollary 3.3, the integral equation

$$\begin{aligned} w(t) &= g(t) + \int_{t_0}^{t} C^{-1}T(t-s)B(s)w(s)ds \\ &= g(t) + \int_{t_0}^{t} C^{-1}T(t-s)h(s,w(s))ds \end{aligned} \tag{3.10}$$

for $t \in [t_0,T']$ has an unique solution $w \in C([t_0,T']:X)$. Moreover, from hypothesis and definition of $B(s)$, we have that

$$f(s,u(s+h))-f(s,u(s))=B(s)[u(s+h)-u(s)]+\varepsilon_1(s,h) \tag{3.11}$$

and

$$f(s,u(s+H))-f(s,u(s+h))=\frac{\partial}{\partial s}f(s,u(s+h))+\varepsilon_2(s,h) \tag{3.12}$$

where $\frac{1}{h}\varepsilon_i(s,h)|\to 0$ as $h\to 0$ for each $i=1,2$. If $w_h(t)=\frac{1}{h}[u(t+h)-u(t)]-w(t)$, then from the definition of u, (3.9), (3.10), (3.11) and (3.12), we have that

$$\begin{aligned}
w_h(t) &= \frac{1}{h}\{C^{-1}T(t+h-t_0)u_0+\int_{t_0}^{t+h}C^{-1}T(t+h-s)f(s,u(s))ds\} \\
&\quad -\frac{1}{h}\{C^{-1}T(t-t_0)u_0+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds\} \\
&\quad -g(t)-\int_{t_0}^{t}C^{-1}T(t-s)B(s)w(s)ds \\
&= \frac{1}{h}\{C^{-1}T(t+h-t_0)u_0+\int_{t_0}^{t+h}C^{-1}T(t+h-s)f(s,u(s))ds\} \\
&\quad -\frac{1}{h}\{C^{-1}T(t-t_0)u_0+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds\} \\
&\quad -C^{-1}T(t-t_0)f(t_0,u(t_0))-C^{-1}AT(t-t_0)u_0 \\
&\quad -\int_{t_0}^{t}C^{-1}T(t-s)\frac{\partial}{\partial s}f(s,u(s))ds-\int_{t_0}^{t}C^{-1}T(t-s)B(s)w(s)ds \\
&= \frac{1}{h}[C^{-1}T(t+h-t_0)u_0-C^{-1}T(t-t_0)u_0]-C^{-1}AT(t-t_0)u_0 \\
&\quad +\frac{1}{h}\int_{t_0}^{t_0+h}C^{-1}T(t+h-s)f(s,u(s))ds-C^{-1}T(t-t_0)f(t_0,u(t_0)) \\
&\quad +\frac{1}{h}\int_{t_0+h}^{t+h}C^{-1}T(t+h-s)f(s,u(s))ds-\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds \\
&\quad -\int_{t_0}^{t}C^{-1}T(t-s)\frac{\partial}{\partial s}f(s,u(s))ds-\int_{t_0}^{t}C^{-1}T(t-s)B(s)w(s)ds
\end{aligned}$$

Since $\int_{t_0+h}^{t+h}C^{-1}T(t+h-s)f(s,u(s))ds=\int_{t_0}^{t}C^{-1}T(t-s)f(s+h,u(s+h))ds$, we have that

$$\begin{aligned}
&\frac{1}{h}\int_{t_0+h}^{t+h}C^{-1}T(t+h-s)f(s,u(s))ds-\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds \\
&=\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)[f(s+h,u(s+h))-f(s,u(s+h))]ds \\
&\quad +\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)[f(s,u(s+h))-f(s,u(s))]ds \\
&=\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)\left[\frac{\partial}{\partial s}f(s,u(s+h))h+\varepsilon_2(s,h)\right]ds \\
&\quad +\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)[B(s)(u(s+h)-u(s))+\varepsilon_1(s,h)]ds.
\end{aligned}$$

