國立臺灣師範大學數學系教學碩士班碩士論文

指導教授： 林延輯 博士

Permutations with 0 or 1 fixed point in hyperoctahedral groups

研 究 生： 周鈺偵

中 華 民 國 108 年6月

致謝

這三年來邊工作邊念書的時光很踏實，雖然選擇了暑期在職教學碩士班，但仍希望自己能完成做純數研究的夢想，於是選擇了做組合學研究，雖然這條路很辛苦，但也因為有太多人的幫忙得以走完，謝謝所有曾經幫助過我的人。

我要特別感謝指導教授林延輯老師，在我就讀教碩班的第一個暑假結束後，我拜託老師擔任我的指導教授，然而念教學碩士班做純數的人少之又少，而我又沒有任何組合學的基礎，大學時期修的離散課程也是很久遠的事。很感謝林延輯老師願意耐心從頭教導我，兩年來念組合學課本、寫習題、旁聽組合學課程、學習寫程式、看一些文章討論想法到慢慢地有基礎後可以做點東西，在老師的身邊學習到的不只是專業的知識，更多的是身為「老師」在教導學生時的恩威並施、正向積極、站在學生的角度思考和不降低標準的細心陪伴你成長，希望我也能成為這樣的老師。

另外，我也要特別感謝游森棚老師，除了讓我旁聽組合學的課程，並在我研究論文的期間給了很重要的想法，在論文口試時也給予實質意見與肯定讓我的論文更完整。

感謝口試時徐祥峻老師對論文的建議，感謝溫雅婷學妹在論文前後和當天體貼的幫忙，感謝數學系女排的教練和學妹在我的碩班生涯陪伴我打球讓我更有動力念書。

最後感謝我的家人總是無條件地陪伴我做任何事，無論遇到什麼挫折總是可以回到家充個電再出發，謝謝你們成就今天的我，希望自己能持續進步，讓自己能更好。

鈺偵

108.07.03

Contents

Permutations with 0 or 1 fixed point in hyperoctahedral groups

Chou Yu-Jen

Abstract

In this thesis, we extend the work of fixed points on the permutations of $[n]$ in two directions: firstly, we discuss the fixed points problems of hyperoctahedral groups B_n; secondly, elements in B_n can be thought the letters are painted by two colors, it can be generalized with r colors. Moreover, we discuss the fixed point problems in the subsets *alternating permutations* of B_n and *strictly decreasing permutations* of $\mathfrak{S}_n^{(r)}$. After removing all fixed points and standardizing the remaining letters, we focus on colored permutations with 0 or 1 fixed point. We obtain combinatorial correspondence between derangements and elements with exactly one fixed point together with their recursions and generating functions.

Keywords. Derangements, hyperoctahedral groups, alternating permutations, colored permutations

1 Introduction

For any positive integer n, denote the set $\{1, 2, \ldots, n\}$ by $[n]$. Given a permutation σ on $[n]$, a label $k \in [n]$ is called a *fixed point* of σ if and only if $\sigma(k) = k$. A permutation of $[n]$ is called a *derangement* if it has no fixed point. Let d_n denote the number of derangements of $[n]$. It is well-known (cf. [Tuc80]) that the sequence d_n satisfies an easier recursion

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad n \geq 2, \tag{1}$$

and also a harder recursion

$$d_n = n d_{n-1} + (-1)^n, \qquad n \geq 1. \tag{2}$$

Combinatorial proofs of the latter identity (2) were given by Remmel [Rem83] and Wilf [Wil84] independently. And the generating function of d_n is also well-known [WG16]:

$$f(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!} = \frac{\mathrm{e}^{-x}}{1-x} \tag{3}$$

Ordinary permutations on $[n]$ can be thought as painted by only one color. Bagno ([Bag04, BG06]) generalizes this notion that allows r colors to be painted on permutations, which results in the *colored permutations*. In this article, we would like to generalize the notion of derangements to the colored permutations.

