國立臺灣師範大學物理所碩士論文

Current-Current correlator in Graphene

石墨烯中電子電子交互作用之研究

吳治緯

指導教授：高賢忠

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摘　要

在本文中，我們利用量子場論計算了Graphene電子間交互作用對current-current correlator造成的一階修正。這個結果可以用來計算Graphene的conductivity 。

關鍵詞：石墨稀、量子場論、微擾理論

Abstract

In this paper, we use quantum field theory to calculate current-current correlator in Graphene to first order in electron-electron interaction. The result can be used to calculate the correction to the conductivity of Graphene.

Keyword： Graphene、QFT、Perturbation theory

Contents

1 Introduction **1**

2 Vacuum polarization **4**

3 First order correction of perturbation to current-current correlator **9**

4 Conclusion **23**

Reference **24**

Appendices **25**

A Tensor analysis for egg diagram **25**

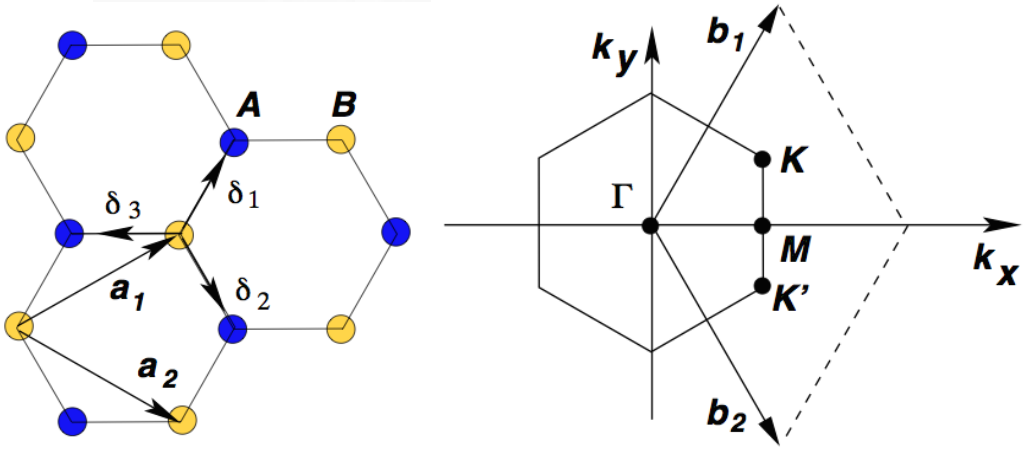
B Tensor analysis for self energy diagram **46**

Chapter 1

Introduction

石墨稀(Graphene)是碳原子用六邊形的形式排列成蜂巢晶格結構(如圖1.1[2])且厚度只有一顆碳原子的二維平面材料，碳原子間用1個s軌域和兩個p軌域所形成鍵結長度為1.42$\AA$的σ鍵以及剩下來1個未混成的sp_z軌域所形成的π鍵來作連接。

在圖1.1(a)中，向量$\vec{a}_1 = \frac{a}{2}(3, \sqrt{3})$,$\vec{a}_2 = \frac{a}{2}(3, -\sqrt{3})$是Graphene的基本晶格向量，而在圖1.1(b)所描述的是它倒置晶格的first Brillouin zone，其形狀剛好也是六邊形。圖中的$\vec{b}_1$,$\vec{b}_2$是它的倒置晶格向量，由於倒置晶格向量與基本晶格向量之間滿足$\vec{b}_i \cdot \vec{a}_j = 2\pi\delta_{ij}$的關係，所以$\vec{b}_1$,$\vec{b}_2$分別等於$\frac{2\pi}{3a}(1, \sqrt{3})$和$\frac{2\pi}{3a}(1, -\sqrt{3})$。



(a) (b)

Brillouin zone的K點以及K'點為Dirac point，它們在倒置晶格的位置分別為$(\frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a})$以及$(\frac{2\pi}{3a}, -\frac{2\pi}{3\sqrt{3}a})$。

電子在某一碳原子及其臨近的原子間跳躍的tight binding Hamiltonian為

$$H = -t \sum_{\langle i,j \rangle, \sigma} (a^{\dagger}_{\sigma,i} b_{\sigma,j} + h.c). \tag{1.1}$$

上式中的$a^{\dagger}_{\sigma,i}$為電子的creation operator，而下標的i所對應的是它的位置$\mathbf{R}_i$(圖1.1(a)中藍點A的位置)，$b_{\sigma,j}$為annihilation operator，它所對應的位置為圖1.1(a)中黃點B的位置，σ則是電子的spin，t為電子跳到最鄰近原子的跳躍能，其值約為2.8eV，而±號分別為上π鍵以及下π^*鍵。

而這個Hamiltonian所產生的能帶(energy band)E為

$$\begin{aligned} E_{\pm}(\mathbf{k}) &= \pm t\sqrt{3 + f(\mathbf{k})}, \\ f(\mathbf{k}) &= 2cos(\sqrt{3}k_y a) + 4cos(\frac{\sqrt{3}}{2}k_y a)cos(\frac{3}{2}k_x a). \end{aligned} \tag{1.2}$$

如果對它的能帶結構作圖(圖1.2[2]) 可以發現在Dirac point附近會產生圓錐形的

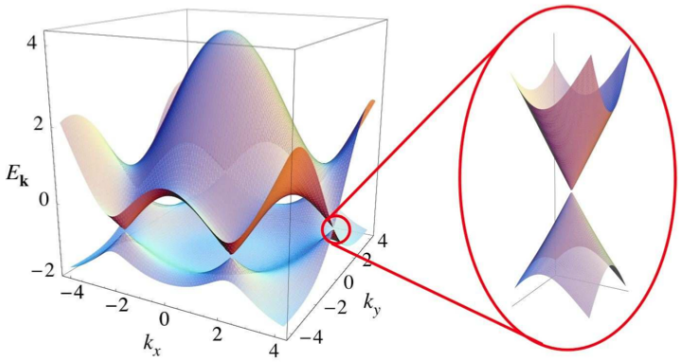


Figure 1.2

能帶結構，而它的dispersion relation為

$$\begin{aligned} E_{\pm}(\mathbf{q}) &\approx \pm v_F |\mathbf{q}| + O(q^2), \\ \mathbf{k} &= \mathbf{q} + \mathbf{K}. \end{aligned} \tag{1.3}$$

其中$\mathbf{q}$是在Dirac point附近的動量，而v_F是Fermi velocity，其值約為光速c的三百分之一。上式的Energy dispersion與一般的自由電子的$\frac{q^2}{2m}$不同，類似于一個相對論性的無質量粒子。

此外由於Graphene的單位晶格中有兩個原子A跟B；且對Hamiltonian中電子的operator在Dirac point K以及K'附近作傅立葉變換。

$$a_n = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{R}_i} a(\mathbf{k}). \tag{1.4}$$

則a(b)位置上的場$a_n(b_n)$會近似兩個新的場$a_{1,n}(b_{1,n})$與$a_{2,n}(b_{2,n})$的和

$$\begin{aligned} a_n &\sim \mathrm{e}^{-i\mathbf{K}\cdot\mathbf{R}_n} a_{1,n} + \mathrm{e}^{-i\mathbf{K}'\cdot\mathbf{R}_n} a_{2,n}, \\ b_n &\sim \mathrm{e}^{-i\mathbf{K}\cdot\mathbf{R}_n} b_{1,n} + \mathrm{e}^{-i\mathbf{K}'\cdot\mathbf{R}_n} b_{2,n}. \end{aligned} \tag{1.5}$$

利用上式，再經過一些計算可以把Hamiltonian改寫成massless的Dirac like Hamiltonian

$$H \simeq -iv_F \int dxdy[\hat{\Psi}_1^\dagger(\mathbf{r})\sigma\cdot\nabla\hat{\Psi}_1(\mathbf{r}) + \hat{\Psi}_2^\dagger(\mathbf{r})\sigma^\dagger\cdot\nabla\hat{\Psi}_2(\mathbf{r})]. \tag{1.6}$$

上式中的$\hat{\Psi}_i^\dagger = (a_i^\dagger, b_i^\dagger)$，i=1,2 分別代表Dirac point K以及K'，而σ為Pauli matrix。這是二次量子化之後的形式，如果用一次量子化去看，電子的wave function $\psi(\vec{r})$會遵守2維的Dirac equation。

$$E\psi(\mathbf{r}) = v_F\sigma\cdot\nabla\psi(\mathbf{r}). \tag{1.7}$$

由於這些特性，K點和K'兩點被稱作為Dirac point，而電子在Graphene中也因此為速度是v_F且massless的Dirac fermion。Graphene中會產生許多像是Klein paradox等現象，因此使研究者感到興趣。

由於在Dirac point 上面幾乎沒有density of states，因此幾乎沒有Screening effect，造成我們在探討它的Transport時需要考慮長距離的Coulomb interaction。又因為fermi velocity是光速的$\frac{1}{300}$所以它的coupling constant α_g約等於$\frac{2.2}{\epsilon_r}$，ϵ_r為介電常數[6]。根據理論所預期Graphene的conductivity為$\sigma_0 = \frac{\mathrm{e}^2}{4\hbar}$，但這個數值從來沒有被測量到，實際所測量到的結果都比σ_0大了一些。會產生這樣的差異的原因就是歸因于Coulomb interaction，而σ_0與σ的比值為

$$\frac{\sigma}{\sigma_0} = 1 + C\alpha_g \tag{1.8}$$

上式中的C為某常數。而根據文獻，常數C會因為不同的momentum cuffoff以及不同的方法如Kubo formula,polarization operator,kinetic equation.等方法去算的話會得到不同的結果[5][6]。

Chapter 2

Vacuum polarization

上一章提到電子在Graphene中為Dirac fermion，所以我們可以利用量子場論探討Graphene的物理。這一章我們利用量子場論來計算current-current correlator。首先我們考慮最低階的效應，在這一階的計算中完全不需要考慮電子間的交互作用，即

$$\begin{aligned}&\langle 0|T(J_\mu(y)J_\nu(0))|0\rangle\\ =&\langle 0|T(\bar{\psi}_a(y)(\gamma_\mu)_{ab}\psi_b(y)\bar{\psi}_{a'}(0)(\gamma_\nu)_{a'b'}\psi_{b'}(0))|0\rangle.\end{aligned} \tag{2.1}$$

我們在這裡使用Wick's theorem來作計算，而可能有的contraction為

$$\begin{aligned}&\bar{\psi}_a(y)(\gamma_\mu)_{ab}\psi_b(y)\bar{\psi}_{a'}(0)(\gamma_\nu)_{a'b'}\psi_{b'}(0)\\ +&\bar{\psi}_a(y)(\gamma_\mu)_{ab}\psi_b(y)\bar{\psi}_{a'}(0)(\gamma_\nu)_{a'b'}\psi_{b'}(0).\end{aligned} \tag{2.2}$$

(2.1)式中的$\langle 0|T(\bar{\psi}(x)_i\psi(y)_j)|0\rangle$可以用fermion的propagator$S(y-x)_{ij}$變換示來表示。之後我們利用傅立葉變換將它轉換到動量空間

$$\begin{aligned}&\langle 0|S(-y)_{b'a}(\gamma_\mu)_{ab}S(y)_{ba'}(\gamma_\nu)_{a'b'}|0\rangle+\langle 0|S(0)_{ba}(\gamma_\mu)_{ab}S(0)_{b'a'}(\gamma_\nu)_{a'b'}|0\rangle\\ =&-\mathrm{Tr}\{S(-y)\gamma_\mu S(y)\gamma_\nu\}+\mathrm{Tr}\{S(0)\gamma_\mu\}\{S(0)\gamma_\nu\},\\ =&-\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}\mathrm{e}^{i(p-k)\cdot y}\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu\tilde{S}(p)\gamma_\nu\}\\ &+\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu\}\{\tilde{S}(p)\gamma_\nu\}.\end{aligned} \tag{2.3}$$

因為整個propagator還是坐標空間，所以必須再對它作一次傅立葉變換$\tilde{S}(q)_{ij} = \int d^3y \mathrm{e}^{iq\cdot y} S(y)_{ij}$，即

$$
\begin{aligned}
&-\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\int d^3y \mathrm{e}^{iq\cdot y}\mathrm{e}^{i(p-k)\cdot y}\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu \tilde{S}(p)\gamma_\nu\} \\
&+\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\int d^3y \mathrm{e}^{iq\cdot y}\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu\}\{\tilde{S}(p)\gamma_\nu\}, \\
=&-\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\int d^3y \mathrm{e}^{i(q+p-k)\cdot y}\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu \tilde{S}(p)\gamma_\nu\} \\
&+\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\int d^3y \mathrm{e}^{iq\cdot y}\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu\}\{\tilde{S}(p)\gamma_\nu\}.
\end{aligned}
\tag{2.4}
$$

$$
\tag{2.5}
$$

利用$\frac{1}{(2\pi)^3}\int d^3y \mathrm{e}^{iq\cdot y} = \delta^3(q)$，上式可寫成

$$
\begin{aligned}
&-\frac{1}{(2\pi)^3}\int d^3k \int d^3p\ \delta^3(q+p-k)\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu \tilde{S}(p)\gamma_\nu\} \\
&+\frac{1}{(2\pi)^3}\int d^3k \int d^3p\ \delta^3(q)\mathrm{Tr}\{\tilde{S}(k)\gamma_\mu\}\{\tilde{S}(p)\gamma_\nu\}, \\
=&-\frac{1}{(2\pi)^3}\int d^3k\ \mathrm{Tr}\{\tilde{S}(k)\gamma_\mu \tilde{S}(k-q)\gamma_\nu\} \\
&+\frac{1}{(2\pi)^3}\int d^3k \int d^3p\ \mathrm{Tr}\{\tilde{S}(k)\gamma_\mu\}\{\tilde{S}(p)\gamma_\nu\}\delta^3(q).
\end{aligned}
\tag{2.6}
$$

Fermion的propagator $\tilde{S}(k)$原本的形式為$\frac{i}{\not{k}-m+i\varepsilon}$。其中$\not{k} = k^\lambda\gamma_\lambda$，$\gamma_\lambda$為Dirac matrices，而$i\varepsilon$則是真空對真空的boundary condition，將它帶入上式後我們得到

$$
\begin{aligned}
&-\frac{1}{(2\pi)^3}\int d^3k\ \mathrm{Tr}\{\frac{i}{\not{k}-m+i\varepsilon}\gamma_\mu \frac{i}{\not{k}-\not{q}-m+i\varepsilon}\gamma_\nu\} \\
&+\frac{1}{(2\pi)^3}\int d^3k \int d^3p\ \mathrm{Tr}\{\frac{i}{\not{k}-m+i\varepsilon}\gamma_\mu\}\mathrm{Tr}\{\frac{i}{\not{p}-m+i\varepsilon}\gamma_\nu\}\delta^3(q).
\end{aligned}
\tag{2.7}
$$

因為$i\epsilon$只是決定路徑積分繞過極點$k_0 = \pm(\mathbf{k}^2+m^2)^{\frac{1}{2}}$的方式，所以整理式子的時候可以將它暫時省略，等到整理完再放回來。整理後，上式可化為

$$
\begin{aligned}
&-\frac{1}{(2\pi)^3}\int d^3k\ \mathrm{Tr}\{\frac{i}{(\not{k}-m)}\frac{(\not{k}+m)}{(\not{k}+m)}\gamma_\mu \frac{i}{(\not{k}-\not{q}-m)}\frac{(\not{k}-\not{q}+m)}{(\not{k}-\not{q}+m)}\gamma_\nu\} \\
&+\int \frac{d^3k}{(2\pi)^3}d^3p\ \mathrm{Tr}\{\frac{i}{(\not{k}-m)}\frac{(\not{k}+m)}{(\not{k}+m)}\gamma_\mu\}\mathrm{Tr}\{\frac{i}{(\not{p}-m)}\frac{(\not{p}+m)}{(\not{p}+m)}\gamma_\nu\}\delta^3(q), \\
=&-\frac{1}{(2\pi)^3}\int d^3k \frac{-\mathrm{Tr}\{(\not{k}+m)\gamma_\mu(\not{k}-\not{q}+m)\gamma_\nu\}}{(k^2-m^2)[(k-q)^2-m^2]} \\
&+\frac{1}{(2\pi)^3}\int d^3k \int d^3p\ \frac{-\mathrm{Tr}\{(\not{k}+m)\gamma_\mu\}\mathrm{Tr}\{(\not{p}+m)\gamma_\nu\}}{(k^2-m^2)(p^2-m^2)}\delta^3(q).
\end{aligned}
\tag{2.8}
$$

又因為奇數個Dirac matrix的trace總是零，所以上式變成

$$
\begin{aligned}
&-\frac{1}{(2\pi)^3}\int d^3k\frac{-\mathrm{Tr}\{\not{k}\gamma_\mu(\not{k}-\not{q})\gamma_\nu+m^2\gamma_\mu\gamma_\nu\}}{(k^2-m^2)[(k-q)^2-m^2]}\\
&+\frac{1}{(2\pi)^3}\int d^3k\int d^3p\ \frac{-(\mathrm{Tr}\{\not{k}\gamma_\mu\}+\mathrm{Tr}\{\not{p}\gamma_\nu\})}{(k^2-m^2)(p^2-m^2)}\delta^3(q).
\end{aligned}
\tag{2.9}
$$

除此之外，根據Dirac matrix取trace的特性

$$
\begin{cases}
\mathrm{Tr}\{\gamma_\lambda\ \gamma_\mu\}=4\eta_{\lambda\mu},\\
\mathrm{Tr}\{\gamma_\lambda\ \gamma_\mu\ \gamma_\rho\ \gamma_\nu\}=4(\eta_{\lambda\mu}\eta_{\rho\nu}+\eta_{\lambda\nu}\eta_{\rho\mu}-\eta_{\lambda\rho}\eta_{\mu\nu}),
\end{cases}
\tag{2.10}
$$

所以(2.9)取完trace變成

$$
\begin{aligned}
&-\frac{4}{(2\pi)^3}\int d^3k\frac{-[m^2\eta_{\mu\nu}+k_\mu(k-q)_\nu+k_\nu(k-q)_\mu-k\cdot(k-q)\eta_{\mu\nu}]}{(k^2-m^2)[(k-q)^2-m^2]}\\
&+\frac{4}{(2\pi)^3}\int d^3k\int d^3p\ \frac{-(k_\mu+p_\nu)}{(k^2-m^2)(p^2-m^2)}\delta^3(q).
\end{aligned}
\tag{2.11}
$$

上式中由於第二項的積分式是奇函數，而且這裏的積分都是從負無窮大到正無窮大，所以此項的積分會等於零。接著我們引進了Feynman trick讓計算變得比較容易

$$
\frac{1}{AB}=\int_0^1 d\alpha\frac{1}{[\alpha A+(1-\alpha)B]^2}.
\tag{2.12}
$$