This implies that

$$\begin{aligned}
w_h(t) &= \frac{1}{h}[C^{-1}T(t+h-t_0)u_0-C^{-1}T(t-t_0)u_0]-C^{-1}AT(t-t_0)u_0 \\
&\quad +\frac{1}{h}\int_{t_0}^{t_0+h}C^{-1}T(t+h-s)f(s,u(s))ds-C^{-1}T(t-t_0)f(t_0,u(t_0)) \\
&\quad +\frac{1}{h}\int_{t_0}^{t}C^{-1}T(t-s)[\varepsilon_1(s,h)+\varepsilon_2(s,h)]ds \\
&\quad +\int_{t_0}^{t}C^{-1}T(t-s)\left[\frac{\partial}{\partial s}f(s,u(s+h))-\frac{\partial}{\partial s}f(s,u(s))\right]ds \\
&\quad +\int_{t_0}^{t}C^{-1}T(t-s)B(s)\left[\frac{u(s+h)-u(s)}{h}-w(s)\right]ds
\end{aligned} \tag{3.13}$$

Since f is continuously differentiable and the fact that

$$\frac{1}{h}\int_{t_0}^{t} C^{-1}T(t-s)[\varepsilon_1(s,h)+\varepsilon_2(s,h)]ds \leqslant \|C^{-1}\|MT\left[\frac{\varepsilon_1(s,h)}{h}+\frac{\varepsilon_2(s,h)|}{h}\right],$$

we have

$$\lim_{h\to 0}\left|\frac{\partial}{\partial s}f(s,u(s+h))-\frac{\partial}{\partial s}f(s,u(s))\right| =0$$

and

$$\lim_{h\to 0}\frac{1}{h}\left|\int_{t_0}^{t} C^{-1}T(t-s)[\varepsilon_1(s,h)+\varepsilon_2(s,h)]ds\right|=0$$

From above estimates and the facts

$$\lim_{h\to 0}\frac{1}{h}[C^{-1}T(t+h-t_0)u_0-C^{-1}T(t-t_0)u_0]=C^{-1}AT(t-t_0)u_0$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h} C^{-1}(t+h-s)f(s,u(s))ds=C^{-1}T(t-t_0)f(t_0,u(t_0)),$$

we have that each norm of the first six terms on the right-hand side of (3.13) tends to zero as $h\to 0$. Moreover, since $w_h(s)=\frac{1}{h}[u(s+h)-u(s)]-w(s)$, we have $|w_h(t)| \leqslant \varepsilon(h)+K\int_{t_0}^{t}|w_h(s)|\,ds$, where $\lim_{h\to 0}\varepsilon(h)=0$ and $K=\|C^{-1}\|M\sup\{\|B(s)\|:s\in[t_0,T]\}$. Form Gronwall's inequality, we have

$$|w_h(t)| \leqslant \varepsilon(h)\exp\left(\int_{t_0}^{t}Kds\right)\leqslant \varepsilon(h)\exp(KT)$$

and thus $\lim_{h\to 0}|w_h(t)|=0$. This implies that $u(t)$ is differentiable on $[t_0,T']$ and $\frac{d}{dt}u(t)=w(t)$. Since $w\in C([t_0,T']:X)$, u is continuously differentiable on $[t_0,T']$. Also since f, u are continuously differentiable, $s\mapsto f(s,u(s))$ is a continuously differentiable function on $[t_0,T']$. From Lemma 2.10, $v(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds$ is the classical solution of the initial value problem:

$$\begin{cases}\frac{d}{dt}v(t)=Av(t)+f(t,u(t)) & 0\leqslant t_0<t<T'\\ v(t_0)=u_0\end{cases}$$

Since u is the mild solution of the initial value problem (1.3), $u(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^{t}C^{-1}T(t-s)f(s,u(s))ds$ for all $t\in[t_0,T']$ and hence u is a classical solution of the initial value problem on $[t_0,T']$. This theorem is completely proved now.

From Theorem 3.2. and Theorem 3.6., one can obtain the following Corollary immediately.

Corollary 3.7. Suppose that A generates a C-semigroup $\{T(t):t\geqslant 0\}$ on X. If $f:[t_0,T]\times X\to R(C)$ is continuously differentiable in the uniformly operator topology from $[t_0,T]\times X$ into $R(C)$ and f is uniformly Lipschitz continuous on X, then for all $u_0\in C(D(A))$, the initial value problem (1.3) has a unique classical solution on $[t_0,T]$.

Corollary 3.8. Suppose A generates a C-semigroup $\{T(t):t\geqslant 0\}$ on X and $R(C)=X$. If $f:[t_0,T]\times X\to R(C)$ is continuously differentiable in the uniformly operator topology from $[t_0,T]\times X$ into $R(C)$, then for every $u_0\in C(D(A))$ there exist a t_{max}, $t_0<t_{max}\leqslant T$ such that the initial value problem (1.3) has a uniue classical solution on $[t_0,t_{max})$.