We first introduce the groups in B_n and $\mathfrak{S}_n^{(r)}$. Let B_n be the hyperoctahedral group of order n, that is, elements in B_n are permutations σ on the set $\{\pm 1, \pm 2, \pm 3, \cdots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$, we use the notation $\bar{\imath} := -i$ in this article [FH05]. In particular, elements of B_n can be thought as permutations whose letters are painted by either of two colors. Therefore when r colors are available to paint the letters, we get the *wreath product* $\mathfrak{S}_n^{(r)}$ of the cyclic group C_r of order r with the permutation group $\mathfrak{S}_n$ (cf. [Bag04]). We call an index $i \in [n]$ is called a *fixed point* of $\sigma \in B_n$ or $\sigma \in \mathfrak{S}_n^{(r)}$ if $\sigma(i) = i$ and i is uncolored, (in the case of B_n, we will refer the first color as uncolored.) A permutation σ without fixed points in B_n or $\mathfrak{S}_n^{(r)}$ is still called a derangement. For a permutation σ in B_n or $\mathfrak{S}_n^{(r)}$, we denote the set of fixed points of σ by $\mathsf{FIX}(\sigma)$, and $\mathsf{fix}(\sigma)$ is its cardinality.

There are two ways to represent the elements in B_n and $\mathfrak{S}_n^{(r)}$ in this paper. First is the *window notation* and the other is *cycle notation*. In the window notation, permutations in B_n can be written as $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i = \sigma(i)$; and we can also denote $\sigma \in B_n$ with cycle notation introduced by Reiner [Rei93]: each cycle $(l_1, l_2, \ldots, l_k, \ldots, l_j)$ means $\sigma(|l_i|) = l_{(i+1)}, 1 \leq i \leq j-1$ and $\sigma(|l_j|) = l_1$ where $l_1, l_2, \ldots, l_j \in \{\pm 1, \pm 2, \ldots, \pm n\}$, $|l_j| \neq |l_k|$ if $j \neq k$.

For example, consider the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & \bar{1} & \bar{4} & \bar{5} \end{pmatrix} \in B_5.$$

Its window notation is $\sigma = 3\ 2\ \bar{1}\ \bar{4}\ \bar{5} \in B_5$, and its cycle notation is written by $\sigma = (\bar{1}\ 3)(2)(\bar{4})(\bar{5})$. Its only fixed point is 2.

As like as elements in B_n, we also can denote the elements in $\mathfrak{S}_n^{(r)}$ with window notation and cycle notation. Elements in $\mathfrak{S}_n^{(r)}$ will be written in the window notation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, where each σ_i is a pair (c_i, j_i) with $c \in C_r$ and $j \in [n]$. We can also write $\sigma \in \mathfrak{S}_n^{(r)}$ in the cycle notation: each cycle $(l_1, l_2, \cdots, l_k, \cdots, l_j)$ means $\sigma(|l_i|) = l_{(i+1)}, 1 \leq i \leq j-1$ and $\sigma(|l_j|) = l_1$ where $l_1, l_2, \cdots, l_j$ on $[n]$ with r colors, $|l_j| \neq |l_k|$ if $j \neq k$.

For example, consider the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & \bar{\bar{1}} & \bar{4} & \bar{5} \end{pmatrix} \in \mathfrak{S}_5^{(3)}.$$

Its window notation is $\sigma = 3\ 2\ \bar{\bar{1}}\ \bar{4}\ \bar{5} \in \mathfrak{S}_5^{(3)}$, and its cycle notation is $\sigma = (\bar{\bar{1}}\ 3)(2)(\bar{4})(\bar{5})$. Its only fixed point is 2.