在(2.11)中，我們把$(k-q)^2-m^2$和k^2-m^2分別設為A和B，再運用上式後可將(2.11)式化成

$$
\begin{aligned}
&-\frac{4}{(2\pi)^3}\int d^3k\int_0^1 d\alpha\frac{-[m^2\eta_{\mu\nu}+k_\mu(k-q)_\nu+k_\nu(k-q)_\mu-k\cdot(k-q)\eta_{\mu\nu}]}{\{\alpha[(k-q)^2-m^2+i\epsilon]+(1-\alpha)(k^2-m^2+i\epsilon)\}^2}\\
=&-\frac{4}{(2\pi)^3}\int_0^1 d\alpha\int d^3k\frac{-[m^2\eta_{\mu\nu}+k_\mu(k-q)_\nu+k_\nu(k-q)_\mu-k\cdot(k-q)\eta_{\mu\nu}]}{[(k-\alpha q)^2+\alpha(1-\alpha)q^2-m^2+i\epsilon]^2}.
\end{aligned}
\tag{2.13}
$$

再來做變數變換$k'=k-\alpha q$使計算可以更簡化，又因為整個積分是球對稱，所以$k_\mu k_\nu=\dfrac{1}{3}k^2\eta_{\mu\nu}$，(2.13)就會變成

$$
\begin{aligned}
&-\frac{4}{(2\pi)^3}\int_0^1 d\alpha\int d^3k'\frac{-\{m^2\eta_{\mu\nu}+\frac{2}{3}(k')^2\eta_{\mu\nu}+2\alpha(\alpha-1)q_\nu q_\mu-[(k')^2+\alpha(\alpha-1)q^2]\eta_{\mu\nu}\}}{[(k')^2-\alpha(1-\alpha)q^2-m^2+i\epsilon]^2},\\
=&-\frac{4}{(2\pi)^3}\int_0^1 d\alpha\int d^3k\frac{-2\alpha(\alpha-1)q_\nu q_\mu+[\frac{1}{3}k^2+\alpha(\alpha-1)q^2-m^2]\eta_{\mu\nu}}{[k^2-\alpha(1-\alpha)q^2-m^2+i\epsilon]^2},\\
=&-\frac{4}{(2\pi)^3}\int_0^1 d\alpha\int d^3k\{\frac{\eta_{\mu\nu}}{3[k^2-\alpha(1-\alpha)q^2-m^2+i\epsilon]}\\
&\qquad+\frac{[4\alpha(\alpha-1)q^2-2m^2]\eta_{\mu\nu}-6\alpha(\alpha-1)q_\nu q_\mu}{3[k^2-\alpha(1-\alpha)q^2-m^2+i\epsilon]^2}\}.
\end{aligned}
\tag{2.14}
$$

除了路徑繞著極點的複變積分之外，我們也可以使用Wick rotation將整個積分的空間從Minkowski space轉換到Euclidean space，即

$$k_0 = ik_3 \Rightarrow \begin{cases} k^2 = k_0^2 - (k_1^2 + k_2^2) = -(k_1^2 + k_2^2 + k_3^2) = -k_E^2, \\ q^2 = -q_E^2, \\ d^3k = id^3k_E, \\ \eta_{\mu\nu} = -\delta_{\mu\nu}, \end{cases} \tag{2.15}$$

上式中的k_E和q_E為propagator在Euclidean space中的動量，而$\delta_{\mu\nu}$為Euclidean metric。如此一來整個積分式變得比較簡單，在經過變換後(2.14)裡面的$i\epsilon$就可以移除，因此整個積分式子就變成

$$\begin{aligned} -\frac{4i}{(2\pi)^3}\int_0^1 d\alpha \int d^3k_E \{ & \frac{\delta_{\mu\nu}}{3[k_E^2 - \alpha(1-\alpha)q_E^2 + m^2]} \\ & + \frac{[4\alpha(\alpha-1)q_E^2 + 2m^2]\delta_{\mu\nu} - 6\alpha(\alpha-1)q_\nu q_\mu}{3[k_E^2 - \alpha(1-\alpha)q_E^2 + m^2]^2}\}. \end{aligned} \tag{2.16}$$

在之前提到，這裡的積分都是球對稱的積分，所以d^3k_E可以用n維球體的球面面積與球體體積的關係式來化簡

$$d^nR = S_n(R)dR \quad , \quad S_n(R) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}R^{n-1}. \tag{2.17}$$

上式中的$\Gamma(\frac{n}{2})$為gamma function，依照這裡的問題帶n=3進去可以得到$d^3k = 4\pi k^2 dk$，所以(2.16)就變成

$$\begin{aligned} -\frac{2i}{(\pi)^2}\int_0^1 d\alpha \int dk_E k_E^2 \{ & \frac{\delta_{\mu\nu}}{3[k_E^2 - \alpha(1-\alpha)q_E^2 + m^2]} \\ & + \frac{[4\alpha(\alpha-1)q_E^2 + 2m^2]\delta_{\mu\nu} - 6\alpha(\alpha-1)q_\nu q_\mu}{3[k_E^2 - \alpha(1-\alpha)q_E^2 + m^2]^2}\}. \end{aligned} \tag{2.18}$$

由於這裡對k_E是從零積分到正無窮大，上式中第一項會因為分母跟分子都是二次方而得到發散的結果，所以我們必須對它做Regularization。在這裡我們選擇Pauli-Villar Regularization來做計算以保護系統中規範對稱性。Pauli-Villar Regularization就是在場裡面再加上一個虛擬且帶有質量Λ的粒子，引

入Regularization之後上式就變成

$$
\begin{aligned}
&-\frac{2i}{(\pi)^2}\int_0^1 d\alpha \int dk_E k_E^2 \{\frac{\delta_{\mu\nu}}{3[k_E^2-\alpha(1-\alpha)q_E^2+m^2]}-\frac{\delta_{\mu\nu}}{3[k_E^2-\alpha(1-\alpha)q_E^2+\Lambda^2]} \\
&\qquad +\frac{[4\alpha(\alpha-1)q_E^2+2m^2]\delta_{\mu\nu}}{3[k_E^2-\alpha(1-\alpha)q_E^2+m^2]^2}-\frac{[4\alpha(\alpha-1)q_E^2+2\Lambda^2]\delta_{\mu\nu}}{3[k_E^2-\alpha(1-\alpha)q_E^2+\Lambda^2]^2} \\
&\qquad -\frac{6\alpha(\alpha-1)q_\nu q_\mu}{3[k_E^2-\alpha(1-\alpha)q_E^2+m^2]^2}+\frac{6\alpha(\alpha-1)q_\nu q_\mu}{3[k_E^2-\alpha(1-\alpha)q_E^2+\Lambda^2]^2}\}, \\
&=-\frac{i}{\pi}\int_0^1 d\alpha\{\frac{1}{[m^2-\alpha(\alpha-1)q_E^2]^{\frac{1}{2}}}-\frac{1}{[\Lambda^2-\alpha(\alpha-1)q_E^2]^{\frac{1}{2}}}\}\alpha(\alpha-1)(q_E^2\delta_{\mu\nu}-q_\mu q_\nu).
\end{aligned}
\tag{2.19}
$$

最後再將Λ取極限到無限大，我們可以得到

$$
\begin{aligned}
&\frac{i}{\pi}\int_0^1 d\alpha\{\frac{1}{[m^2-\alpha(\alpha-1)q_E^2]^{\frac{1}{2}}}\}\alpha(\alpha-1)(q_\mu q_\nu-q_E^2\delta_{\mu\nu}), \\
=&\frac{i}{\pi}\{\frac{(4m^2-q_E^2)\cot^{-1}(\frac{2m}{q_E})-2mq_E}{4q_E^3}\}(q_\mu q_\nu-q_E^2\delta_{\mu\nu}).
\end{aligned}
\tag{2.20}
$$

Chapter 3

First order correction of perturbation to current-current correlator

在上一章所計算的current-current correlator是最低階的，而這一章我們將進一步計算它的下一階修正，這個時候電子間的交互作用會開始產生影響。因為Graphene中的交互作用單純只是電子間的Coulomb interaction，因此它的位能既為

$$L_I = -\frac{1}{2}\int d^2x d^2y \bar{\psi}(\vec{x})\gamma_0\psi(\vec{x})\frac{\mathrm{e}^2}{|\vec{x}-\vec{y}|}\bar{\psi}(\vec{y})\gamma_0\psi(\vec{y}), \tag{3.1}$$

所以第一階的perturbation correction就變成

$$\begin{aligned}
&\langle iL_I J(z)J(0)\rangle \\
=&-\frac{i}{2}\mathrm{e}^2\langle\int d^2x d^2y \bar{\psi}(\vec{x})\gamma_0\psi(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\bar{\psi}(\vec{y})\gamma_0\psi(\vec{y})J_\mu(z)J_\nu(0)\rangle, \\
=&-\frac{i}{2}\mathrm{e}^2\langle\int d^2x d^2y \bar{\psi}(\vec{x})\gamma_0\psi(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\bar{\psi}(\vec{y})\gamma_0\psi(\vec{y})\bar{\psi}(z)\gamma_\mu\psi(z)\bar{\psi}(0)\gamma_\nu\psi(0)\rangle.
\end{aligned} \tag{3.2}$$

上式中把$\hbar$設為1。如果使用Wick's theorem的話，會有許多組合需要去配對。為了利用我們在QED所學到的經驗，我們引進Hubbard-Stratonovich transformation(HST)。

首先將Coulomb potential轉換到動量空間，所以對它作傅立葉轉換

$$
\begin{aligned}
&-\frac{1}{2}\int d^2xd^2y\bar{\psi}(\vec{x})\gamma_0\psi(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\bar{\psi}(\vec{y})\gamma_0\psi(\vec{y}),\\
=&-\frac{1}{4\pi}\int d^2kd^2xd^2y\rho(\vec{x})\frac{1}{|\vec{k}|}\mathrm{e}^{i\vec{k}\cdot(\vec{x}-\vec{y})}\rho(\vec{y}),\\
=&-\frac{1}{4\pi}\int d^2k\frac{1}{|\vec{k}|}\mathrm{e}^{i\vec{k}\cdot(\vec{x}-\vec{y})}\int d^2x\frac{d^2p}{(2\pi)^2}\tilde{\rho}(\vec{p})\mathrm{e}^{i\vec{p}\cdot\vec{x}}\int d^2y\frac{d^2q}{(2\pi)^2}\tilde{\rho}(\vec{q})\mathrm{e}^{i\vec{q}\cdot\vec{y}},\\
=&-\frac{1}{4\pi}\int d^2k\tilde{\rho}(-\vec{k})\frac{1}{|\vec{k}|}\tilde{\rho}(\vec{k}),\\
=&-\frac{1}{4\pi}\int d^2k\tilde{\rho}^*(\vec{k})\frac{1}{|\vec{k}|}\tilde{\rho}(\vec{k}).
\end{aligned}
\tag{3.3}
$$

因為上式中的$\tilde{\rho}$是一個real field，所以$\tilde{\rho}(-\vec{k}) = \tilde{\rho}^*(\vec{k})$。之後再引進一個輔助場$\phi$並進行配方，即

$$
\begin{aligned}
&(\tilde{\phi}^*-\frac{\tilde{\rho}^*}{|\vec{k}|})|\vec{k}|(\tilde{\phi}-\frac{\tilde{\rho}}{|\vec{k}|}) = \tilde{\phi}^*|\vec{k}|\tilde{\phi}-2\tilde{\phi}^*\tilde{\rho}+\tilde{\rho}^*\frac{1}{|\vec{k}|}\tilde{\rho},\\
\Rightarrow&(\tilde{\phi}^*-\frac{\tilde{\rho}^*}{|\vec{k}|})|\vec{k}|(\tilde{\phi}-\frac{\tilde{\rho}}{|\vec{k}|})-\tilde{\rho}^*\frac{1}{|\vec{k}|}\tilde{\rho} = \tilde{\phi}^*|\vec{k}|\tilde{\phi}-2\tilde{\phi}^*\tilde{\rho}.
\end{aligned}
\tag{3.4}
$$

最後將上式帶回到partition function Z 即

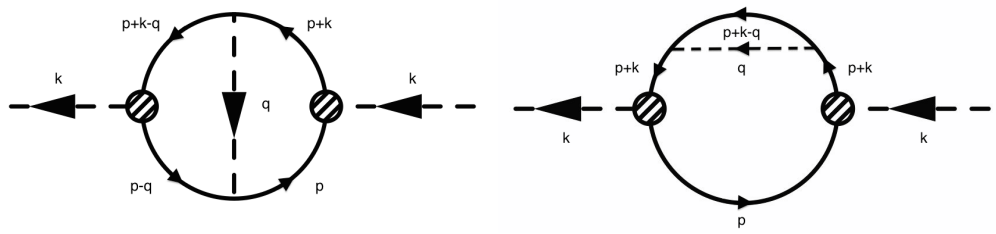
$$
\begin{aligned}
Z =& \int D[\tilde{\phi}]D[\tilde{\phi}^*]\mathrm{e}^{-\frac{i}{4\pi}\int d^2k\{(\tilde{\phi}^*-\frac{\tilde{\rho}^*}{|\vec{k}|})|\vec{k}|(\tilde{\phi}-\frac{\tilde{\rho}}{|\vec{k}|})-\tilde{\rho}^*\frac{1}{|\vec{k}|}\tilde{\rho}\}},\\
=&\int D[\tilde{\phi}]D[\tilde{\phi}^*]\mathrm{e}^{-\frac{i}{4\pi}\int d^2k(\tilde{\phi}^*|\vec{k}|\tilde{\phi}-2\tilde{\phi}^*\tilde{\rho})},\\
\Rightarrow&\int D[\tilde{\phi}]D[\tilde{\phi}^*]\mathrm{e}^{-\frac{i}{4\pi}\int d^2k(\tilde{\phi}^*|\vec{k}|\tilde{\phi}-2\tilde{\phi}^*\tilde{\rho})} = N\int D[\tilde{\phi}]D[\tilde{\phi}^*]\mathrm{e}^{\frac{i}{4\pi}\int d^2k\{\tilde{\rho}^*\frac{1}{|\vec{k}|}\tilde{\rho}\}}.
\end{aligned}
\tag{3.5}
$$

上式中的N為某一常數。

以上的步驟就是HST，因為這樣的關係使得(3.3)中的$-\tilde{\rho}^*(\vec{k})\frac{1}{|\vec{k}|}\tilde{\rho}(\vec{k})$ 可以用$(\tilde{\phi}^*|\vec{k}|\tilde{\phi}-2\tilde{\phi}^*\tilde{\rho})$代換， 再將它換回坐標空間，則$(\tilde{\phi}^*|\vec{k}|\tilde{\phi}-2\tilde{\phi}^*\tilde{\rho})$會等於$(\phi^*|\nabla|\phi-\frac{i}{\pi}\phi^*\bar{\psi}\gamma_0\psi)$，而Lagrangian density就會變成

$$
\mathcal{L} = \bar{\psi}\gamma_0\partial^0\psi+v_F\bar{\psi}\gamma_i\partial^i\psi+\frac{i}{4\pi}\phi^*|\nabla|\phi-\frac{i}{2\pi}\phi\bar{\psi}\gamma_0\psi. \tag{3.6}
$$

在這個新的Larangian下，J-J correlator的一階修正為 $\langle 0|(\phi\bar{\psi}\gamma_0\psi)^2J_\mu(x)J_\nu(0)|0\rangle$，而它對應了兩個Feynman diagrams(圖3.1)。



(a) Egg diagram (b) Self energy

Figure 3.1

首先看Egg diagram，從Feynman diagrams可以很容易的得到它所對應的算式為

$$\int \frac{d^3p}{(2\pi)^3}\ \mathrm{Tr}\{\frac{i}{\not{P}_a}\gamma_\nu\frac{i}{\not{P}_b}\gamma_0\frac{i}{\not{P}_c}\gamma_\mu\frac{i}{\not{P}_d}\gamma_0\},$$
$$\begin{cases} P_a = p, \\ P_b = p+k, \\ P_c = p+k-q, \\ P_d = p-q, \end{cases} \tag{3.7}$$

在這裡我們暫時只考慮對p的積分。之後依照式子(2.9)同樣的模式對上式做整理，同乘一樣的分母使$\not{P}$移動到分子

$$\int \frac{d^3p}{(2\pi)^3}\ \mathrm{Tr}\{\frac{\not{P}_a\gamma_\nu\not{P}_b\gamma_0\not{P}_c\gamma_\mu\not{P}_d\gamma_0}{P_a^2P_b^2P_c^2P_d^2}\}. \tag{3.8}$$

取trace後會產生relativistic,semi-relativistic, non-relativistic三個部分。

Relativistic part:

$$\int \frac{d^3p}{(2\pi)^3}\frac{4[(P_a\cdot P_b)(P_c\cdot P_d)+(P_a\cdot P_d)(P_b\cdot P_c)-(P_a\cdot P_b)(P_c\cdot P_d)]}{P_a^2P_b^2P_c^2P_d^2}. \tag{3.9}$$

Semi-relativistic part:

$$\int \frac{d^3p}{(2\pi)^3}\frac{8[\omega_a\omega_d(P_b\cdot P_c)+\omega_c\omega_d(P_a\cdot P_b)+\omega_b\omega_c(P_a\cdot P_d)]}{P_a^2P_b^2P_c^2P_d^2}, \tag{3.10}$$

上式中的ω_a為P_a的時間分量，且$\omega_a\omega_d(P_b\cdot P_c)+\omega_c\omega_d(P_a\cdot P_b)+\omega_b\omega_c(P_a\cdot P_d)$可改寫為

$$\begin{aligned}&\omega_a\omega_d(P_b\cdot P_c)+\omega_c\omega_d(P_a\cdot P_b)+\omega_b\omega_c(P_a\cdot P_d),\\ =&\frac{1}{2}[\omega_c(\omega_b+\omega_d)P_a^2+\omega_d(\omega_a+\omega_c)P_b^2\\ &+\omega_a(\omega_b+\omega_d)P_c^2+\omega_b(\omega_a+\omega_c)P_d^2\\ &-k^2(\omega_a\omega_b+\omega_c\omega_d)-q^2(\omega_a\omega_d+\omega_b\omega_c)].\end{aligned}\tag{3.11}$$

上式中的$k^2\equiv|\vec{k}|^2, q^2\equiv|\vec{q}|^2$

Non-relativistic part:

$$\int\frac{d^3p}{(2\pi)^3}\frac{32\omega_a\omega_b\omega_c\omega_d}{P_a^2P_b^2P_c^2P_d^2},\tag{3.12}$$

而上式的$\omega_a\omega_b\omega_c\omega_d$也可以改寫為

$$\begin{aligned}&\omega_a\omega_b\omega_c\omega_d\\ =&\omega_p^4+2(\omega_k-\omega_q)\omega_p^3+[\omega_k^2-3\omega_k\omega_q+\omega_q^2]\omega_p^2+[-\omega_k\omega_q(\omega_k-\omega_q)]\omega_p.\end{aligned}\tag{3.13}$$

從這三個部分可以發現它們分別為四種積分

$$\begin{aligned}I_i(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{p_i}{P_a^2P_b^2P_c^2P_d^2},\\ I_{ij}(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{p_ip_j}{P_a^2P_b^2P_c^2P_d^2},\\ I_{ijm}(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{p_ip_jp_m}{P_a^2P_b^2P_c^2P_d^2},\\ I_{ijmn}(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{p_ip_jp_mp_n}{P_a^2P_b^2P_c^2P_d^2}.\end{aligned}\tag{3.14}$$

由於P_a,P_b,P_c,P_d中的p會被積分掉而剩下k跟q，所以$I_i,I_{ij},I_{ijm},I_{ijmn}$分別會變成k跟q所組成的一階到四階且完全對稱的張量。