Proof. Since f is continuously differentiable, $B(t,u)=\frac{\partial}{\partial u}f(s,u)$ is continuous on $[t_0,T]\times X$. Thus, for each $\alpha>0$, $\sup\{\|B(t,u)\|:t\in[t_0,T],|u|\leqslant\alpha\}$ is finite, and we denote it by $L(\alpha)$. From the Mean value theorem for operators (see [17, A. Wouk, 265~267]),

$$\begin{aligned} |f(t,u)-f(t,v)| &\leq \|B(t, \theta u+(1-\theta)v)\| \, |u-v| \\ &\leq L(\alpha)|u-v| \end{aligned}$$

for all $|u|$, $|v| \leq \alpha$, $0<\theta<1$. This implies that f is locally Lipschitz continuous in u uniformly on $[t_0,T]$. This corollary is followed from Corollary 3.5. and Theorem 3.6.

Theorem 3.9. Suppose A generates a C-semigroup $\{T(t):t \geq 0\}$ on a reflexive Banach space X. If f satisfies the Lipschitz condition in both variables from $[t_0,T] \times X$ into $R(C)$, then for all $u_0 \in C(D(A))$, the initial value problem (1.3) has the unique strong solution on $[t_0,T]$ which is the mild solution of the initial value problem (1.3).

Proof. Since f satisfy the inequality (1.7), f is continuous in t on $[t_0,T]$ and satisfies uniformly Lipschitz condition (with constant L) on X. From Theorem 3.2, the initial value problem (1.3) has a unique mild solution $u \in C([t_0,T]:X)$. Since

$$|f(t,u(t))-f(t_0,u(t_0))| \leq L(|t-t_0|+\|u-u(t_0)\|_\infty)$$

and $u \in C([t_0,T]:X)$, one can easily obtain that

$$|f(t,u(t))| \leq |f(t_0,u(t_0))|+L(|T-t_0|+\|u-u(t_0)\|_\infty)$$

is finite. Thus, there is a $0<N<\infty$ such that $|f(t,u(t))| \leq N$ for all $t \in [t_0,T]$. For every $h \in (0,t-t_0)$, we have that

$$\begin{aligned} u(t+h)-u(t) &= C^{-1}T(t+h-t_0)u_0 + \int_{t_0}^{t_0+h} C^{-1}T(t+h-s)f(s,u(s))ds + \int_{t_0+h}^{t+h} C^{-1}T(t+h-s)f(s,u(s))ds \\ &\quad - [C^{-1}T(t-t_0)u_0 + \int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds] \\ &= C^{-1}T(t+h-t_0)u_0 - C^{-1}T(t-t_0)u_0 + \int_{t_0}^{t_0+h} C^{-1}T(t+h-s)f(s,u(s))ds \\ &\quad + \int_{t_0}^{t} C^{-1}T(t-s)[f(s+h,u(s+h))-f(s,u(s))]ds. \end{aligned}$$

Since $u_0 \in C(D(A))$, there is a constant $\theta \in [0,1]$ such that

$$C^{-1}T(t+h-t_0)u_0 - C^{-1}T(t-t_0)u_0 = C^{-1}T(t-t_0+\theta h)Au_0 h.$$

Thus, one may obtain

$$\begin{aligned} |u(t+h)-u(t)| &= |C^{-1}T(t-t_0+\theta h)Au_0 h| + \|C^{-1}\| \int_{t_0}^{t_0+h} \|T(t+h-s)\| \, |f(s,u(s))| ds \\ &\quad + \|C^{-1}\| \int_{t_0}^{t} \|T(t-s)\| \, |f(s+h,u(s+h))-f(s,u(s))| ds \\ &\leq h\|C^{-1}\|M|Au_0| + h\|C^{-1}\|MN + \|C^{-1}\|ML \int_{t_0}^{t} (h+|u(s+h)-u(s)|)ds \\ &\leq h\|C^{-1}\|M|Au_0| + h\|C^{-1}\|MN + h\|C^{-1}\|ML(T-t_0) + \|C^{-1}\|ML\int_{t_0}^{t}|u(s+h)-u(s)|ds, \end{aligned}$$

where $M=\sup\{\|T(t)\|:t \in [0,T]\}$. Set

$$K=\|C^{-1}\|M|Au_0| + \|C^{-1}\|MN + \|C^{-1}\|ML(T-t_0),$$

then

$$|u(t+h)-u(t)| \leq Kh + \|C^{-1}\| ML \int_{t_0}^{t}|u(s+h)-u(s)|ds.$$

From Gronwall's inequality,

$$\begin{aligned} |u(t+h)-u(t)| &\leq Kh\exp(\int_{t_0}^{t} \|C^{-1}\| MLds) \\ &= Kh\exp(\|C^{-1}\|ML(t-t_0)) \\ &\leq [K\exp(\|C^{-1}\|MLT)]h. \end{aligned}$$