因此(3.14)中的積分出來會成為以下的組合

$$
\begin{aligned}
I_i(\vec{k},\vec{q}) &= A_1(\vec{k},\vec{q})k_i + A_2(\vec{k},\vec{q})q_i,\\
I_{ij}(\vec{k},\vec{q}) &= C_1(\vec{k},\vec{q})\delta_{ij} + C_2(\vec{k},\vec{q})k_ik_j + C_3(\vec{k},\vec{q})q_iq_j + C_4(\vec{k},\vec{q})(k_iq_j + q_ik_j),\\
I_{ijm}(\vec{k},\vec{q}), &= E_1(\vec{k},\vec{q})(\delta_{ij}k_m + \delta_{im}k_j + \delta_{jm}k_i)\\
&+ E_2(\vec{k},\vec{q})(\delta_{ij}q_m + \delta_{im}q_j + \delta_{jm}q_i)\\
&+ E_3(\vec{k},\vec{q})(k_ik_jk_m) + E_4(\vec{k},\vec{q})(q_iq_jq_m)\\
&+ E_5(\vec{k},\vec{q})(k_ik_jq_m + k_iq_jk_m + q_ik_jk_m)\\
&+ E_6(\vec{k},\vec{q})(q_iq_jk_m + q_ik_jq_m + k_iq_jq_m)\\
I_{ijmn}(\vec{k},\vec{q}), &= F_1(\vec{k},\vec{q})(\delta_{ij}\delta_{mn}) + F_2(\vec{k},\vec{q})(\delta_{ij}k_mk_n) + F_3(\vec{k},\vec{q})(\delta_{ij}q_mq_n) + F_4(\vec{k},\vec{q})(\delta_{ij}k_mq_n)\\
&+ F_5(\vec{k},\vec{q})(k_ik_jk_mk_n) + F_6(\vec{k},\vec{q})(q_iq_jq_mq_n)\\
&+ F_7(\vec{k},\vec{q})(k_ik_jk_mq_n) + F_8(\vec{k},\vec{q})(q_iq_jq_mk_n)\\
&+ F_9(\vec{k},\vec{q})(k_ik_jq_mq_n).
\end{aligned}
\tag{3.15}
$$

上式中的$A.C.E.F$分別為係數，而這些係數可以利用張量分析來求得(詳細過程見Appendix A)。

而self-energy diagram與egg diagram類似，從圖3.1中可以發現將egg diagram中的P_d替換成P_b就變成self-energy diagram，它所對應的算式就變成

$$
\int \frac{d^3p}{(2\pi)^3}\ \mathrm{Tr}\{\frac{i}{\not{P}_a}\gamma_\nu\frac{i}{\not{P}_b}\gamma_0\frac{i}{\not{P}_c}\gamma_0\frac{i}{\not{P}_b}\gamma_\mu\},
\begin{cases}
P_a = p,\\
P_b = p + k,\\
P_c = p + k - q.
\end{cases}
\tag{3.16}
$$

取trace後一樣也有relativistic,semi-relativistic, non-relativistic三個部分 Relativistic part:

$$
\int \frac{d^3p}{(2\pi)^3}\frac{4[2(P_a\cdot P_b)(P_b\cdot P_c) - P_b^2(P_a\cdot P_c)]}{P_a^2P_b^4P_c^2}. \tag{3.17}
$$

Semi-relativistic part:

$$
\int \frac{d^3p}{(2\pi)^3}\frac{16[\omega_a\omega_b(P_b\cdot P_c) + \omega_b\omega_c(P_a\cdot P_b)]}{P_a^2P_b^4P_c^2}, \tag{3.18}
$$

上式的$\omega_a\omega_b(P_b\cdot P_c)+\omega_b\omega_c(P_a\cdot P_b)$可改寫為

$$\begin{aligned}
&\omega_a\omega_b(P_b\cdot P_c)+\omega_b\omega_c(P_a\cdot P_b),\\
=&\frac{1}{4}[\omega_c(\omega_b+\omega_d)P_a^2+\omega_d(\omega_a+\omega_c)P_b^2\\
&+\omega_a(\omega_b+\omega_d)P_c^2+\omega_b(\omega_a+\omega_c)P_d^2\\
&-k^2(\omega_a\omega_b+\omega_c\omega_d)-q^2(\omega_a\omega_d+\omega_b\omega_c)].
\end{aligned}\tag{3.19}$$

Non-relativistic part:

$$\int\frac{d^3p}{(2\pi)^3}\frac{32\omega_a\omega_b^2\omega_c}{P_a^2P_b^4P_c^2}.\tag{3.20}$$

因此所需要的積分為

$$\begin{aligned}
J_i(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{(P_b)_i}{P_a^2P_b^4P_c^2},\\
J_{ij}(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{(P_b)_i(P_b)_j}{P_a^2P_b^4P_c^2},\\
J_{ijm}(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{(P_b)_i(P_b)_j(P_b)_m}{P_a^2P_b^4P_c^2},\\
J_{ijmn}(\vec{k},\vec{q})&=\int\frac{d^3p}{(2\pi)^3}\frac{(P_b)_i(P_b)_j(P_b)_m(P_b)_n}{P_a^2P_b^4P_c^2}.
\end{aligned}\tag{3.21}$$

而一階到四階張量，為以下的組合

$$\begin{aligned}
J_i(\vec{k},\vec{q})&=B_1(\vec{k},\vec{q})k_i+B_2(\vec{k},\vec{q})q_i,\\
J_{ij}(\vec{k},\vec{q})&=D_1(\vec{k},\vec{q})\delta_{ij}+D_2(\vec{k},\vec{q})k_ik_j+D_3(\vec{k},\vec{q})q_iq_j+D_4(\vec{k},\vec{q})(k_iq_j+q_ik_j),\\
J_{ijm}(\vec{k},\vec{q})&=G_1(\vec{k},\vec{q})(\delta_{ij}k_m+\delta_{im}k_j+\delta_{jm}k_i)\\
&+G_2(\vec{k},\vec{q})(\delta_{ij}q_m+\delta_{im}q_j+\delta_{jm}q_i)\\
&+G_3(\vec{k},\vec{q})(k_ik_jk_m)+G_4(\vec{k},\vec{q})(q_iq_jq_m)\\
&+G_5(\vec{k},\vec{q})(k_ik_jq_m+k_iq_jk_m+q_ik_jk_m)\\
&+G_6(\vec{k},\vec{q})(q_iq_jk_m+q_ik_jq_m+k_iq_jq_m),\\
J_{ijmn}(\vec{k},\vec{q})&=H_1(\vec{k},\vec{q})(\delta_{ij}\delta_{mn})+H_2(\vec{k},\vec{q})(\delta_{ij}k_mk_n)+H_3(\vec{k},\vec{q})(\delta_{ij}q_mq_n)+H_4(\vec{k},\vec{q})(\delta_{ij}k_mq_n)\\
&+H_5(\vec{k},\vec{q})(k_ik_jk_mk_n)+H_6(\vec{k},\vec{q})(q_iq_jq_mq_n)\\
&+H_7(\vec{k},\vec{q})(k_ik_jk_mq_n)+H_8(\vec{k},\vec{q})(q_iq_jq_mk_n)\\
&+H_9(\vec{k},\vec{q})(k_ik_jq_mq_n).
\end{aligned}\tag{3.22}$$

這些係數一樣可以用張量分析來計算(詳細過程見Appendix B)。

最後(3.9)∼(3.11)和(3.15)∼(3.17)都可以用我們所引進的I, J張量來表示。egg diagram的relativistic,semi-relativistic以及non-relativistic部分分別為

Relativistic part:

$$
\begin{aligned}
&T_{re}(\vec{k},\vec{q}),\\
=&4I_{\delta\delta}(\vec{k},\vec{q})+8I_{\delta k}(\vec{k},\vec{q})-8I_{\delta q}(\vec{k},\vec{q})+4(k^2+q^2-\vec{k}\cdot\vec{q})I_{\delta}(\vec{k},\vec{q})\\
&-8I_{kq}(\vec{k},\vec{q})+4q^2I_k(\vec{k},\vec{q})-4k^2I_q(\vec{k},\vec{q}).
\end{aligned}
\tag{3.23}
$$

Semi-relativistic part:

$$
\begin{aligned}
&\delta_{ij}\{T_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 32I_{\delta\delta}(\vec{k},\vec{q}) + 64I_{\delta k}(\vec{k},\vec{q}) - 64I_{\delta q}(\vec{k},\vec{q}) + 16I_{\delta}(\vec{k},\vec{q}) \\
&+ 16I_{kk}(\vec{k},\vec{q}) + 16I_{qq}(\vec{k},\vec{q}) - 64I_{kq}(\vec{k},\vec{q}) + 16(q^2 - \vec{k}\cdot\vec{q})I_k(\vec{k},\vec{q}) \\
&- 16(k^2 - \vec{k}\cdot\vec{q})I_q(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&k_ik_j\{T_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 32I_{\delta kk}(\vec{k},\vec{q}) + 32I_{\delta k}(\vec{k},\vec{q}) + 32I_{kkk}(\vec{k},\vec{q}) \\
&- 32I_{kkq}(\vec{k},\vec{q}) + 8(k^2 - \vec{k}\cdot\vec{q})^2 I_{\delta}(\vec{k},\vec{q}) + 8(3k^2 + q^2 - 6\vec{k}\cdot\vec{q})I_{kk}(\vec{k},\vec{q}) \\
&+ 16(-2k^2 + \vec{k}\cdot\vec{q})I_{kq}(\vec{k},\vec{q}) + 8[k^2q^2 - 3k^2(\vec{k}\cdot\vec{q}) + 2(\vec{k}\cdot\vec{q})^2]I_k(\vec{k},\vec{q}) \\
&- 8k^2(k^2 - \vec{k}\cdot\vec{q})I_q(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&q_iq_j\{T_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 32I_{\delta qq}(\vec{k},\vec{q}) - 32I_{\delta q}(\vec{k},\vec{q}) - 32I_{qqq}(\vec{k},\vec{q}) \\
&+ 32I_{kqq}(\vec{k},\vec{q}) + 8(q^2 - \vec{k}\cdot\vec{q})I_{\delta}(\vec{k},\vec{q}) + 8(3k^2 + k^2 - 6\vec{k}\cdot\vec{q})I_{qq}(\vec{k},\vec{q}) \\
&+ 16(-2q^2 + \vec{k}\cdot\vec{q})I_{kq}(\vec{k},\vec{q}) - 8[k^2q^2 - 3q^2(\vec{k}\cdot\vec{q}) + 2(\vec{k}\cdot\vec{q})^2]I_q(\vec{k},\vec{q}) \\
&+ 8q^2(q^2 - \vec{k}\cdot\vec{q})I_k(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_iq_j + q_ik_j)\{T_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 64I_{\delta kq}(\vec{k},\vec{q}) \\
&- 32(q^2 - \vec{k}\cdot\vec{q})I_{\delta k}(\vec{k},\vec{q}) + 32(k^2 - \vec{k}\cdot\vec{q})I_{\delta q}(\vec{k},\vec{q}) \\
&+ 64I_{kkq}(\vec{k},\vec{q}) - 64I_{kqq}(\vec{k},\vec{q}) \\
&- 16(k^2 - \vec{k}\cdot\vec{q})(q^2 - \vec{k}\cdot\vec{q})I_{\delta}(\vec{k},\vec{q}) \\
&+ 16(-2q^2 + \vec{k}\cdot\vec{q})I_{kk}(\vec{k},\vec{q}) + 16(-2k^2 + \vec{k}\cdot\vec{q})I_{qq}(\vec{k},\vec{q}) + 32(k^2q^2 - 3\vec{k}\cdot\vec{q})I_{kq}(\vec{k},\vec{q}) \\
&- 16[k^2q^2 - 2q^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2]I_k(\vec{k},\vec{q}) + 16[k^2q^2 - 2q^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2]I_q(\vec{k},\vec{q}).
\end{aligned}
\tag{3.24}
$$

Non-relativistic part:

$$
\begin{aligned}
&(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})\{T_{nr}(\vec{k},\vec{q})\}_{ijmn}\\
&=96I_{\delta\delta}(\vec{k},\vec{q})+192I_{\delta k}(\vec{k},\vec{q})-192I_{\delta q}(\vec{k},\vec{q})\\
&+32(k^2+q^2-3\cdot q)I_{\delta}(\vec{k},\vec{q})\\
&+64I_{kk}(\vec{k},\vec{q})+64I_{qq}(\vec{k},\vec{q})-192I_{kq}(\vec{k},\vec{q})\\
&+32(q^2-2\vec{k}\cdot\vec{q})I_k(\vec{k},\vec{q})-32(k^2-2\vec{k}\cdot\vec{q})I_q(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(\delta_{ij}k_mk_n+k_ik_j\delta_{mn}+\delta_{im}k_jk_n+k_ik_m\delta_{jn}+\delta_{in}k_jk_m+k_ik_n\delta_{jm})\{T_{nr}(\vec{k},\vec{q})\}_{ijmn}\\
&=192I_{\delta kk}(\vec{k},\vec{q})-192(k^2-\vec{k}\cdot\vec{q})I_{\delta k}(\vec{k},\vec{q})\\
&+192I_{kkk}(\vec{k},\vec{q})-192I_{kkq}(\vec{k},\vec{q})\\
&+32[k^4-3k^2(\vec{k}\cdot\vec{q})+(\vec{k}\cdot\vec{q})^2]I_{\delta}(\vec{k},\vec{q})\\
&+32(5k^2+q^2-9\vec{k}\cdot\vec{q})I_{kk}(\vec{k},\vec{q})-64(3k^2-2\vec{k}\cdot\vec{q})I_{kq}(\vec{k},\vec{q})\\
&+32[k^2q^2-5k^2(\vec{k}\cdot\vec{q})+3(\vec{k}\cdot\vec{q})^2]I_k(\vec{k},\vec{q})-32k^2(k^2-2\vec{k}\cdot\vec{q})I_q(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(\delta_{ij}q_mq_n+q_iq_j\delta_{mn}+\delta_{im}q_jq_n+q_iq_m\delta_{jn}+\delta_{in}q_jq_m+q_iq_n\delta_{jm})\{T_{nr}(\vec{k},\vec{q})\}_{ijmn}\\
&=192I_{\delta qq}(\vec{k},\vec{q})-192(q^2-\vec{k}\cdot\vec{q})I_{\delta q}(\vec{k},\vec{q})\\
&-192I_{qqq}(\vec{k},\vec{q})+192I_{kqq}(\vec{k},\vec{q})\\
&+32[q^4-3q^2(\vec{k}\cdot\vec{q})+(\vec{k}\cdot\vec{q})^2]I_{\delta}(\vec{k},\vec{q})\\
&+32[5q^2+k^2-9(\vec{k}\cdot\vec{q})]I_{qq}(\vec{k},\vec{q})-64[k^2q^2-5q^2(\vec{k}\cdot\vec{q})+3(k\cdot q)^2]I_{kq}(\vec{k},\vec{q})\\
&-32q^2(q^2-2\vec{k}\cdot\vec{q})I_q(\vec{k},\vec{q})+32I_k(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&[(k_mq_n+q_nk_m)\delta_{ij}+(k_iq_j+q_ik_j)\delta_{mn}+(k_jq_n+q_ik_n)\delta_{im}\\
&+(k_iq_m+q_ik_m)\delta_{jn}+(k_jq_m+q_jk_m)\delta_{in}+(k_iq_n+q_ik_n)\delta_{jm}]\{T_{nr}(\vec{k},\vec{q})\}_{ijmn}\\
&=384I_{\delta kq}(\vec{k},\vec{q})-192(q^2-\vec{k}\cdot\vec{q})I_{\delta k}(\vec{k},\vec{q})+192(k^2-\vec{k}\cdot\vec{q})I_{\delta q}(\vec{k},\vec{q})\\
&+384I_{kkq}(\vec{k},\vec{q})-384I_{kqq}(\vec{k},\vec{q})\\
&-32[3k^2q^2-2k^2(\vec{k}\cdot\vec{q})-2q^2(\vec{k}\cdot\vec{q})+3(\vec{k}\cdot\vec{q})^2]I_{\delta}(\vec{k},\vec{q})\\
&-64(3q^2-2\vec{k}\cdot\vec{q})I_{kk}(\vec{k},\vec{q})-64(3k^2-2\vec{k}\cdot\vec{q})I_{qq}(\vec{k},\vec{q})+192(k^2+q^2-3\vec{k}\cdot\vec{q})I_{kq}(\vec{k},\vec{q})
\end{aligned}
$$

$$
\begin{aligned}
&- 32[k^2q^2 - 5q^2(\vec{k}\cdot\vec{q}) + 4(\vec{k}\cdot\vec{q})^2]I_k(\vec{k},\vec{q}) \\
&+ 32[k^2q^2 - 5k^2(\vec{k}\cdot\vec{q}) + 4(\vec{k}\cdot\vec{q})^2]I_q(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_ik_jk_mk_n)\{T_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 32I_{kkkk}(\vec{k},\vec{q}) + 64(k^2 - \vec{k}\cdot\vec{q})I_{kkk}(\vec{k},\vec{q}) \\
&+ 32[k^4 - 3k^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2]I_{kk}(\vec{k},\vec{q}) \\
&- 32k^2(\vec{k}\cdot\vec{q})(k^2 - \vec{k}\cdot\vec{q})I_k(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(q_iq_jq_mq_n)\{T_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 32I_{qqqq}(\vec{k},\vec{q}) - 64(q^2 - \vec{k}\cdot\vec{q})I_{qqq}(\vec{k},\vec{q}) \\
&+ 32[q^4 - 3q^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2]I_{qq}(\vec{k},\vec{q}) \\
&+ 32q^2(\vec{k}\cdot\vec{q})(q^2 - \vec{k}\cdot\vec{q})I_q(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_ik_jk_mq_n + k_ik_jq_mk_n + k_iq_jk_mk_n + q_ik_jk_mk_n)\{T_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 128I_{kkkq}(\vec{k},\vec{q}) \\
&- 64(q^2 - \vec{k}\cdot\vec{q})I_{kkk}(\vec{k},\vec{q}) + 192(k^2 - \vec{k}\cdot\vec{q})I_{kkq}(\vec{k},\vec{q}) \\
&- 32[3k^2q^2 - 2k^2(\vec{k}\cdot\vec{q}) - 2q^2(\vec{k}\cdot\vec{q}) + 3(\vec{k}\cdot\vec{q})^2]I_{kk}(\vec{k},\vec{q}) + 64[k^4 - 3k^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2]I_{kq}(\vec{k},\vec{q}) \\
&- 32[k^4q^2 - 2k^2q^2(\vec{k}\cdot\vec{q}) + 2k^2(\vec{k}\cdot\vec{q})^2 - (\vec{k}\cdot\vec{q})^3]I_k(\vec{k},\vec{q}) - 32k^2(\vec{k}\cdot\vec{q})(k^2 - \vec{k}\cdot\vec{q})I_qk(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(q_iq_jq_mk_n + q_iq_jk_mq_n + q_ik_jq_mq_n + k_iq_jq_mq_n)\{T_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 128I_{kqqq}(\vec{k},\vec{q}) \\
&+ 64(k^2 - \vec{k}\cdot\vec{q})I_{qqq}(\vec{k},\vec{q}) - 192(q^2 - \vec{k}\cdot\vec{q})I_{kqq}(\vec{k},\vec{q}) \\
&- 32[3k^2q^2 - 2k^2(\vec{k}\cdot\vec{q}) - 2q^2(\vec{k}\cdot\vec{q}) + 3(\vec{k}\cdot\vec{q})^2]I_{qq}(\vec{k},\vec{q}) \\
&+ 64[q^4 - 3q^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2]I_{kq}(\vec{k},\vec{q}) + 32[q^4k^2 - 2k^2q^2(\vec{k}\cdot\vec{q}) + 2q^2(\vec{k}\cdot\vec{q})^2 - (\vec{k}\cdot\vec{q})^3]I_q(\vec{k},\vec{q}) \\
&+ 32q^2(\vec{k}\cdot\vec{q})(q^2 - \vec{k}\cdot\vec{q})I_k(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_ik_jq_mq_n + q_iq_jk_mk_n + k_iq_jk_mq_n + q_ik_jq_mk_n + k_iq_jq_mk_n + q_ik_jk_mq_n)\{T_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 192 I_{kkqq}(\vec{k},\vec{q}) \\
&- 192(q^2 - \vec{k}\cdot\vec{q}) I_{kkq}(\vec{k},\vec{q}) + 192(k^2 - \vec{k}\cdot\vec{q}) I_{kqq}(\vec{k},\vec{q}) \\
&+ 32[q^4 - 3q^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2] I_{kk}(\vec{k},\vec{q}) + 32[k^4 - 3k^2(\vec{k}\cdot\vec{q}) + (\vec{k}\cdot\vec{q})^2] I_{qq}(\vec{k},\vec{q}) \\
&- 64[3k^2q^2 - 2k^2(\vec{k}\cdot\vec{q}) - 2q^2(\vec{k}\cdot\vec{q}) + 3(\vec{k}\cdot\vec{q})^2] I_{kq}(\vec{k},\vec{q}) \\
&+ 32[k^2q^4 - 2k^2q^2(\vec{k}\cdot\vec{q}) - 2q^2(\vec{k}\cdot\vec{q}) - (\vec{k}\cdot\vec{q})^3] I_k(\vec{k},\vec{q}) \\
&- 32[q^2k^4 - 2k^2q^2(\vec{k}\cdot\vec{q}) + 2k^2(\vec{k}\cdot\vec{q}) - (\vec{k}\cdot\vec{q})^3] I_q(\vec{k},\vec{q}).
\end{aligned}
\tag{3.25}
$$

而self energy diagram的relativistic,semi-relativistic以及non-relativistic分別為

Relativistic part:

$$
\begin{aligned}
&V_{re}(\vec{k},\vec{q}) \\
=&4J_{\delta\delta}(\vec{k},\vec{q}) - 4J_{\delta k}(\vec{k},\vec{q}) - 4J_{\delta q}(\vec{k},\vec{q}) - 4\vec{k}\cdot\vec{q}J_{\delta}(\vec{k},\vec{q}) + 8J_{kq}(\vec{k},\vec{q}).
\end{aligned}
\tag{3.26}
$$

Semi-relativistic part:

$$
\begin{aligned}
&\delta_{ij}\{V_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 32J_{\delta\delta}(\vec{k},\vec{q}) - 32J_{\delta k}(\vec{k},\vec{q}) - 32J_{\delta q}(\vec{k},\vec{q}) + 32J_{kq}(\vec{k},\vec{q}), \\
\\
&k_i k_j\{V_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 32J_{\delta kk}(\vec{k},\vec{q}) - 16(k^2 + \vec{k}\cdot\vec{q})J_{\delta k}(\vec{k},\vec{q}) \\
&- 16J_{kkk}(\vec{k},\vec{q}) - 16J_{kkq}(\vec{k},\vec{q}) \\
&+ 16\vec{k}\cdot\vec{q}J_{kk}(\vec{k},\vec{q}) + 16k^2J_{kq}(\vec{k},\vec{q}), \\
\\
&q_i q_j\{V_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 32J_{\delta qq}(\vec{k},\vec{q}) - 16(q^2 + \vec{k}\cdot\vec{q})J_{\delta q}(\vec{k},\vec{q}) \\
&- 16J_{qqq}(\vec{k},\vec{q}) - 16J_{kqq}(\vec{k},\vec{q}) \\
&+ 16\vec{k}\cdot\vec{q}J_{qq}(\vec{k},\vec{q}) + 16q^2J_{kq}(\vec{k},\vec{q}), \\
\\
&(k_i q_j + q_i k_j)\{V_{sr}(\vec{k},\vec{q})\}_{ij} \\
&= 64J_{\delta kq}(\vec{k},\vec{q}) \\
&- 16(q^2 - \vec{k}\cdot\vec{q})J_{\delta k}(\vec{k},\vec{q}) - 16(k^2 + \vec{k}\cdot\vec{q})J_{\delta q}(\vec{k},\vec{q}) \\
&- 32J_{kkq}(\vec{k},\vec{q}) - 32J_{kqq}(\vec{k},\vec{q}) \\
&- 16q^2J_{kk}(\vec{k},\vec{q}) + 16k^2J_{qq}(\vec{k},\vec{q}) + 32\vec{k}\cdot\vec{q}J_{kq}(\vec{k},\vec{q}).
\end{aligned}
\tag{3.27}
$$

Non-relativistic part:

$$
\begin{aligned}
&(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 96J_{\delta\delta}(\vec{k},\vec{q}) - 96J_{\delta k}(\vec{k},\vec{q}) - 96J_{\delta q}(\vec{k},\vec{q}) \\
&+ 32\vec{k}\cdot\vec{q}J_{\delta}(\vec{k},\vec{q}) + 64J_{kq}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(\delta_{ij}k_m k_n + k_i k_j \delta_{mn} + \delta_{im}k_j k_n + k_i k_m \delta_{jn} + \delta_{in}k_j k_m + k_i k_n \delta_{jm})\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 192J_{\delta kk}(\vec{k},\vec{q}) - 96(k^2 + \vec{k}\cdot\vec{q})J_{\delta k}(\vec{k},\vec{q}) \\
&- 96J_{kkk}(\vec{k},\vec{q}) - 96J_{kkq}(\vec{k},\vec{q}) \\
&+ 32k^4(\vec{k}\cdot\vec{q})J_{\delta}(\vec{k},\vec{q}) \\
&+ 96\vec{k}\cdot\vec{q}J_{kk}(\vec{k},\vec{q}) + 64k^2J_{kq}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(\delta_{ij}q_m q_n + q_i q_j \delta_{mn} + \delta_{im}q_j q_n + q_i q_m \delta_{jn} + \delta_{in}q_j q_m + q_i q_n \delta_{jm})\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 192J_{\delta qq}(\vec{k},\vec{q}) - 96(q^2 + \vec{k}\cdot\vec{q})J_{\delta q}(\vec{k},\vec{q}) \\
&- 96J_{qqq}(\vec{k},\vec{q}) - 96J_{kqq}(\vec{k},\vec{q}) \\
&+ 32q^2(\vec{k}\cdot\vec{q})J_{\delta}(\vec{k},\vec{q}) \\
&+ 96(\vec{k}\cdot\vec{q})J_{qq}(\vec{k},\vec{q}) + 64q^2J_{kq}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&[(k_m q_n + q_n k_m)\delta_{ij} + (k_i q_j + q_i k_j)\delta_{mn} + (k_j q_n + q_i k_n)\delta_{im} \\
&+ (k_i q_m + q_i k_m)\delta_{jn} + (k_j q_m + q_j k_m)\delta_{in} + (k_i q_n + q_i k_n)\delta_{jm}]\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 384J_{\delta kq}(\vec{k},\vec{q}) - 96(q^2 + \vec{k}\cdot\vec{q})J_{\delta k}(\vec{k},\vec{q}) - 96(k^2 - \vec{k}\cdot\vec{q})J_{\delta q}(\vec{k},\vec{q}) \\
&- 192J_{kkq}(\vec{k},\vec{q}) - 192J_{kqq}(\vec{k},\vec{q}) \\
&- 32[3k^2q^2 + (\vec{k}\cdot\vec{q})^2]J_{\delta}(\vec{k},\vec{q}) \\
&+ 64q^2J_{kk}(\vec{k},\vec{q}) + 64k^2J_{qq}(\vec{k},\vec{q}) + 192\vec{k}\cdot\vec{q}J_{kq}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_i k_j k_m k_n)\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 32 J_{kkk}(\vec{k},\vec{q}) - 32(k^2 + \vec{k}\cdot\vec{q}) J_{kkk}(\vec{k},\vec{q}) \\
&+ 32 k^2 (\vec{k}\cdot\vec{q}) J_{kk}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(q_i q_j q_m q_n)\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 32 J_{qqqq}(\vec{k},\vec{q}) - 32(q^2 + \vec{k}\cdot\vec{q}) J_{qqq}(\vec{k},\vec{q}) \\
&+ 32 q^2 (\vec{k}\cdot\vec{q}) J_{qq}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_i k_j k_m q_n + k_i k_j q_m k_n + k_i q_j k_m k_n + q_i k_j k_m k_n)\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 128 J_{kkkq}(\vec{k},\vec{q}) \\
&- -32(q^2 + \vec{k}\cdot\vec{q}) J_{kkk}(\vec{k},\vec{q}) - 96(k^2 + \vec{k}\cdot\vec{q}) J_{kkq}(\vec{k},\vec{q}) \\
&- 32[k^2 q^2 + (\vec{k}\cdot\vec{q})^2] J_{kk}(\vec{k},\vec{q}) + 64 k^2 (\vec{k}\cdot\vec{q}) J_{kq}(\vec{k},\vec{q}) \\
&- 32[k^4 q^2 - 2k^2 q^2 (\vec{k}\cdot\vec{q}) + 2k^2 (\vec{k}\cdot\vec{q})^2 - (\vec{k}\cdot\vec{q})^3] J_k(\vec{k},\vec{q}) - 32 k^2 (\vec{k}\cdot\vec{q})(k^2 - \vec{k}\cdot\vec{q}) J_q k(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(q_i q_j q_m k_n + q_i q_j k_m q_n + q_i k_j q_m q_n + k_i q_j q_m q_n)\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 128 J_{kqqq}(\vec{k},\vec{q}) \\
&- 32(k^2 + \vec{k}\cdot\vec{q}) J_{qqq}(\vec{k},\vec{q}) - 96(q^2 + \vec{k}\cdot\vec{q}) J_{kqq}(\vec{k},\vec{q}) \\
&+ 32[k^2 q^2 + (\vec{k}\cdot\vec{q})^2] J_{qq}(\vec{k},\vec{q}) + 64 q^2 (\vec{k}\cdot\vec{q}) J_{kq}(\vec{k},\vec{q}),
\end{aligned}
$$

$$
\begin{aligned}
&(k_i k_j q_m q_n + q_i q_j k_m k_n + k_i q_j k_m q_n + q_i k_j q_m k_n + k_i q_j q_m k_n + q_i k_j k_m q_n)\{V_{nr}(\vec{k},\vec{q})\}_{ijmn} \\
&= 192 J_{kkqq}(\vec{k},\vec{q}) \\
&- 96(q^2 + \vec{k}\cdot\vec{q}) J_{kkq}(\vec{k},\vec{q}) - 96(k^2 + \vec{k}\cdot\vec{q}) J_{kqq}(\vec{k},\vec{q}) \\
&+ 32 q^2 (\vec{k}\cdot\vec{q}) J_{kk}(\vec{k},\vec{q}) + 32 k^2 (\vec{k}\cdot\vec{q}) J_{qq}(\vec{k},\vec{q}) \\
&+ 64[k^2 q^2 + (\vec{k}\cdot\vec{q})^2] J_{kq}(\vec{k},\vec{q}).
\end{aligned}
\tag{3.28}
$$

Chapter 4

Conclusion

在本篇論文中我們利用了量子場論來計算在Graphene中的current-current correlator。我們在計算最低階的效應時，電子間的交互作用完全沒有貢獻。得到$\langle J_\mu J_\nu \rangle = \frac{C}{q}(q_\mu q_\nu - q^2\delta_{\mu\nu})$這個結果，其中C為某一常數。 在計算一階修正時，電子間的交互作用開始發揮影響，會因此產生Egg 跟Self energy 兩種Feynman diagram，最後我們可以利用(3.7)以及(3.16)所得到的結果再加上對光子的動量q作積分後，可以得到一階微擾後的coorelator，而我們可以利用這個correlator去計算出Graphene的conductivity。我們可以由$\rho\rho$ correlator計算出Graphene電導的一階修正，並且利用particle conservation condition，即

$$\sigma(\omega) = \lim_{q\to 0} \frac{ie^2\omega}{q^2}\langle J_0(x)J_0(x)\rangle. \tag{4.1}$$

得到結果為$\sigma = C_2\sigma_0\alpha_g$，其中$\alpha_g$為graphene中的coupling constant，常數C_2則約等於0.01[4]。 另一方面，也可以經由Kubo formula用JJ correlator來計算電導：

$$\sigma(\omega) = \frac{1}{\omega}(\langle J_i(x)J_i(0)\rangle - \langle J_i(0)J_i(0)\rangle). \tag{4.2}$$

而得到的結果為$\sigma = (2C_1 - C_2)\sigma_0\alpha_g$，其中常數$C_1$約等於0.51[4]。兩種計算方法之所以得到不一樣的結果，主要是因為在Graphene中電子的等效質量為零，系統幾乎沒有disorder，而且Coulomb interaction又是長程作用力。以上這三個因素綜合起來會破壞電導在長波極限下的解析性質。這個預測可以透過實驗Corbino disk[4]來驗證是否正確。

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Appendix A

Tensor analysis for egg diagram

在第三章提到egg diagram的propagator所產生的積分可以用張量分析來計算，本附錄是其詳細的計算過程。首先

$$
\begin{aligned}
I_k(\vec{k},\vec{q}) \equiv k_i I_i(\vec{k},\vec{q}) &= \int \frac{d^3p}{(2\pi)^3}\frac{\vec{k}\cdot\vec{p}}{P_a^2P_b^2P_c^2P_d^2} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{P_b^2-P_a^2-k^2}{2P_a^2P_b^2P_c^2P_d^2} \\
&= \int \frac{d^3p}{(2\pi)^3}\{\frac{1}{2P_a^2P_c^2P_d^2}-\frac{1}{2P_b^2P_c^2P_d^2}-\frac{k^2}{2P_a^2P_b^2P_c^2P_d^2}\}.
\end{aligned}
\tag{A.1}
$$

上式的積分可以利用留數定理去計算，即

$$
\begin{aligned}
&\int \frac{d^3p}{(2\pi)^3}\{\frac{1}{2P_a^2P_c^2P_d^2}-\frac{1}{2P_b^2P_c^2P_d^2}-\frac{k^2}{2P_a^2P_b^2P_c^2P_d^2}\} \\
=&\frac{\pi^3}{2kq|\vec{k}-\vec{q}|}-\frac{\pi^3}{2kq|\vec{k}+\vec{q}|} \\
&-(k^2)[\frac{\pi^3}{2kq(\vec{k}\cdot\vec{q})|\vec{k}-\vec{q}|}-\frac{\pi^3}{2kq(\vec{k}\cdot\vec{q})|\vec{k}+\vec{q}|}].
\end{aligned}
\tag{A.2}
$$

用同樣的方法來算$I_q(\vec{k},\vec{q})$

$$
I_q(\vec{k},\vec{q}) \equiv q_i I_i(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{q}\cdot\vec{p}}{P_a^2P_b^2P_c^2P_d^2}. \tag{A.3}
$$

因為$I_q(\vec{k},\vec{q})$會等於

$$
\begin{aligned}
I_q(\vec{k},\vec{q}) =& -I_k(\vec{q},\vec{k}) \\
=&\frac{-\pi^3}{2kq|\vec{k}-\vec{q}|}+\frac{\pi^3}{2kq|\vec{k}+\vec{q}|}+(q^2)[\frac{\pi^3}{2kq(\vec{k}\cdot\vec{q})|\vec{k}-\vec{q}|}-\frac{\pi^3}{2kq(\vec{k}\cdot\vec{q})|\vec{k}+\vec{q}|}].
\end{aligned}
\tag{A.4}
$$

有了I_k, I_q後可以利用(3.15)中的$I_i(\vec{k},\vec{q}) = A_1(\vec{k},\vec{q})k_i + A_2(\vec{k},\vec{q})q_i$來做比對即可知道係數$A_1, A_2$為多少，即

$$\begin{cases} I_k = A_1 k^2 + A_2(\vec{k}\cdot\vec{q}), \\ I_q = A_1(\vec{k}\cdot\vec{q}) + A_2 q^2, \end{cases}$$
$$\Rightarrow \begin{cases} A_1 = \dfrac{q^2 I_k - (\vec{k}\cdot\vec{q})I_q}{k^2q^2 - (\vec{k}\cdot\vec{q})^2} = \dfrac{-\pi^3}{2kq(\vec{k}\cdot\vec{q})}[\dfrac{1}{\vec{k}-\vec{q}} - \dfrac{1}{|\vec{k}+\vec{q}|}], \\ A_2 = \dfrac{k^2 I_q - (\vec{k}\cdot\vec{q})I_k}{k^2q^2 - (\vec{k}\cdot\vec{q})^2} = \dfrac{\pi^3}{2kq(\vec{k}\cdot\vec{q})}[\dfrac{1}{\vec{k}-\vec{q}} - \dfrac{1}{|\vec{k}+\vec{q}|}]. \end{cases} \tag{A.5}$$

其他階張量的係數C,E,F亦可用同樣的方法來計算。

$$\begin{aligned} I_\delta(\vec{k},\vec{q}) &\equiv \delta_{ij} I_{ij}(\vec{k},\vec{q}) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{p^2}{P_a^2P_b^2P_c^2P_d^2} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{P_b^2P_c^2P_d^2} \\ &= \frac{\pi^3}{kq|\vec{k}+\vec{q}|}. \end{aligned} \tag{A.6}$$

I_{kk}, I_{qq}, I_{kq}也是一樣

$$\begin{aligned} I_{kk}(\vec{k},\vec{q}) &\equiv k_i k_j I_{ij}(\vec{k},\vec{q}) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})^2}{P_a^2P_b^2P_c^2P_d^2}, \\ I_{qq}(\vec{k},\vec{q}) &\equiv q_i q_j I_{ij}(\vec{k},\vec{q}) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{q}\cdot\vec{p})^2}{P_a^2P_b^2P_c^2P_d^2}, \\ I_{kq}(\vec{k},\vec{q}) &\equiv k_i q_j I_{ij}(\vec{k},\vec{q}) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{P_a^2P_b^2P_c^2P_d^2}. \end{aligned} \tag{A.7}$$

為了使計算簡單化，所以可以將$(\vec{k}\cdot\vec{p})^2$化簡為$P_a^2\sim P_d^2$的組合，其中因為$\vec{k}\cdot\vec{p}=(P_b-P_a)\cdot P_a=(P_c-P_d)\cdot P_a$所以

$$\begin{aligned}(\vec{k}\cdot\vec{p})^2 =&(-\frac{1}{2})[(P_a-P_b)^2+P_a^2-P_b^2](-\frac{1}{2})[(P_a-P_c)^2-(P_a-P_d)^2-P_c^2+P_d^2]\\ =&(\frac{1}{4})[k^2+P_a^2-P_b^2][(k-q)^2-q^2-P_c^2+P_d^2]\\ =&(\frac{1}{4})[k^2(k^2-2\vec{k}\cdot\vec{q})+(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)+k^2(-P_c^2+P_d^2)\\ &-P_a^2P_c^2+P_a^2P_d^2+P_b^2P_c^2-P_b^2P_d^2].\end{aligned} \tag{A.8}$$