Thus u satisfy uniformly Lipschitz condition with Lipschitz constant $Kexp(\|C^{-1}\|MLT)$ and hence

$$
\begin{aligned}
&|f(t_1,u(t_1))-f(t_2,u(t_2))| \\
&\leqslant L(|t_1-t_2|-|u(t_1)-u(t_2)|) \\
&\leqslant L(|t_1-t_2|-[Kexp(\|C^{-1}\| MLT)]|t_1-t_2|) \\
&=(L+L[Kexp(\|C^{-1}\| MLT)]|t_1-t_2|.
\end{aligned}
$$

It implies that the function $t \mapsto f(t,u(t))$ satisfies uniformly Lipschitz condition on $[t_0,T]$. Let $f(t)$ be defined as $f(t,u(t))$ where u is the mild solution of the initial value problem (1.3), then $f(t)$ satisfies the conditions of Corollary 2.3. Thus, the initial value problem

$$
\begin{cases} \frac{d}{dt}v(t)=Av(t)+f(t,u(t)) \\ v(t_0)=u_0 \end{cases} \tag{3.14}
$$

has a unique strong solution v on $[t_0,T]$ satisfying

$$v(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds.$$

However, u is the unique mild solution of the initial value problem (1.4), we have that

$$u(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds.$$

and hence $u(t)=v(t)$ on $[t_0,T]$. Since $v(t)$ is the strong solution of the initial value problem (3.14) on $[t_0,T]$ and $u(t)=v(t)$ on $[t_0,T]$, $u(t)$ is a continuous function on $[t_0,T]$ and satisfies $u(t)\in D(A)$ a.e. on $[t_0,T]$, $u(t_0)=u_0$ and $u'(t)=Au(t)+f(t,u(t))$ a.e. on $[t_0,T]$. This implies that u is the unique strong solution of the initial value problem on $[t_0,T]$ and Theorem 3.9. is completely proved now.

Let A be the infinitesimal generator of a C-semigroup $\{T(t):t\geqslant 0\}$ on a Banach space X. We endow the domain of A with the graph norm $|\cdot|_A$, that is, for every $x\in D(A)$, we define $|x|_A=|x|+|Ax|$. It is well-known that $D(A)$ with the norm $|\cdot|_A$ is a Banach space. We denote it by Y. The completeness of Y is a direct consequence of the closeness of A. Clearly, $Y\subset X$ and $T(t)(D(A))\subset D(A)$, and hence $\{T(t):t\geqslant 0\}$ is a C-semigroup on Y.

Theorem 3.10. Suppose that A generates a C-semigroup $\{T(t):t\geqslant 0\}$ on a Banach space X and $C(Y)$ is a closed subspace of Y. If $f:[t_0,T]\times Y\rightarrow C(Y)$ satisfies the uniformly Lipschitz condition in Y, and for each $y\in Y$, $f(t,y)$ is continuous from $[t_0,T]$ into $C(Y)$, then for all $u_0\in C(D(A))$, the initial value problem (1.3) has a unique classical solution on $[t_0,T]$.

Proof. We apply Theorem 3.2 on the Banach space Y, then we can obtain a function $u\in C([t_0,T]:Y)$ satisfying following equation

$$u(t)=C^{-1}T(t-t_0)u_0+\int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds \tag{3.15}$$

in Y. Let $g(s)=f(s,u(s))$, where $u(s)$ is defined as (3.15). From our assumptions, it follows that $g(s)\in C(D(A))$, for all $s\in[t_0,T]$ and

$$
\begin{aligned}
|g(t)-g(s)|_A &= |f(t,u(t))-f(s,u(s))|_A \\
&\leqslant |f(t,u(t))-f(t,u(s))|_A+|f(t,u(s))-f(s,u(s))|_A \\
&\leqslant L|u(t)-u(s)|_A+|f(t,u(s))-f(s,u(s))|_A
\end{aligned}
$$