將上式帶入(A.7)中的第一個積分式可以得到

$$\begin{aligned}I_{kk} =&\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\{\frac{k^2(k^2-2\vec{k}\cdot\vec{q})}{P_a^2P_b^2P_c^2P_d^2}+\frac{k^2-2\vec{k}\cdot\vec{q}}{P_b^2P_c^2P_d^2}-\frac{k^2-2\vec{k}\cdot\vec{q}}{P_a^2P_c^2P_d^2}-\frac{k^2}{P_a^2P_b^2P_d^2}+\frac{k^2}{P_a^2P_b^2P_c^2}\\ &-\frac{1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_d^2}+\frac{1}{P_a^2P_c^2}\}\\ =&\frac{1}{4}\{k^2(k^2-2\vec{k}\cdot\vec{q})\frac{\pi^3}{kq(\vec{k}\cdot\vec{q})}[-\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}]+(k^2-2\vec{k}\cdot\vec{q})\frac{\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]\\ &+k^2\frac{\pi^3}{kq}[-\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}]+\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{q}+\frac{1}{q}-\frac{1}{|\vec{k}-\vec{q}|}]\}\\ =&\frac{1}{4}\{(k^4)\frac{\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}-\vec{q}|}-\frac{1}{|\vec{k}+\vec{q}|}]\\ &+2(\vec{k}\cdot\vec{q}-k^2)\frac{\pi^3}{kq}[\frac{1}{|\vec{k}-\vec{q}|}-\frac{1}{|\vec{k}+\vec{q}|}]\\ &+\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}+\frac{2}{q}]\}.\end{aligned} \tag{A.9}$$

而I_{qq}與I_q類似，將$I_{kk}(\vec{k},\vec{q})$中的k,q對調可以得到I_{qq}，

$$I_{kq}(\vec{q},\vec{k})=\int\frac{d^3p}{(2\pi)^3}\frac{(\vec{q}\cdot\vec{p})^2}{P_a^2P_b^2P_c^2P_d^2}=I_{qq}(\vec{k},\vec{q}). \tag{A.10}$$

所以

$$\begin{aligned}I_{qq} =&\frac{1}{4}\{(q^4)\frac{\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}-\vec{q}|}-\frac{1}{|\vec{k}+\vec{q}|}]\\ &+2(\vec{k}\cdot\vec{q}-q^2)\frac{\pi^3}{kq}[\frac{1}{|\vec{k}-\vec{q}|}-\frac{1}{|\vec{k}+\vec{q}|}]\\ &+\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}+\frac{2}{k}]\}.\end{aligned} \tag{A.11}$$

同樣地在(A.7)第三個積分式中的$(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})$也可以簡化成$P_a^2\sim P_d^2$的組合，其中因為$\vec{q}\cdot\vec{p}=(P_a-P_d)\cdot P_a=(P_b-P_c)\cdot P_a$，所以

$$\begin{aligned}(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})=&(-\frac{1}{2})[(P_a-P_b)^2+P_a^2-P_b^2](\frac{1}{2})[(P_a-P_d)^2+P_a^2+P_d^2],\\ =&(-\frac{1}{4})[k^2+P_a^2-P_b^2][q^2+P_a^2-P_d^2],\\ =&(-\frac{1}{4})[k^2q^2+q^2(P_a^2-P_b^2)+k^2(P_a^2-P_d^2)\\ &+P_a^4-P_a^2P_b^2-P_a^2P_d^2+P_b^2P_d^2].\end{aligned}\tag{A.12}$$

代回(A.7)中第三個積分式

$$\begin{aligned}I_{kq}=&-\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\{\frac{k^2q^2}{P_a^2P_b^2P_c^2P_d^2}+\frac{q^2}{P_b^2P_c^2P_d^2}-\frac{q^2}{P_a^2P_c^2P_d^2}+\frac{k^2}{P_b^2P_c^2P_d^2}-\frac{k^2}{P_a^2P_b^2P_c^2}\\ &+\frac{P_a^2}{P_b^2P_c^2P_d^2}-\frac{1}{P_c^2P_d^2}-\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_c^2}\},\\ =&-\frac{1}{4}\{k^2q^2\frac{\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}-\vec{q}|}-\frac{1}{|\vec{k}+\vec{q}|}]\\ &+q^2\frac{\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]\\ &+k^2\frac{\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]\\ &+\pi^3[-\frac{2\vec{k}\cdot\vec{q}}{kq|\vec{k}+\vec{q}|}+\frac{1}{k}+\frac{1}{q}-\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{k}-\frac{1}{q}+\frac{1}{|\vec{k}-\vec{q}|}]\},\\ =&-\frac{1}{4}\{(k^2q^2)\frac{\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}-\vec{q}|}-\frac{1}{|\vec{k}+\vec{q}|}]\\ &+\frac{\pi^3}{kq}[\frac{|\vec{k}-\vec{q}|^2}{|\vec{k}+\vec{q}|}-\frac{(k^2+q^2)}{|\vec{k}-\vec{q}|}]\\ &+\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]\}.\end{aligned}\tag{A.13}$$

最後跟解A_1,A_2的方法一樣，因為$I_{ij}(\vec{k},\vec{q})=C_1(\vec{k},\vec{q})\delta_{ij}+C_2(\vec{k},\vec{q})k_ik_j+C_3(\vec{k},\vec{q})q_iq_j+C_4(\vec{k},\vec{q})(k_iq_j+q_ik_j)$，所以可以利用先前所算出來的$I_\delta,I_{kk},I_{qq},I_{kq}$來做比對進一步取得係數C，則

$$\begin{cases}I_\delta=3C_1+k^2C_2+q^2C_3+2\vec{k}\cdot\vec{q}C_4,\\ I_{kk}=k^2C_1+k^4C_2+(\vec{k}\cdot\vec{q})^2C_3+2k^2(\vec{k}\cdot\vec{q})C_4,\\ I_{qq}=q^2C_1+(\vec{k}\cdot\vec{q})^2C_2+q^4C_3+2q^2(\vec{k}\cdot\vec{q})C_4,\\ I_{kq}=\vec{k}\cdot\vec{q}C_1+2k^2(\vec{k}\cdot\vec{q})C_2+q^2C_3+[k^2q^2+(\vec{k}\cdot\vec{q})^2]C_4.\end{cases}\tag{A.14}$$

以上是一階以及二階張量的部分，而三階和四階所用的方法也亦同

$$
\begin{aligned}
I_{\delta k}(\vec{k},\vec{q}) \equiv & \delta_{ij}k_m I_{ijm}(\vec{k},\vec{q}) \\
= & \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})}{P_b^2P_c^2P_d^2}.
\end{aligned}
\tag{A.15}
$$

這裡將動量p轉變為$-p-k+q$，則

$$
\begin{cases}
P_b \to -p+q = -P_d, \\
P_c \to -p = -P_a, \\
P_d \to -p-k = -P_b.
\end{cases}
\tag{A.16}
$$

而(A.15)會轉變成

$$
\begin{aligned}
I_{\delta k}(\vec{k},\vec{q}) = & \int \frac{d^3p}{(2\pi)^3}\frac{k\cdot(-p-k+q)}{P_a^2P_b^2P_d^2}, \\
= & \int \frac{d^3p}{(2\pi)^3}\{-\frac{\vec{k}\cdot\vec{p}}{P_a^2P_b^2P_d^2}-\frac{k\cdot k}{P_a^2P_b^2P_d^2}+\frac{\vec{k}\cdot\vec{q}}{P_a^2P_b^2P_d^2}\}, \\
= & \int \frac{d^3p}{(2\pi)^3}\{\frac{k^2+P_a^2-P_b^2}{P_a^2P_b^2P_d^2}+(\vec{k}\cdot\vec{q}-k^2)\frac{1}{P_a^2P_b^2P_d^2}\}, \\
= & \pi^3\{\frac{-1}{2q}+\frac{1}{2|\vec{k}+\vec{q}|}+\frac{k}{2q|\vec{k}+\vec{q}|}+\frac{\vec{k}\cdot\vec{q}-k^2}{kq|\vec{k}+\vec{q}|}\}, \\
= & \pi^3\{\frac{-1}{2q}+\frac{1}{2|\vec{k}+\vec{q}|}+\frac{2\vec{k}\cdot\vec{q}-k^2}{2kq|\vec{k}+\vec{q}|}\}.
\end{aligned}
\tag{A.17}
$$

相同地將k跟q的位置對調，$I_{\delta q}(\vec{k},\vec{q}) \equiv \delta_{ij}q_m I_{ijm}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{q}\cdot\vec{p}}{P_b^2P_c^2P_d^2} = -I_{\delta k}(\vec{q},\vec{k})$。依照同樣的方法可以得到$I_{kkk}, I_{qqq}, I_{kkq}, I_{kqq}$而分別為

$$
\begin{cases}
I_{kkk} = & -\frac{1}{8}\{\frac{-k^5\pi^3}{q(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
& +\frac{[3k^4-6(\vec{k}\cdot\vec{q})k^2+4(\vec{k}\cdot\vec{q})^2]\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
& +3(k^2-\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{-1}{|\vec{k}-\vec{q}}+\frac{2}{q}]\}, \\
I_{qqq} = & -I_{kkk}(\vec{q},\vec{k}) \\
& =\frac{1}{8}\{\frac{-q^5\pi^3}{q(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
& +\frac{[3q^4-6(\vec{k}\cdot\vec{q})q^2+4(\vec{k}\cdot\vec{q})^2]\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
& +3(q^2-\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{-1}{|\vec{k}-\vec{q}|}+\frac{2}{q}]\}, \\
I_{kkq} = & \frac{1}{8}\{\frac{-k^3q\pi^3}{(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]-\frac{4(k^2-\vec{k}\cdot\vec{q})\vec{k}\cdot\vec{q}\pi^3}{kq|\vec{k}+\vec{q}|} \\
& +\frac{[k^4-2(k^2q^2)-2q^2(\vec{k}\cdot\vec{q})]\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
& +2(k^2-\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
& +(q^2-\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{-1}{|\vec{k}-\vec{q}|}+\frac{2}{q}]\}, \\
I_{kqq} = & -I_{kkq}(\vec{q},\vec{k}), \\
& =-\frac{1}{8}\{\frac{-kq^3\pi^3}{(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]-\frac{4(q^2-\vec{k}\cdot\vec{q})\vec{k}\cdot\vec{q}\pi^3}{kq|\vec{k}+\vec{q}|} \\
& +\frac{[q^4-2(k^2q^2)-2q^2(\vec{k}\cdot\vec{q})]\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
& +2(q^2-\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
& +(k^2-\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{-1}{|\vec{k}-\vec{q}|}+\frac{2}{k}]\}.
\end{cases}
\tag{A.18}
$$

從(3.15)式可知第四階需要$I_{\delta\delta}, I_{\delta kk}, I_{\delta qq}, I_{\delta kq}, I_{kkkk}, I_{qqqq}, I_{kqqq}, I_{kkqq}, I_{kkkq}$， 首先是$I_{\delta\delta}$，依照前面的方法將$I_{\delta\delta}$定義為$\delta_{ij}\delta_{mn}I_{ijmn}(\vec{k},\vec{q})$，則

$$\begin{aligned} I_{\delta\delta} \equiv& \delta_{ij}\delta_{mn}I_{ijmn}(\vec{k},\vec{q}) \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{P_a^2}{P_b^2P_c^2P_d^2}, \\ =& \pi^3[\frac{-2\vec{k}\cdot\vec{q}}{kq|\vec{k}+\vec{q}|}+\frac{1}{k}+\frac{1}{q}-\frac{1}{|\vec{k}+\vec{q}|}]. \end{aligned} \tag{A.19}$$

第二是$I_{\delta kk}(\vec{k},\vec{q})$

$$\begin{aligned} I_{\delta kk}(\vec{k},\vec{q}) \equiv& \delta_{ij}k_mk_nI_{ijmn}(\vec{k},\vec{q}), \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})^2}{P_b^2P_c^2P_d^2}. \end{aligned} \tag{A.20}$$

因為$(\vec{k}\cdot\vec{p})^2 = (\frac{1}{4})[(k-q)^2-q^2-P_c^2+P_d^2]^2$，帶入上式後積分

$$\begin{aligned} I_{\delta kk} =& (k^2-2\vec{k}\cdot\vec{q})^2\frac{\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}] \\ &+\pi^3[\frac{-2q^2+3\vec{k}\cdot\vec{q}}{|\vec{k}+\vec{q}|}+\frac{3q^3-3\vec{k}\cdot\vec{q}}{|\vec{k}|}]. \end{aligned} \tag{A.21}$$

而$I_{\delta qq}(\vec{k},\vec{q})$則會等於$I_{\delta kk}(\vec{q},\vec{k})$。第四個$I_{\delta kq}$的積分會用到的$(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})$可以變成

$$\begin{aligned} \vec{k}\cdot\vec{p} =& (P_c-P_d)P_a, \qquad (P_b-P_c)P_a \\ (\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p}) =& (-\frac{1}{2})^2[(P_a-P_c)^2-(P_a-P_d)^2-P_c^2+P_d^2] \\ &\times[(P_a-P_b)^2-(P_a-P_c)^2-P_b^2+P_c^2] \\ =& (\frac{1}{4})[(k^2-2\vec{k}\cdot\vec{q})-P_c^2+P_d^2][(2\vec{k}\cdot\vec{q}-q^2)-P_b^2+P_c^2] \\ =& \frac{1}{4}\{(k^2-2\vec{k}\cdot\vec{q})(2\vec{k}\cdot\vec{q}-q^2)+(k^2-2\vec{k}\cdot\vec{q})(-P_b^2+P_c^2) \\ &+(2\vec{k}\cdot\vec{q}-q^2)(-P_c^2+P_d^2)-[2q\cdot(p+k)-q^2](-P_c^2+P_d^2)\}. \end{aligned} \tag{A.22}$$

$I_{\delta kq}$因為帶入上式積分後

$$
\begin{aligned}
I_{\delta kq} =& \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})}{P_b^2P_c^2P_d^2} \\
=&\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\{\frac{(k^2-2\vec{k}\cdot\vec{q})(2\vec{k}\cdot\vec{q}-q^2)}{P_b^2P_c^2P_d^2}+(k^2-2\vec{k}\cdot\vec{q})\frac{(-P_b^2+P_c^2)}{P_b^2P_c^2P_d^2} \\
&+(2\vec{k}\cdot\vec{q}-q^2)\frac{(-P_c^2+P_d^2)}{P_b^2P_c^2P_d^2}-[2q\cdot(p+k)]\frac{(-P_c^2+P_d^2)}{P_b^2P_c^2P_d^2}\} \\
=&\frac{1}{4}\{(k^2-2\vec{k}\cdot\vec{q})(2\vec{k}\cdot\vec{q}-q^2)[\frac{\pi^3}{kq|\vec{k}+\vec{q}|}]+(k^2-2\vec{k}\cdot\vec{q})[-\frac{\pi^3}{k}-\frac{\pi^3}{|\vec{k}+\vec{q}|}] \\
&+(2\vec{k}\cdot\vec{q}-q^2)[-\frac{\pi^3}{|\vec{k}+\vec{q}|}+\frac{\pi^3}{q}]+(q^2-2\vec{k}\cdot\vec{q})[-\frac{\pi^3}{|\vec{k}+\vec{q}|}+\frac{\pi^3}{q}] \\
&+(-2q)[-\frac{(q+k-2k)\pi^3}{2|\vec{k}+\vec{q}|}+\frac{(q-2k)\pi^3}{q}]\} \\
=&\frac{1}{4}\{(k^2-2\vec{k}\cdot\vec{q})(2\vec{k}\cdot\vec{q}-q^2)[\frac{\pi^3}{kq|\vec{k}+\vec{q}|}] \\
&+[\frac{(k^2+q^2-3\vec{k}\cdot\vec{q})\pi^3}{|\vec{k}+\vec{q}|}-\frac{(k^2-2\vec{k}\cdot\vec{q})\pi^3}{k}] \\
&-\frac{(q^2-2\vec{k}\cdot\vec{q})\pi^3}{q}.
\end{aligned}
\tag{A.23}
$$

I_{kkkk}積分中的$(\vec{k}\cdot\vec{p})^4$可以用以下的方式替換

$$
\begin{aligned}
(\vec{k}\cdot\vec{p})^4 =&(-\frac{1}{2})^2[(P_a-P_b)^2+P_a^2-P_b^2]^2(-\frac{1}{2})^2[(P_a-P_c)^2-(P_a-P_d)^2-P_c^2-P_d^2]^2 \\
=&\frac{1}{16}[k^2+P_a^2-P_b^2]^2[(k-q)^2-q^2-P_c^2+P_d^2]^2 \\
=&\frac{1}{16}[k^4+2k^2(P_a^2-P_b^2)+(P_a^4+P_b^4-2P_a^2P_b^2)] \\
&\times[(k^2-2\vec{k}\cdot\vec{q})^2+2(k^2-2\vec{k}\cdot\vec{q})(-P_c^2+P_d^2)+(P_c^4+P_d^4-2P_c^2P_d^2)] \\
=&\frac{1}{16}[k^4(k^2-2\vec{k}\cdot\vec{q})^2+2k^2(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)+2k^4(k^2-2\vec{k}\cdot\vec{q})(-P_c^2+P_d^2) \\
&+(k^2-2\vec{k}\cdot\vec{q})^2(P_a^4+P_b^4-2P_a^2P_b^2) \\
&+k^4(P_c^4+P_d^4-2P_c^2P_d^2) \\
&+4k^2(k^2-2\vec{k}\cdot\vec{q})(-P_a^2P_c^2+P_a^2P_d^2+P_b^2P_c^2-P_b^2P_d^2) \\
&+2(k^2-2\vec{k}\cdot\vec{q})(-P_c^2+P_d^2)(P_a^2-P_b^2)^2 \\
&+2k^2(-P_c^2+P_d^2)^2(P_a^2-P_b^2) \\
&+(-P_c^2+P_d^2)^2(P_a^2-P_b^2)^2].
\end{aligned}
\tag{A.24}
$$