for any fixed $s\in[t_0,T]$. Since $u\in C([t_0,T]:Y)$ and for each $y\in Y$, $f(t,y)$ is continuous from $[t_0,T]$ into $C(Y)$, we have that $\lim_{t\to s}|g(t)-g(s)|_A=0$. From the definition of the graph norm, we have

$$|g(t)-g(s)|\leqslant|g(t)-g(s)|_A$$

and hence $g(s)$ is continuous on $[t_0,T]$. Since $C(D(A)) \subset D(A)$ and $g(s) \in C(D(A))$ for all $s \in [t_0,T]$, this implies

$$|AC^{-1}g(t)-AC^{-1}g(s)| \leqslant \|C^{-1}\| \, |Ag(t)-Ag(s)| \leqslant \|C^{-1}\| \, |g(t)-g(s)|_A.$$

Thus, $|AC^{-1}g(t)-AC^{-1}g(s)|$ converge to zero as $t \to s$ and hence $AC^{-1}g(s)$ is continuous on $[t_0,T]$. From Corollary 2.4, the initial value problem

$$\begin{cases} \frac{d}{dt}v(t)=Av(t)+g(t) \\ v(t_0)=u_0 \end{cases} \tag{3.16}$$

has a unique classical solution v on $[t_0,T]$ and hence v is the mild solution of the initial value problem (3.16) on $[t_0,T]$. This implies that,

$$\begin{aligned} v(t) &= C^{-1}T(t-t_0)u_0 + \int_{t_0}^{t} C^{-1}T(t-s)g(s)ds \\ &= C^{-1}T(t-t_0)u_0 + \int_{t_0}^{t} C^{-1}T(t-s)f(s,u(s))ds = u(t) \end{aligned}$$

Since $u(t)=v(t)$ is the unique classical solution of (3.16) on $[t_0,T]$, $u \in C[t_0,T]$, $u \in C^1(t_0,T)$, $u(t) \in D(A)$ for all $0<t<T$ and $u(t)$ satisfies

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t,u(t)) \\ u(t_0)=u_0 \end{cases}$$

Thus, u is the unique classical solution of (1.3) on $[t_0,T]$. This theorem is completely proved now.

Remark. If in the preceding theorem we assume only that $f:[t_0,T] \times Y \to Y$ satisfies locally Lipschitz condition in Y, uniformly in $[t_0,T]$ and $C(Y)=Y$, one can follow from Corollary 3.5 that the initial value problem (1.3) possesses a classical solution on a maximal interval $[t_0,t_{max})$ for every $u_0 \in C(D(A))$. Moreover, if $t_{max}<T$ then $\lim_{t \uparrow t_{max}} (|u(t)|+|Au(t)|)=\infty$.
In this situation, one may have noted that $|u(t)|$ is bounded on $[t_0,t_{max})$ and $|Au(t)| \to \infty$ as $t \uparrow t_{max}$.

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C-半群與半線性方程式

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本文是考慮抽象半線性微分方程式

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t,u(t)) & 0\leqslant t_0<t<T \\ u(t_0)=u_0 & u_0\in C(D(A)) \end{cases} \tag{0.1}$$

其中A是一個在 Banach 空間X上之C-半群的生成元，$f:[t_0,T]\times X\rightarrow X$為一個函數。我們給予函數$f$某些適當的條件，使得以上之抽象半線性方程式 (0.1) 有唯一的古典解、強解或弱解。我們也找出弱解存在的最大時間範圍，並探討此解在趨近邊界時的行為；此外，我們也證明了解對初值條件的連續性。為了證明這些結果，我們先證明 (0.1) 所對應的非齊次方程式

$$\begin{cases} \frac{d}{dt}u(t)=Au(t)+f(t) & 0\leqslant t_0<t<T \\ u(t_0)=u_0 & u_0\in C(D(A)) \end{cases} \tag{0.2}$$

在給予非齊次項函數C某些適當的條件，使得以上之抽象非齊次微分方程式 (0.2) 有唯一的古點解、強解或弱解。本文最大之特色是無需假設這個C-半群是指數有界 (exponential bounded)。

關鍵詞：C-正則半群　指數有界C-正則半群　抽象非齊次微分方程式　抽象半線性微分方程式