積分後的I_{kkkk}為

$$
\begin{aligned}
I_{kkkk} =& k_i k_j k_m k_n I_{ijmn}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})^4}{P_a^2P_b^2P_c^2P_d^2} \\
=& \frac{1}{16}\{k^4(k^2-2\vec{k}\cdot\vec{q})^2\frac{-\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+2k^2(k^2-2\vec{k}\cdot\vec{q})^2\frac{\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}]+2k^4(k^2-2\vec{k}\cdot\vec{q})^2\frac{\pi^3}{kq}[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
&+(k^2-2\vec{k}\cdot\vec{q})^2\pi^3[\frac{-2\vec{k}\cdot\vec{q}}{kq|\vec{k}+\vec{q}|}+\frac{1}{k}+\frac{1}{q}-\frac{1}{|\vec{k}+\vec{q}|}+\frac{2\vec{k}\cdot\vec{q}}{kq|\vec{k}-\vec{q}|}+\frac{1}{k}+\frac{1}{q}-\frac{1}{|\vec{k}-\vec{q}|}-\frac{2}{k}] \\
&+k^4\pi^3[\frac{-2\vec{k}\cdot\vec{q}}{kq|\vec{k}+\vec{q}|}+\frac{1}{k}+\frac{1}{q}-\frac{1}{|\vec{k}+\vec{q}|}+\frac{2\vec{k}\cdot\vec{q}}{kq|\vec{k}-\vec{q}|}+\frac{1}{k}+\frac{1}{q}-\frac{1}{|\vec{k}-\vec{q}|}-\frac{2}{k}] \\
&+4k^2(k^2-2\vec{k}\cdot\vec{q})\pi^3[\frac{-1}{|\vec{k}+\vec{q}|}+\frac{2}{q}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+2(k^2-2\vec{k}\cdot\vec{q})(\vec{k}\cdot\vec{q})\pi^3[\frac{1}{|\vec{k}-\vec{q}|}+\frac{2}{q}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+2k^2(\vec{k}\cdot\vec{q})\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}+\frac{2}{q}+\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\pi^3\frac{2k^4-k^2q^2-(\vec{k}\cdot\vec{q})^2}{q}\} \\
=& \frac{1}{16}\{\frac{-k^4(k^2-\vec{k}\cdot\vec{q})^2\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\frac{-8(\vec{k}\cdot\vec{q})(k^2-\vec{k}\cdot\vec{q})^2\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{\vec{k}-\vec{q}}] \\
&+2[3k^4-6(\vec{k}\cdot\vec{q})k^2+4(\vec{k}\cdot\vec{q})^2]\pi^3 \\
&\quad\times[-\frac{\pi^3}{|\vec{k}+\vec{q}|}-\frac{\pi^3}{|\vec{k}-\vec{q}|}+\frac{2\pi^3}{q}] \\
&+\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2}[\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\frac{\pi^3[2k^4-k^2q^2-(\vec{k}\cdot\vec{q})^2]}{q}\}.
\end{aligned}
\tag{A.25}
$$

(A.24)式中最後一項$(P_a^2-P_b^2)^2(-P_c^2+P_d^2)^2$，的積分比先前的複雜，所以先對它拆解來簡化

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(P_a^2-P_b^2)^2(-P_c^2+P_d^2)^2}{P_a^2P_b^2P_c^2P_d^2}\\
=&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(P_a^2-P_b^2)(-P_c^2-P_d^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_d^2}-\frac{1}{P_a^2P_c^2}\}\\
=&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(-2k\cdot(p-q)-k^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_d^2}-\frac{1}{P_a^2P_c^2}\}.
\end{aligned}
\tag{A.26}
$$

其中對$\frac{-1}{P_b^2P_d^2}$和$\frac{1}{P_a^2P_d^2}$這兩項的積分使用了Feynman trick使得

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)[-2k\cdot(p-q)-k^2]\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\}\\
&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)[-2k\cdot(p-q)-k^2]\\
&\times\{\frac{-1}{\{[p+\alpha k-(1-\alpha)q]^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})\}^2}\\
&+\{\frac{-1}{\{[p-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}\}.
\end{aligned}
\tag{A.27}
$$

由於單獨一項從負無限大積分到無限大會得到發散的結果，所以這裡把積分的上下限從無限大改成某個值Λ。這樣在計算過程中就可以放心地做變數的平移，等到最後再令Λ趨近於無限大，上式就變為

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_{-\Lambda-\alpha k+(1-\alpha)q_1}^{\Lambda-\alpha k+(1-\alpha)q_1}dp_1\int_{-\Lambda+(1-\alpha)q_2}^{\Lambda+(1-\alpha)q_2}dp_2\int_{-\Lambda}^{\Lambda}dp_3\\
&\quad\{-2k\cdot\{[p-\alpha k+(1-\alpha)q]-k^2\}\{-2k\cdot[p-q-\alpha k+(1-\alpha)q]-k^2\}\\
&\quad\frac{-1}{\{[p+\alpha(1-\alpha)(k^2+q^2++2\vec{k}\cdot\vec{q})]\}^2}\\
&\quad+\{\int_0^1 d\alpha\int_{-\Lambda+(1-\alpha)q_1}^{\Lambda+(1-\alpha)q_1}dp_1\int_{-\Lambda+(1-\alpha)q_2}^{\Lambda+(1-\alpha)q_2}dp_2\int_{-\Lambda}^{\Lambda}dp_3\\
&\quad\{-2k\cdot\{[p+(1-\alpha)q]-k^2\}\{-2k\cdot[p-q+(1-\alpha)q]-k^2\}\frac{1}{\{[p^2+\alpha(1-\alpha)q^2]\}^2}\}.
\end{aligned}
\tag{A.28}
$$

之後再將積分的上下限做拆解

$$
\begin{aligned}
&\int_{-\Lambda-\alpha k+(1-\alpha)q_1}^{\Lambda-\alpha k+(1-\alpha)q_1}dp_1\int_{-\Lambda+(1-\alpha)q_2}^{\Lambda+(1-\alpha)q_2}dp_2\\
=&\int_{-\Lambda}^{\Lambda}dp_1\int_{-\Lambda}^{\Lambda}dp_2+\int_{\Lambda}^{\Lambda-\alpha k+(1-\alpha)q_1}dp_1\int_{-\Lambda+(1-\alpha)q_2}^{-\Lambda}dp_2\\
+&\int_{-\Lambda-\alpha k+(1-\alpha)q_1}^{-\Lambda}dp_1\int_{\Lambda}^{\Lambda+(1-\alpha)q_2}dp_2+\int_{-\Lambda-\alpha k+(1-\alpha)q_1}^{-\Lambda}dp_1\int_{-\Lambda+(1-\alpha)q_2}^{-\Lambda}dp_2
\end{aligned}
\tag{A.29}
$$

因此積分會變成

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}dp_1\int_{-\Lambda}^{\Lambda}dp_2\int_{-\Lambda}^{\Lambda}dp_3\\
&\quad\{(-2\vec{k}\cdot\vec{p})^2+[(2\alpha-1)k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(2\alpha-1)k^2+2\alpha k\cdot q]\}\\
&\quad\{\frac{-1}{[p^2+\alpha(1-\alpha)(k^2+q^q+2\vec{k}\cdot\vec{q})]^2}\}\\
&\quad+\{\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}dp_1\int_{-\Lambda}^{\Lambda}dp_2\int_{-\Lambda}^{\Lambda}dp_3\\
&\quad\{(-2\vec{k}\cdot\vec{p})^2+[-k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][-k^2+2\alpha\vec{k}\cdot\vec{q}]\}\\
&\quad\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\\
&\quad+\int_0^1\alpha\int_{-\Lambda}^{\Lambda}dp_3\{(1-1-1+1)[-\alpha k+(1-\alpha)q_1][(1-\alpha)q_2]\\
&\quad\frac{(-2k\Lambda)^2}{[2\Lambda^2+p_3^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\}\\
&\quad+\int_0^1\alpha\int_{-\Lambda}^{\Lambda}dp_3\{(1-1-1+1)[-\alpha k+(1-\alpha)q_1][(1-\alpha)q_2]\\
&\quad\frac{(-2k\Lambda)^2}{[2\Lambda^2+p_3^2+\alpha(1-\alpha)q]^2}\}.
\end{aligned}
\tag{A.30}
$$

帶回到(A.28)之後

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_0^{\Lambda}4\pi p^2dp\{\frac{-1}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\}\\
&\quad\{\frac{4}{3}k^2p^2+[(2\alpha-1)k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(2\alpha-1)k^2+2\alpha\vec{k}\cdot\vec{q}]\}\\
&\quad+\int_0^1 d\alpha\int_0^{\Lambda}4\pi p^2dp\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\{\frac{4}{3}k^2p^2+[-k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][-k^2+2\alpha\vec{k}\cdot\vec{q}]\}.
\end{aligned}
\tag{A.31}
$$

因為

$$
\begin{cases}
\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{\frac{p^2}{(p^2+M_1^2)^2}-\frac{p^2}{(p^2+M_2^2)^2}\}=3\pi^3(-M_1+M_2),\\
\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{\frac{1}{(p^2+M^2)^2}\}=\frac{\pi}{M}.
\end{cases}
\tag{A.32}
$$

所以(A.31)式變成

$$
\begin{aligned}
&\int_0^1 d\alpha 4\pi\{4\pi^2k^2[\sqrt{\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})}-\sqrt{\alpha(1-\alpha)q^2}] \\
&\quad -\pi^2\frac{[(2\alpha-1)k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(2\alpha-1)k^2+2\alpha\vec{k}\cdot\vec{q}]}{\sqrt{\alpha(1-\alpha)(k^2+q^2 2\vec{k}\cdot\vec{q})}} \\
&\quad +\pi^2\frac{[-k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(-k^2+2\alpha\vec{k}\cdot\vec{q}]}{\sqrt{(1-\alpha)q^2}})\} \\
&=\frac{k^2\pi^3}{2}[|\vec{k}-\vec{q}|-q]-\frac{\pi^3[k^4-2(\vec{k}\cdot\vec{q})k^2-(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}-\vec{q}|}+\frac{\pi^3[2k^4-(\vec{k}\cdot\vec{q})^2]}{2q}.
\end{aligned}
\tag{A.33}
$$

再來是用一樣的方法算$\frac{1}{P_b^2P_c^2}-\frac{1}{P_a^2P_c^2}$項的積分

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)[-2k\cdot(p-q)-k^2]\{\frac{1}{P_b^2P_c^2}-\frac{1}{P_a^2P_c^2}\} \\
&=\frac{k^2\pi^3}{2}[|\vec{k}-\vec{q}|-q]-\frac{\pi^3[k^4-2(\vec{k}\cdot\vec{q})k^2-(\vec{k}\cdot\vec{q})^2}{2|\vec{k}+\vec{q}|}+\frac{\pi^3[2k^4-(\vec{k}\cdot\vec{q})^2]}{2q}.
\end{aligned}
\tag{A.34}
$$

最後把兩項加起來，則

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(P_a^2-P_b^2)^2(-P_c^2+P_d^2)^2}{P_a^2P_b^2P_c^2P_d^2} \\
&=\frac{k^2\pi^3}{2}[|\vec{k}+\vec{q}|-q]-\frac{\pi^3[k^4+2(\vec{k}\cdot\vec{q})k^2-(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}+\vec{q}|}+\frac{\pi^3[2k^4-(\vec{k}\cdot\vec{q})^2]}{2q} \\
&\quad+\frac{k^2\pi^3}{2}[|\vec{k}-\vec{q}|-q]-\frac{\pi^3[k^4-2(\vec{k}\cdot\vec{q})k^2-(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}+\vec{q}|}+\frac{\pi^3[2k^4-(\vec{k}\cdot\vec{q})^2]}{2q} \\
&=\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2}\{\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}\}+\frac{\pi^3[2k^4-k^2q^2-(\vec{k}\cdot\vec{q})^2]}{q}.
\end{aligned}
\tag{A.35}
$$

第六個$I_{qqqq}(\vec{k},\vec{q})$

$$
\begin{aligned}
I_{qqqq} =& q_i q_j q_m q_n I_{ijmn}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{q}\cdot\vec{p})^4}{P_a^2 P_b^2 P_c^2 P_d^2} \\
=& I_{kkkk}(\vec{q},\vec{k}) \\
=& \frac{1}{16}\{-q^4(q^2-2\vec{k}\cdot\vec{q})^2\frac{\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\frac{-8(\vec{k}\cdot\vec{q})(q^2-\vec{k}\cdot\vec{q})^2\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+2[3q^4-6(\vec{k}\cdot\vec{q})q^2+4(\vec{k}\cdot\vec{q})^2]\pi^3[-\frac{\pi^3}{|\vec{k}-\vec{q}|}+\frac{2\pi^3}{k}-\frac{\pi^3}{|\vec{k}-\vec{q}|}] \\
&+\frac{k^2q^2+(\vec{k}\cdot\vec{q})^2]\pi^3}{2}[\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\pi^3\frac{2q^4-k^2q^2-(\vec{k}\cdot\vec{q})^2}{k}\}.
\end{aligned}
\tag{A.36}
$$

第七個I_{kkkq}中的$(\vec{k}\cdot\vec{p})^3(\vec{q}\cdot\vec{p})$為可拆解為

$$
\begin{aligned}
(\vec{k}\cdot\vec{p})^3(\vec{q}\cdot\vec{p}) =&(-\frac{1}{2})^2[(P_a-P_b)^2+P_a^2-P_b^2]^2(-\frac{1}{2})[(P_a-P_c)^2-(P_a-P_d)^2-P_c^2+P_d^2]\\
&\times(\frac{1}{2})[(P_a-P_d)^2+P_a^2-P_d^2]^2\\
=&-\frac{1}{16}[k^2+P_a^2-P_b^2]^2[(k-q)^2-q^2-P_c^2+P_d^2][q^2+P_a^2-P_d^2]\\
=&-\frac{1}{16}[k^4+2k^2(P_a^2-P_b^2)+(P_a^4+P_b^4-2P_a^2P_b^2)]\\
&\times[(q^2(k^2-2\vec{k}\cdot\vec{q})+(k^2-2\vec{k}\cdot\vec{q})(P_a^2+P_d^2)+q^2(-P_c^2+P_d^2)\\
&+(-P_a^2P_c^2+P_a^2P_d^2+P_c^2P_d^2-P_d^4)]\\
=&-\frac{1}{16}[k^4q^2(k^2-2\vec{k}\cdot\vec{q})+2k^2q^2(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)\\
&+k^4(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_d^2)\\
&+k^4q^2(-P_c^2+P_d^2)+q^2(k^2-2\vec{k}\cdot\vec{q})(P_a^4+P_b^4-2P_a^2P_b^2)\\
&+2k^2(k^2-2\vec{k}\cdot\vec{q})(P_a^4-P_a^2P_b^2-P_a^2P_d^2+P_b^2P_d^2)\\
&+2k^2q^2(-P_a^2P_c^2+P_a^2P_d^2+P_b^2P_c^2-P_b^2P_d^2)\\
&+k^4(-P_a^2P_c^2+P_a^2P_d^2+P_c^2P_d^2-P_d^4)\\
&+(k^2-2\vec{k}\cdot\vec{q})(P_a^2+P_d^2)(P_a^2-P_b^2)^2+q^2(-P_c^2+P_d^2)^2(P_a^2-P_b^2)\\
&+2k^2(P_a^2-P_d^2)(-P_c^2+P_d^2)(P_a^2-P_b^2)\\
&+(P_a^2-P_d^2)(-P_c^2+P_d^2)(P_a^2-P_b^2)^2].
\end{aligned}
\tag{A.37}
$$

跟I_{kkkk}一樣，在(A.37)中的最後一項$(P_a^2-P_d^2)(-P_c^2+P_d^2)(P_a^2-P_b^2)$也要用(A.26)∼(A.35)的方法去做積分，首先是將它化簡成四個相加的積分

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(P_a^2-P_d^2)(-P_c^2+P_d^2)(P_a^2-P_b^2)}{P_a^2P_b^2P_c^2P_d^2}\\
=&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(P_a^2-P_b^2)(P_a^2-P_d^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{Pb^2P_c^2}+\frac{1}{P_a^2P_d^2}+\frac{1}{P_a^2P_c^2}\}\\
=&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_d^2}+\frac{1}{P_a^2P_c^2}\}.
\end{aligned}
\tag{A.38}
$$

再來是先對$\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}$項用Feynman trick

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\\
&\times\{\frac{-1}{\{[p+\alpha k-(1-\alpha)q]^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})\}^2}\\
&+\frac{1}{\{[p-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{-4(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})+[-k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(1-2\alpha)q^2]\\
&\times\{\frac{-1}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\}\\
&+\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{-4(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})+[-k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(1-2\alpha)q^2]\}\\
&\times\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}.
\end{aligned}
\tag{A.39}
$$

之後再對動量的上下限先設個任意數Λ之後再取極限到無限大

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_0^{\Lambda}4\pi p^2dp\{\frac{-1}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\}\\
&\quad\{-\frac{4}{3}(\vec{k}\cdot\vec{q})p^2+[(2\alpha-1)k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(1-2\alpha)q^2+2\alpha\vec{k}\cdot\vec{q}]\}\\
&\quad+\int_0^1 d\alpha\int_0^{\Lambda}4\pi p^2dp\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\{-\frac{4}{3}(\vec{k}\cdot\vec{q})p^2+[-k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(1-2\alpha)q^2]\}\\
&=\frac{\pi^3[k^2q^2-3(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}+\vec{q}|}.
\end{aligned}
\tag{A.40}
$$

同樣地對於$\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_c^2}$項

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\{\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_c^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\{\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_c^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(-2\vec{k}\cdot\vec{p}-k^2)(2\vec{q}\cdot\vec{p}-q^2)\\
&\times\{\frac{-1}{\{[p+k-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}\\
&-\frac{1}{\{[p+\alpha k-\alpha)q]^2+\alpha(1-\alpha)(k^2+q^2-2\vec{k}\cdot\vec{q})\}^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\\
&\times\{-\frac{4}{3}(\vec{k}\cdot\vec{q})p^2+[k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(1-2\alpha)q^2-2\vec{k}\cdot\vec{q}]\}\\
&-\frac{1}{[p^2+\alpha(1-\alpha)(k^2+q^2-2\vec{k}\cdot\vec{q})]^2}\\
&\{-\frac{4}{3}(\vec{k}\cdot\vec{q})p^2+[(2\alpha-1)k^2-2\alpha\vec{k}\cdot\vec{q}][(2\alpha-1)q^2-2\alpha\vec{k}\cdot\vec{q}]\}\}.
\end{aligned}
\tag{A.41}
$$

同樣取極限後

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_0^{\Lambda}4\pi p^2dp\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\\
&\{-\frac{4}{3}(\vec{k}\cdot\vec{q})p^2+[k^2-2(1-\alpha)\vec{k}\cdot\vec{q}][(1-2\alpha)q^2-2\alpha\vec{k}\cdot\vec{q}]\}\\
&-\{\frac{1}{[p^2+\alpha(1-\alpha)(k^2+q^2-2\vec{k}\cdot\vec{q})]^2}\}\\
&\times\{-\frac{4}{3}(\vec{k}\cdot\vec{q})p^2+[(2\alpha-1)k^2-2\alpha\vec{k}\cdot\vec{q}][(2\alpha-1)q^2-2\alpha\vec{k}\cdot\vec{q}]\}\\
=&\frac{-\pi^3[k^2q^2-(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}-\vec{q}|}-\frac{4\pi^3(\vec{k}\cdot\vec{q})[k^2-\vec{k}\cdot\vec{q}]}{q}.
\end{aligned}
\tag{A.42}
$$

最後再將(A.40)和(A.42)相加得到

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(P_a^2-P_d^2)(-P_c^2+P_d^2)(P_a^2-P_b^2)}{P_a^2P_b^2P_c^2P_d^2}\\
=&\frac{\pi^3[k^2q^2-3(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}+\vec{q}|}\\
&-\frac{\pi^3[k^2q^2-(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}-\vec{q}|}-\frac{4\pi^3(\vec{k}\cdot\vec{q})[k^2-\vec{k}\cdot\vec{q}]}{q}.
\end{aligned}
\tag{A.43}
$$

最後I_{kqqq}則等於

$$
\begin{aligned}
I_{kkkq} =& k_i k_j k_m q_n I_{ijmn}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})^3(\vec{q}\cdot\vec{p})}{P_a^2P_b^2P_c^2P_d^2} \\
=& -\frac{1}{16}\{-k^4q^2(k^2-2\vec{k}\cdot\vec{q})\frac{-\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\frac{-\pi^3}{kq}[k^6+k^4q^2-2k^4(\vec{k}\cdot\vec{q})-6k^2q^2(\vec{k}\cdot\vec{q})+4q^2(\vec{k}\cdot\vec{q})^2][\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+[3k^2q^2-3k^2(\vec{k}\cdot\vec{q})-3q^2(\vec{k}\cdot\vec{q})+2(\vec{k}\cdot\vec{q})^2]\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|+\frac{2}{q}}] \\
&+[-3k^4+6k^2(\vec{k}\cdot\vec{q})-4(\vec{k}\cdot\vec{q})^2]\pi^3[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\frac{-4(\vec{k}\cdot\vec{q})(k^2-\vec{k}\cdot\vec{q})(k^2-2\vec{k}\cdot\vec{q})\pi^3}{kq|\vec{k}+\vec{q}|}-\frac{2k^4(\vec{k}\cdot\vec{q})\pi^3}{kq|\vec{k}-\vec{q}|} \\
&+\frac{\pi^3[k^2q^2-3(\vec{k}\cdot\vec{q})^2]}{|\vec{k}+\vec{q}|}-\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{|\vec{k}-\vec{q}|} \\
&+\frac{-4\pi^3(\vec{k}\cdot\vec{q})[k^2-\vec{k}\cdot\vec{q}]}{q}.
\end{aligned}
\tag{A.44}
$$

而$I_{kqqq}(\vec{k},\vec{q}) = I_{kkkq}(\vec{q},\vec{k})$。最後一個是$I_{kkqq}$，它的$(\vec{k}\cdot\vec{p})^2(\vec{q}\cdot\vec{p})^2$可改寫為

$$
\begin{aligned}
(\vec{k}\cdot\vec{p})^2(\vec{q}\cdot\vec{p})^2 =&(-\frac{1}{2})[(P_a-P_b)^2+P_a^2-P_b^2]^2(-\frac{1}{2})[(P_a-P_c)^2-(P_a-P_d)^2-P_c^2+P_d^2]\\
&\times(\frac{1}{2})[(P_a-P_d)^2+P_a^2-P_d^2](\frac{1}{2})[(P_a-P_c)^2-(P_a-P_b)^2+P_b^2-P_c^2]\\
=&\frac{1}{16}[k^2+P_a^2-P_b^2][(k-q)^2-q^2-P_c^2+P_d^2]\\
&\times[q^2+P_a^2-P_d^2][(k-q)^2-k^2+P_b^2-P_c^2]\\
=&\frac{1}{16}[k^4+2k^2(P_a^2-P_b^2)+(P_a^4+P_b^4-2P_a^2P_b^2)]\\
&\times[q^2(k^2-2\vec{k}\cdot\vec{q})+(k^2-2\vec{k}\cdot\vec{q})(P_a^2+P_d^2)+q^2(-P_c^2+P_d^2)\\
&+(-P_a^2P_c^2+P_a^2P_d^2+P_c^2P_d^2-P_d^4)]\\
=&\frac{1}{16}[k^2(k^2-2\vec{k}\cdot\vec{q})+(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)\\
&+k^2(-P_c^2+P_d^2)(P_a^2+P_b^2)(-P_c^2+P_d^2)]\\
&\times[q^2(q^2-2\vec{k}\cdot\vec{q})+(q^2-2\vec{k}\cdot\vec{q})(P_a^2-P_d^2)q^2(P_b^2-P_c^2)+(P_a^2-P_d^2)(P_b^2-P_c^2)]\\
=&\frac{1}{16}[k^2q^2(q^2-2\vec{k}\cdot\vec{q})(q^2-2\vec{k}\cdot\vec{q})\\
&+q^2(k^2-2\vec{k}\cdot\vec{q})(q^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)\\
&+k^2q^2(k^2-2\vec{k}\cdot\vec{q})(P_b^2-P_c^2)+k^2q^2(q^2-2\vec{k}\cdot\vec{q})(-P_c^2+P_d^2)\\
&+k^2(k^2-2\vec{k}\cdot\vec{q})(q^2-2\vec{k}\cdot\vec{q})(P_a^2-P_d^2)\\
&+q^2(q^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)(-P_c^2+P_d^2)\\
&+(k^2-2\vec{k}\cdot\vec{q})(q^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)(P_a^2-P_d^2)\\
&+k^2(q^2-2\vec{k}\cdot\vec{q})(-P_c^2+P_d^2)(P_a^2-P_d^2)\\
&+q^2(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)(P_b^2-P_c^2)\\
&+k^2q^2(-P_c^2-P_d^2)(P_b^2-P_c^2)\\
&+k^2(k^2-2\vec{k}\cdot\vec{q})(P_b^2-P_c^2)(P_b^2-P_c^2)\\
&+(k^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)(P_a^2-P_d^2)(P_b^2-P_c^2)\\
&+k^2(-P_c^2+P_d^2)(P_a^2-P_d^2)(P_b^2-P_c^2)\\
&+(q^2-2\vec{k}\cdot\vec{q})(P_a^2-P_b^2)(-P_c^2+P_c^2)(P_b^2-P_c^2)\\
&+q^2(P_a^2-P_b^2)(-P_c^2+P_d^2)(P_b^2-P_c^2)\\
&+(P_a^2-P_b^2)(-P_c^2+P_d^2)(P_a^2-P_d^2)(P_b^2-P_c^2)].
\end{aligned}
\tag{A.45}
$$

而它的最後一項$(P_a^2-P_b^2)(-P_c^2+P_d^2)(P_a^2-P_d^2)(P_b^2-P_c^2)$的積分則化簡為

$$
\begin{aligned}
&\int_{\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(P_a^2-P_b^2)(-P_c^2+P_d^2)(P_a^2-P_d^2)(P_b^2-P_c^2)}{P_a^2P_b^2P_c^2P_d^2}\\
=&\int_{\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(P_a^2-P_d^2)(P_b^2-P_c^2)\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_d^2}-\frac{1}{P_a^2P_c^2}\}\\
=&\int_{\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(2\vec{q}\cdot\vec{p}-q^2)[2q\cdot(p+k)-q^2]\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}+\frac{1}{P_a^2P_d^2}-\frac{1}{P_a^2P_c^2}\}.
\end{aligned}
\tag{A.46}
$$

依照前面的方法，先看Feynman trick用在$\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}$項的積分

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(2\vec{q}\cdot\vec{p}-q^2)[2q\cdot(p+k)-q^2]\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_b^2P_c^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(2\vec{q}\cdot\vec{p}-q^2)[2q\cdot(p+k)-q^2]\\
&\times\{\frac{-1}{\{[p+\alpha k-(1-\alpha)q]^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})\}^2}\\
&+\frac{-1}{\{[p-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{2q\cdot[p-\alpha k+(1-\alpha)q]-q^2\}\\
&\times\{2q\cdot[p+k+(1-\alpha)k+(1+\alpha)q]-q^2\}\{\frac{-1}{\{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\}\\
+&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{2q\cdot[p+(1-\alpha)q]-q^2\}\\
&\times\{2q\cdot[p+k+(1+\alpha)q]-q^2\}\{\frac{1}{\{[p^2+\alpha(1-\alpha)q^2]^2}\}\\
=&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{4(\vec{q}\cdot\vec{p})^2[(1-2\alpha)q^2-2\alpha\vec{k}\cdot\vec{q}][(1-2\alpha)q^2+2(1-\alpha)\vec{k}\cdot\vec{q}]\}\\
&\times\{\frac{-1}{\{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\}\\
+&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\{4(\vec{q}\cdot\vec{p})^2+[(1-2\alpha)q^2][(1-2\alpha)q^2+2\vec{k}\cdot\vec{q}]\}\{\frac{1}{\{[p^2+\alpha(1-\alpha)q^2]^2}\}.
\end{aligned}
\tag{A.47}
$$

取極限之後

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_0^\Lambda 4\pi p^2dp\{\frac{-1}{[p^2+\alpha(1-\alpha)(k^2+q^2-2\vec{k}\cdot\vec{q})]^2}\}\\
&\quad\{\frac{4}{3}q^2p^2+[(1-2\alpha)q^2-2\alpha\vec{k}\cdot\vec{q}][(1-2\alpha)q^2+2(1-\alpha)\vec{k}\cdot\vec{q}]\}\\
&\quad+\{\int_0^1 d\alpha\int_0^\Lambda 4\pi p^2dp\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\\
&\quad\times\{\frac{4}{3}q^2p^2+[(1-2\alpha)q^2][(1-2\alpha)q^2+2\alpha\vec{k}\cdot\vec{q}]\}\\
&=\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}+\vec{q}|}.
\end{aligned}
\tag{A.48}
$$

而$\frac{1}{P_b^2P_c^2}-\frac{1}{P_a^2P_c^2}$項

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(2\vec{q}\cdot\vec{p}-q^2)[2q\cdot(p+k)-q^2]\{\frac{1}{P_b^2P_c^2}-\frac{1}{P_a^2P_c^2}\}\\
&=\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(2\vec{q}\cdot\vec{p}-q^2)[2q\cdot(p+k)-q^2]\\
&\quad\times\{\frac{1}{\{[p+k-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}-\frac{1}{\{[p+\alpha k-\alpha q]^2+\alpha(1-\alpha)(k^2+q^2-2\vec{k}\cdot\vec{q})\}^2}\}.
\end{aligned}
\tag{A.49}
$$

取極限之後

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_0^\Lambda 4\pi p^2dp\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\\
&\quad\times\{\frac{4}{3}q^2p^2+[(1-2\alpha)q^2-2\vec{k}\cdot\vec{q}][(1-2\alpha)q^2]\}\\
&\quad-\frac{1}{[p^2+\alpha(1-\alpha)(k^2+q^2-2\vec{k}\cdot\vec{q})]^2}\\
&\quad+\{\int_0^1 d\alpha\int_0^\Lambda 4\pi p^2dp\{\frac{1}{[p^2+\alpha(1-\alpha)q^2]^2}\}\\
&\quad\times\{\frac{4}{3}q^2p^2+[(2\alpha-1)q^2-2\alpha\vec{k}\cdot\vec{q}][(2\alpha-1)q^2+2(1-\alpha)\vec{k}\cdot\vec{q}]\}\\
&=\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}-\vec{q}|}.
\end{aligned}
\tag{A.50}
$$

再將(A.48)和(A.50)的結果加在一起

$$
\begin{aligned}
&\int_{\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(P_a^2-P_b^2)(-P_c^2+P_d^2)(P_a^2-P_d^2)(P_b^2-P_c^2)}{P_a^2P_b^2P_c^2P_d^2}\\
&=\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}+\vec{q}|}+\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2|\vec{k}-\vec{q}|}.
\end{aligned}
\tag{A.51}
$$

最後I_{kkqq}會等於

$$
\begin{aligned}
I_{kkqq} =& k_i k_j q_m q_n I_{ijmn}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{(\vec{k}\cdot\vec{p})^2(\vec{q}\cdot\vec{p})^2}{P_a^2 P_b^2 P_c^2 P_d^2} \\
=& \frac{1}{16}\{k^2q^2(k^2-2\vec{k}\cdot\vec{q})\frac{-\pi^3}{kq(\vec{k}\cdot\vec{q})}[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&-\frac{2\pi^3}{kq}[k^4+q^4-2(k^2+q^2)(\vec{k}\cdot\vec{q})][\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&-4[k^2q^2-k^2(\vec{k}\cdot\vec{q})-q^2(\vec{k}\cdot\vec{q})+(\vec{k}\cdot\vec{q})^2]\pi^3[\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}] \\
&+[k^4-2k^2(\vec{k}\cdot\vec{q})+2(\vec{k}\cdot\vec{q})^2]\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}+\frac{2}{k}] \\
&+[q^4-2q^2(\vec{k}\cdot\vec{q})+2(\vec{k}\cdot\vec{q})^2]\pi^3[-\frac{1}{|\vec{k}+\vec{q}|}-\frac{1}{|\vec{k}-\vec{q}|}+\frac{2}{k}] \\
&-4[k^2q^2-(\vec{k}\cdot\vec{q})k^2-(\vec{k}\cdot\vec{q})q^2]\frac{(\vec{k}\cdot\vec{q})\pi^3}{kq}[\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
&+\frac{\pi^3[k^2q^2+(\vec{k}\cdot\vec{q})^2]}{2}[\frac{1}{|\vec{k}+\vec{q}|}+\frac{1}{|\vec{k}-\vec{q}|}] \\
&-\frac{8\pi^3(\vec{k}\cdot\vec{q})^3}{[\vec{k}\cdot\vec{q}]}[\frac{1}{|\vec{k}+\vec{q}|}]\}.
\end{aligned}
\tag{A.52}
$$

Appendix B

Tensor analysis for self energy diagram

附錄A是egg diagram的計算過程，本節是self energy diagram，而 self energy的算法跟egg diagramx類似，首先定義$J_k(\vec{k},\vec{q})$為$k_iJ_i(\vec{k},\vec{q})$

$$J_k(\vec{k},\vec{q}) \equiv k_iJ_i(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{k}\cdot\vec{P}_b}{P_a^2P_b^4P_c^2}. \tag{B.1}$$

為了算這個積分，首先我們先定義一個函數$K(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{P_a^2P_b^2P_d^2}$，其中$P_d = p-q$，而從(A.9)找到相同的積分，所以函數K等於$\frac{\pi^3}{kq|\vec{k}+\vec{q}|}$，進一步對它對$k_i$作微分，則

$$\partial_iK = \frac{\partial K}{\partial k_i} = \int \frac{d^3p}{(2\pi)^3}\frac{-2(P_b)_i}{P_a^2P_b^4P_d^2}. \tag{B.2}$$

除此之外還定義一個函數$L_k(\vec{k},\vec{q})$

$$L_k(\vec{k},\vec{q}) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{\vec{k}\cdot\vec{P}_b}{P_a^2P_b^4P_d^2} = -\frac{1}{2}\{k_i\partial_iK(\vec{k},\vec{q})\}. \tag{B.3}$$

如果將函數L裡面的$\vec{q}$用$\vec{q}-\vec{k}$代換，則$L_k(\vec{k},\vec{q}-\vec{k})$等於

$$L_k(\vec{k},\vec{q}-\vec{k}) = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{k}\cdot\vec{P}_b}{p^2(p+k)^4(p-q+k)^2}. \tag{B.4}$$

因此$J_k(\vec{k},\vec{q})$會等於

$$J_k(\vec{k},\vec{q}) = -\frac{1}{2}\{k_i\partial_iK(\vec{k},\vec{q}-\vec{k})\} = \frac{\pi^3(q^2+\vec{k}\cdot\vec{q})}{kq^3|\vec{k}-\vec{q}|}. \tag{B.5}$$

同樣地對於$J_q(\vec{k},\vec{q})$

$$\begin{aligned}J_q(\vec{k},\vec{q}) \equiv & q_i J_i(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{q}\cdot\vec{P}_b}{P_a^2P_b^4P_d^2}\\ =& J_k(\vec{q},\vec{k})\\ =& \frac{\pi^3(k^2+\vec{k}\cdot\vec{q})}{k^3q|\vec{k}-\vec{q}|}.\end{aligned} \tag{B.6}$$

二階張量有$J_\delta, J_{kk}, J_{qq}, J_{kq}$，首先是$J_\delta(\vec{k},\vec{q})$

$$\begin{aligned}J_\delta(\vec{k},\vec{q}) \equiv & \delta_{ij}J_{ij}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{P_b^2}{P_a^2P_b^4P_c^2}\\ =& \int \frac{d^3p}{(2\pi)^3}\frac{1}{P_a^2P_b^2P_c^2}\\ =& \frac{\pi^3}{kq|\vec{k}-\vec{q}|}.\end{aligned} \tag{B.7}$$

而$J_{kk}(\vec{k},\vec{q})$

$$J_{kk}(\vec{k},\vec{q}) \equiv k_ik_jJ_{ij}(\vec{k},\vec{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^2}{P_a^2P_b^4P_c^2}. \tag{B.8}$$

定義$K_k(\vec{k},\vec{q})$為$\int \frac{d^3p}{(2\pi)^3}\frac{\vec{k}\cdot\vec{P}_b}{P_a^2P_b^4P_d^2}$，對$k_i$的微分則為

$$\partial_iK_k = \int d^3\{\frac{(P_i+2k_i)}{P_a^2P_b^2P_d^2} - 2\frac{(\vec{k}\cdot\vec{P}_b)(P_b)_i}{P_a^2P_b^4P_d^2}\}. \tag{B.9}$$

跟(B.3)類似，將函數$L_{kk}(\vec{k},\vec{q})$定義為$\int \frac{d^3p}{(2\pi)^3}\{\frac{(\vec{k}\cdot\vec{P}_b)^2}{P_a^2P_b^4P_d^2}\}$則

$$\begin{aligned}L_{kk}(\vec{k},\vec{q}) &= -\frac{1}{2}\{k_i\partial_iK_k\} + \frac{1}{2}\{K_k + k^2K\},\\ \Rightarrow J_{kk}(\vec{k},\vec{q}) &= L_{kk}(\vec{k},\vec{q}-\vec{k})\\ &= \frac{\pi^3}{4}\{\frac{1}{|\vec{k}-\vec{q}|} + \frac{k(2q^2+\vec{k}\cdot\vec{q})}{q^3|\vec{k}-\vec{q}|} - \frac{(q^2+\vec{k}\cdot\vec{q})}{q^3}\}.\end{aligned} \tag{B.10}$$

相同地$J_{qq}(\vec{k},\vec{q})$則等於$J_{kk}(\vec{q},\vec{k})$。而二階張量最後的$J_{kq}(\vec{k},\vec{q})$，

$$J_{kq} \equiv k_iq_jJ_{ij}(\vec{q},\vec{k}) = \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)(\vec{q}\cdot\vec{P}_b)}{P_a^2P_b^4P_c^2}. \tag{B.11}$$

而$L_{kq}(\vec{q},\vec{k})$

$$L_{kq}(\vec{k},\vec{q}) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)(\vec{q}\cdot\vec{P}_b)}{P_a^2P_b^4P_d^2} = -\frac{1}{2}\{q_i\partial_iK_k + \frac{1}{2}\{K_q + \vec{k}\cdot\vec{q}K\}\}. \tag{B.12}$$

從上式可以發現將L_{kq}如果將“怎樣”，則

$$\begin{aligned} J_{kq}(\vec{k},\vec{q}) &= L_{kk}(\vec{k},\vec{q}-\vec{k}) + J_{kk}(\vec{k},\vec{q}) \\ &= \frac{\pi^3}{4}\{\frac{1}{|\vec{k}-\vec{q}|} + \frac{k^2+q^2\vec{k}\cdot\vec{q}}{kq|\vec{k}-\vec{q}|} - \frac{1}{k} - \frac{1}{q}\}. \end{aligned} \tag{B.13}$$

最後依照(3.22)裡二階張量$J_{ij}(\vec{k},\vec{q})$的關係，則

$$\begin{cases} J_\delta = 3D_1 + k^2D_2 + q^2D_3 + 2\vec{k}\cdot\vec{q}D_4, \\ J_{kk} = k^2D_1 + k^4D_2 + (\vec{k}\cdot\vec{q})^2D_3 + 2k^2(\vec{k}\cdot\vec{q})D_4, \\ J_{qq} = q^2D_1 + (\vec{k}\cdot\vec{q})^2D_2 + q^4D_3 + 2q^2(\vec{k}\cdot\vec{q})D_4, \\ J_{kq} = \vec{k}\cdot\vec{q}D_1 + k^2(\vec{k}\cdot\vec{q})D_2 + q^2(\vec{k}\cdot\vec{q})D_3 + [k^2q^2 + (\vec{k}\cdot\vec{q})^2]D_4. \end{cases} \tag{B.14}$$

第三階張量

$$\begin{aligned} J_{\delta k}(\vec{k},\vec{q}) \equiv& \delta_{ij}k_mJ_{ijm}(\vec{k},\vec{q}) \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{P_b^2(\vec{k}\cdot\vec{P}_b)}{P_a^2P_b^4P_c^2} \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{\vec{k}\cdot\vec{P}_b}{P_a^2P_b^2P_c^2} \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{k\cdot(p+k)}{P_a^2P_b^2P_c^2}. \end{aligned} \tag{B.15}$$

上式的積分可以從(A.17)找到，所以

$$J_{\delta k}(\vec{k},\vec{q}) = \frac{\pi^3}{2}\{\frac{1}{|\vec{k}-\vec{q}|} + \frac{k}{q|\vec{k}-\vec{q}|} - \frac{1}{q}\}. \tag{B.16}$$

而$J_{\delta q}(\vec{k},\vec{q}) = J_{\delta k}(\vec{q},\vec{k})$。接下來算是$J_{kkk}(\vec{k},\vec{q})$，它的定義是

$$\begin{aligned} J_{kkk}(\vec{k},\vec{q}) \equiv& k_ik_jk_mJ_{ijm}(\vec{q},\vec{k}) \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^3}{P_a^2P_b^4P_c^2}. \end{aligned} \tag{B.17}$$

之後再定義$K_{kk}(\vec{k},\vec{q})$

$$\begin{aligned} K_{kk}(\vec{k},\vec{q}) \equiv& \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^2}{P_a^2P_b^2P_d^2} \\ \Rightarrow \partial_iK_{kk} =& \int \frac{d^3p}{(2\pi)^3}\{\frac{2(\vec{k}\cdot\vec{P}_b)(P_i+2k_i)}{P_a^2P_b^2P_d^2} - 2\frac{(\vec{k}\cdot\vec{P}_b)^2(P_b)_i}{P_a^2P_b^4P_d^2}\}. \end{aligned} \tag{B.18}$$

而函數$L_{kkk}(\vec{k},\vec{q})$

$$\begin{aligned} L_{kkk}(\vec{k},\vec{q}) &\equiv \int d^3 \frac{(\vec{k}\cdot\vec{P}_b)^3}{P_a^2 P_b^4 P_d^2} \\ &= -\frac{1}{2}\{k_i\partial_i K_{kk}\} + \{K_{kk} + k^2 K_k\}. \end{aligned} \tag{B.19}$$

最後$J_{kkk}(\vec{k},\vec{q})$則為

$$\begin{aligned} J_{kkk} &= L_{kkk}(\vec{k},\vec{q}-\vec{k}) \\ &= \frac{\pi^3}{8}\{\frac{3k^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|} + \frac{k^3(3q^2+\vec{k}\cdot\vec{q})}{q^3|\vec{k}-\vec{q}|} - \frac{3k^2+\vec{k}\cdot\vec{q}}{q} - \frac{\vec{k}\cdot\vec{q}(k^2+\vec{k}\cdot\vec{q})}{q^3}\}. \end{aligned} \tag{B.20}$$

依照同樣的方法找到其他的三階張量分別等於

$$\begin{cases} J_{qqq} = \frac{\pi^3}{8}\{\frac{3q^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|} + \frac{q^3(3k^2+\vec{k}\cdot\vec{q})}{k^3|\vec{k}-\vec{q}|} - \frac{3q^2+\vec{k}\cdot\vec{q}}{k} - \frac{\vec{k}\cdot\vec{q}(q^2+\vec{k}\cdot\vec{q})}{k^3}\}, \\ J_{kkq} = \frac{\pi^3}{8}\{\frac{2k^2+q^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|} + \frac{k(k^2+2q^2+\vec{k}\cdot\vec{q})}{q|\vec{k}-\vec{q}|} - \frac{k^2+q^2+2\vec{k}\cdot\vec{q}}{q} - 2k\}, \\ J_{kqq} = \frac{\pi^3}{8}\{\frac{2q^2+k^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|} + \frac{q(k^2+2q^2+\vec{k}\cdot\vec{q})}{k|\vec{k}-\vec{q}|} - \frac{k^2+q^2+2\vec{k}\cdot\vec{q}}{k} - 2q\}. \end{cases} \tag{B.21}$$

最後一組的第四階張量有$J_{\delta\delta}, J_{\delta kk}, J_{\delta qq}, J_{\delta kq}, J_{kkkk}, J_{qqqq}, J_{kkkq}, J_{kqqqq}, J_{kkqq}$，首先是$J_{\delta\delta}$

$$\begin{aligned} J_{\delta\delta} \equiv& \delta_{ij}\delta_{mn}J_{ijmn}(\vec{k},\vec{q}) \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{P_b^4}{P_a^2 P_b^4 P_c^2} \\ =& \int \frac{d^3p}{(2\pi)^3}\frac{1}{P_a^2 P_c^2} \\ =& \frac{\pi^3}{|\vec{k}-\vec{q}|}. \end{aligned} \tag{B.22}$$

$J_{\delta kk}, J_{\delta kq}$的積分都可以從egg diagram的計算中找到

$$\begin{cases} J_{\delta kk} &= \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^2}{P_a^2P_b^2P_c^2} \\ &= \frac{\pi^3}{4}\{\frac{2k^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|}+\frac{k^3}{q|\vec{k}-\vec{q}|}-\frac{k^2+\vec{k}\cdot\vec{q}}{q}\}, \\ \\ J_{\delta qq} &= J_{\delta kk}(\vec{q},\vec{k}) \\ &= \frac{\pi^3}{4}\{\frac{2q^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|}+\frac{q^3}{q|\vec{k}-\vec{q}|}-\frac{q^2+\vec{k}\cdot\vec{q}}{k}\}, \\ \\ J_{\delta kq} &= \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)(\vec{q}\cdot\vec{P}_b)}{P_a^2P_b^2P_c^2} \\ &= \frac{\pi^3}{4}\{\frac{2k^2+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|}+\frac{kq+\vec{k}\cdot\vec{q}}{|\vec{k}-\vec{q}|}-k-q\}. \end{cases} \tag{B.23}$$

再來是算$J_{kkkk}(\vec{k},\vec{q})$，首先定義函數$K_{kkk}(\vec{k},\vec{q})$

$$K_{kkk}(\vec{k},\vec{q}) \equiv \int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^3}{P_a^2P_b^2P_d^2}. \tag{B.24}$$

其中將$(\vec{k}\cdot\vec{P}_b)$改寫成$\frac{1}{2}[P_b^2-P_a^2+k^2]$

$$\begin{aligned} \vec{k}\cdot\vec{P}_b = \vec{k}\cdot\vec{p}+k^2 &= \frac{1}{2}[P_b^2-P_a^2-k^2]+k^2 \\ &= \frac{1}{2}[P_b^2-P_a^2+k^2]. \end{aligned} \tag{B.25}$$

所以(B.24)的積分就變成

$$K_{kkk} = \frac{1}{2}\int_1^2\frac{d^3p}{(2\pi)^3}\{\frac{\vec{k}\cdot\vec{p}+k^2}{P_a^2P_d^2}-\frac{(\vec{k}\cdot\vec{p}+k^2)^2}{P_b^2P_d^2}+\frac{k^2(\vec{k}\cdot\vec{p}+k^2)^2}{P_a^2P_b^2P_d^2}\}. \tag{B.26}$$

跟Egg diagram一樣對$\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}$的積分作Feynman trick

$$\begin{aligned} &\int_{\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)^2\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\} \\ =&\int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)(\vec{k}\cdot\vec{p}+k^2) \\ &\{\frac{-1}{\{[p+\alpha k-(1-\alpha)q]^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})\}^2} \\ &+\frac{1}{\{[p-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}\}. \end{aligned} \tag{B.27}$$

再對積分的上下限設定為一個數Λ之後再對它取極限到無限大

$$\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{-(\vec{k}\cdot\vec{p})^2-[(1-\alpha)(k^2+\vec{k}\cdot\vec{q})]^2}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2} + \int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{q})^2+[k^2+(1-\alpha)\vec{k}\cdot\vec{q}]^2}{[p^2+\alpha(1-\alpha)q^2]^2}\}. \tag{B.28}$$

上式積分又等於

$$\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{-\frac{1}{3}k^2p^2-[(1-\alpha)(k^2+\vec{k}\cdot\vec{q})]^2}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2} + \int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{\frac{1}{3}k^2p^2+[k^2+(1-\alpha)\vec{k}\cdot\vec{q}]^2}{[p^2+\alpha(1-\alpha)q^2]^2}\}. \tag{B.29}$$

所以(B.27)變成

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)^2\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\}\\ =&\int_0^1 d\alpha 4\pi\{\pi^2k^2[\sqrt{\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})}-\sqrt{\alpha(1-\alpha)q^2}]\\ &-\pi^2\frac{[(1-\alpha)(k^2+\vec{k}\cdot\vec{q})]^2}{\sqrt{\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})}}+\pi^2\frac{[k^2+(1-\alpha)\vec{k}\cdot\vec{q}]^2}{\sqrt{\alpha(1-\alpha)q^2}}\}.\end{aligned} \tag{B.30}$$

又因為(A.32)的關係，所以積分的解為

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)^2\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\}\\ =&\frac{\pi^3}{8}[|\vec{k}+\vec{q}|-q]\\ &-\frac{\pi^3[3(k^2+\vec{k}\cdot\vec{q})^2]}{8|\vec{k}+\vec{q}|}+\frac{\pi^3[8k^4+8k^2(\vec{k}\cdot\vec{q})+3(\vec{k}\cdot\vec{q})^2]}{8q}.\end{aligned} \tag{B.31}$$

跟之前一樣定義一個函數$L_{kkkk}(\vec{k},\vec{q})$

$$
\begin{aligned}
L_{kkkk}(\vec{k},\vec{q}) \equiv & \int \frac{d^3p}{(2\pi)^3}\{\frac{(\vec{k}\cdot\vec{p})^4}{P_a^2P_b^4P_d^2}\} \\
= & -\frac{1}{2}\{k_i\partial_i K_{kkk}\}+\frac{3}{2}\{K_{kkk}+k^2K_{kk}\}, \\
\Rightarrow J_{kkkk}(\vec{k},\vec{q}) = & L_{kkkk}(\vec{k},\vec{q}-\vec{k}) \\
= & \frac{\pi^3}{32}\{\frac{13k^4+6k^2(\vec{k}\cdot\vec{q})+3(k\cdot)q^2}{|\vec{k}-\vec{q}|} \\
& +\frac{8k^5}{q|\vec{k}-\vec{q}|}+\frac{2k^5(\vec{k}\cdot\vec{q})}{q^3|\vec{k}-\vec{q}|} \\
& -\frac{8k^4+7k^2(\vec{k}\cdot\vec{q})+3(\vec{k}\cdot\vec{q})^2}{q} \\
& -\frac{\vec{k}\cdot\vec{q}[2k^4+2k^2(\vec{k}\cdot\vec{q})+3(\vec{k}\cdot\vec{q})^2]}{q^3} \\
& +k^2q-k^2|\vec{k}-\vec{q}|\}.
\end{aligned}
\tag{B.32}
$$

而$J_{qqqq}(\vec{k},\vec{q})=J_{kkkk}(\vec{q},\vec{k})$。算第三個$J_{kkkq}(\vec{k},\vec{q})$前先定義$L_{kkkq}(\vec{k},\vec{q})$

$$
\begin{aligned}
L_{kkkq}(\vec{k},\vec{q}) \equiv & \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^3(\vec{q}\cdot\vec{P}_b)}{P_a^2P_b^4P_d^2} \\
= & -\frac{1}{2}\{q_i\partial_i K_{kkk}\}+\frac{3}{2}\{K_{kkq}+\vec{k}\cdot\vec{q}K_{kk}\}.
\end{aligned}
\tag{B.33}
$$

上式中的K_{kkq}

$$
\begin{aligned}
K_{kkq} = & \int \frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^2(\vec{q}\cdot\vec{P}_b)}{P_a^2P_b^2P_d^2} \\
= & \frac{1}{2}\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)(\vec{q}\cdot\vec{p}+\vec{k}\cdot\vec{p})\{\frac{1}{P_a^2P_d^2}-\frac{1}{P_b^2P_d^2}+\frac{k^2}{P_a^2P_b^2P_d^2}\}.
\end{aligned}
\tag{B.34}
$$

上式中最後一項的積分為$\frac{1}{2}k^2K_{kq}$，所以將焦點放在另外兩項，首先是對它們作Feynamm trick

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)(\vec{q}\cdot\vec{p}+\vec{k}\cdot\vec{p})\{\frac{-1}{P_b^2P_d^2}+\frac{1}{P_a^2P_d^2}\} \\
= & \int_0^1 d\alpha\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}(\vec{k}\cdot\vec{p}+k^2)(\vec{q}\cdot\vec{p}+\vec{k}\cdot\vec{p}) \\
& \{\frac{-1}{\{[p+\alpha k-(1-\alpha)q]^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})\}^2} \\
& +\frac{1}{\{[p-(1-\alpha)q]^2+\alpha(1-\alpha)q^2\}^2}\}.
\end{aligned}
\tag{B.35}
$$

一樣先對上下限設個數Λ之後再取極限至無限大

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{-(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})-(1-\alpha)^2(k^2+\vec{k}\cdot\vec{q})(q^2+\vec{k}\cdot\vec{q})}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\\
&\quad+\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{p})(\vec{q}\cdot\vec{p})+[k^2+(1-\alpha)\vec{k}\cdot\vec{q}][\vec{k}\cdot\vec{q}+(1-\alpha)q^2]}{[p^2+\alpha(1-\alpha)q^2]^2}\\
&=\lim_{\Lambda\to\infty}\{\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{-\frac{1}{3}(\vec{k}\cdot\vec{q})p^2-(1-\alpha)^2(k^2+\vec{k}\cdot\vec{q})(q^2+\vec{k}\cdot\vec{q})}{[p^2+\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})]^2}\\
&\quad+\int_0^1 d\alpha\int_{-\Lambda}^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{\frac{1}{3}(\vec{k}\cdot\vec{q})p^2+[k^2+(1-\alpha)\vec{k}\cdot\vec{q}][\vec{k}\cdot\vec{q}+(1-\alpha)q^2]}{[p^2+\alpha(1-\alpha)q^2]^2}\\
&=\int_0^1 d\alpha 4\pi\{\pi^2(\vec{k}\cdot\vec{q})[\sqrt{\alpha(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})}-\sqrt{\alpha(1-\alpha)q^2}]\\
&\qquad-\pi^2\frac{(1-\alpha)^2[k^2+\vec{k}\cdot\vec{q}][q^2+\vec{k}\cdot\vec{q}]}{\sqrt{\alpha}(1-\alpha)(k^2+q^2+2\vec{k}\cdot\vec{q})}+\pi^2\frac{[k^2+(1-\alpha)\vec{k}\cdot\vec{q}][\vec{k}\cdot\vec{q}+(1-\alpha)q^2]}{\sqrt{\alpha(1-\alpha)q^2}}\}\\
&=\frac{\pi^3}{8}(k\cdot q)[|\vec{k}+\vec{q}|-q]-\frac{\pi^3[3(k^2+\vec{k}\cdot\vec{q})(q^2+\vec{k}\cdot\vec{q})]}{8|\vec{k}+\vec{q}|}-\frac{\pi^3[4k^2q^2+(8k^2+3q^2)(\vec{k}\cdot\vec{q})+4(\vec{k}\cdot\vec{q})^2]}{8q}.
\end{aligned}
\tag{B.36}
$$

則$J_{kkkq}(\vec{k},\vec{q})$

$$
\begin{aligned}
J_{kkkq}(\vec{k},\vec{q})\equiv&k_ik_jk_mq_nJ_{ijmn}(\vec{k},\vec{q})\\
=&\int\frac{d^3p}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{P}_b)^3(\vec{q}\cdot\vec{P}_b)}{P_a^2P_b^4P_c^2}\\
=&L_{kkkq}(\vec{k},\vec{q})+J_{kkkk}(\vec{k},\vec{q})\\
=&\frac{\pi^3}{32}\{\frac{k^4+14k^2(\vec{k}\cdot\vec{q})+7(\vec{k}\cdot\vec{q})^2}{|\vec{k}-\vec{q}|}\\
&+\frac{2k^3(-2k^2+7\vec{k}\cdot\vec{q})}{q|\vec{k}-\vec{q}|}\\
&+\frac{6^3|\vec{k}-\vec{q}|}{q}-\frac{2k^4+4^2(\vec{k}\cdot\vec{q})+5(\vec{k}\cdot\vec{q})^2}{q}\\
&+(5k^2+2\vec{k}\cdot\vec{q})|\vec{k}-\vec{q}|-(5k^2+2\vec{k}\cdot\vec{q})q-6k^3\}.
\end{aligned}
\tag{B.37}
$$

跟先前一樣$J_{kqqq}(\vec{k},\vec{q})=J_{kkkq}(\vec{q},\vec{k})$。最後算$J_{kkqq}$需要定義$L_{kkqq}(\vec{k},\vec{q})$

$$
\begin{aligned}
L_{kkqq}(\vec{k},\vec{q})\equiv&\int\frac{d^3p}{(2\pi)^3}\{\frac{(\vec{k}\cdot\vec{P}_b)^2(\vec{q}\cdot\vec{P}_b)^2}{P_a^2P_b^4P_d^2}\}\\
=&-\frac{1}{2}\{q_i\partial_iK_{kkq}\}+\frac{1}{2}\{2K_{kkq}+2\vec{k}\cdot\vec{q}K_{kq}+q^2K_{kk}\}.
\end{aligned}
\tag{B.38}
$$

因此$J_{kkqq}(\vec{k},\vec{q})$

$$
\begin{aligned}
J_{kkqq}(\vec{k},\vec{q}) =& L_{kkqq}(\vec{k},\vec{q}-\vec{k}) + 2J_{kkkq}(\vec{k},\vec{q}-\vec{k}) - J_{kkkk}(\vec{k},\vec{q}-\vec{k}) \\
=& \frac{\pi^3}{32}\{\frac{3k^2q^2+(\vec{k}\cdot\vec{q})^2}{|\vec{k}-\vec{q}|} + \frac{2(k^2+q^2+\vec{k}\cdot\vec{q})^2}{|\vec{k}-\vec{q}|} + \frac{2kq(2k^2+2q^2+\vec{k}\cdot\vec{q})}{|\vec{k}-\vec{q}|} \\
& - 2q(2k^2+q^2+3\vec{k}\cdot\vec{q}) - 2k(2q^2+k^2+3\vec{k}\cdot\vec{q})\}.
\end{aligned}
\tag{B.39}
$$