This article begins with focus on derangements in B_n and $\mathfrak{S}_n^{(r)}$. First, we count the numbers of elements with k fixed points in B_n. By removing all fixed points and standardizing [FH08] the remaining letters from a permutation in B_n, we can get a derangement on fewer letters. For instance, let us consider the permutation $\pi = 5\ 2\ 3\ \overline{4}\ \overline{1}\ \in B_5$ with $\mathsf{FIX}(\sigma) = \{2,3\}$. By removing all fixed points, we have $\sigma = 5\ \overline{4}\ \overline{1}$, then we standardize the remaining letters σ, we obtain a derangement $\tau = 3\ \overline{2}\ \overline{1} \in B_3$. With this procedure, we realize that enumeration on derangements is essential.

Formulas similar to Equations (1), (2) and (3) are obtained for hyperoctahedral groups. We prove the easier recursion of the derangement numbers in B_n by using the cycle notation. Similarly, it can be extended to an easier recurrence of the derangement numbers in $\mathfrak{S}_n^{(r)}$. Next, we prove the harder recursion in two different ways, we use the easier recursion to prove and another way is the combinatorial proof by removing all fixed points and standardizing [HX09] the remaining letters to get a bijection. Finally, we get the correspondence between the derangements in B_n and elements with exactly one fixed point in B_n. Then, we extend the results to $\mathfrak{S}_n^{(r)}$ and get the generating function which is similar to the generating function of d_n in the end of section 2.

At the start of this project, we observed that the relationship between the number of derangements in B_n and the number of elements in B_n with $\mathsf{fix}(\sigma) = 1$. Next we consider two subsets of B_n and $\mathfrak{S}_n^{(r)}$. In Section 3, we denote the subset of alternating permutations ("snakes") by $G_{n,k}$, that is, the elements $\sigma \in B_n$ satisfying $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \cdots$ and having k fixed points. We enumerate the permutations with maximal number of fixed points in $G_{n,k}$. We discover an interesting identity between the derangements in B_n and the elements with maximal number of fixed points in G_{2n} and G_{2n+1}. To conclude Section 3, we give a combinatorial bijection for that identity.

Another interesting subset of $\mathfrak{S}_n^{(r)}$ is the subset of strictly decreasing permutations, which we discuss in Section 4. By strict decrease we employ an order $\succ$ on the set $C_r \times [n]$ is found in Bagno and Garber [BG06]. It is easy to see that any strictly decreasing permutation can have at most 1 fixed point, therefore we only need to enumerate the set of all strictly decreasing derangements in $\mathfrak{S}_n^{(r)}$, which is denoted by $\mathcal{S}_{n,0}$, as well as the set of all strictly decreasing permutations with only one fixed point in $\mathfrak{S}_n^{(r)}$, which is denoted by $\mathcal{S}_{n,1}$. We give some enumerative results like the number of elements in $\mathcal{S}_{n,0}$, the connection between $\mathcal{S}_{n,0}$ and $\mathcal{S}_{n+1,1}$, and we give the recursion and the generating function. In the end, we obtain a formula analogous to Equation (2) for the strictly decreasing permutations in $\mathfrak{S}_n^{(r)}$.

The paper is organized as follows. In Section 2, we discuss the derangements in hyperoctahedral groups together with some generalization. In Section 3, we give bijections between alternating permutations in B_{2n} with maximal number of fixed points and the derangements in B_n. In Section 4, we enumerate the strictly decreasing permutations in the wreath product $\mathfrak{S}_n^{(r)}$ and provide their generating functions. We end this paper with some comments and future work in Section 5.

2 Derangements in hyperoctahedral groups

Our task in this section is to extend the easier recursion like Equation (1) and the harder recursion like Equation (2) in B_n and $\mathfrak{S}_n^{(r)}$, finally we have the generating function like Equation (3) in $\mathfrak{S}_n^{(r)}$.

We first classify the permutations in B_n by their numbers of fixed points. Define

$$F_{n,k} := \{\sigma \in B_n : \mathsf{fix}(\sigma) = k\}, \qquad f_{n,k} := |F_{n,k}|, \qquad k = 0, 1, \ldots, n.$$

Elements in $F_{n,0}$ are also known as the *derangements* in B_n. Below is the table of the first few values of $f_{n,k}$:

Table 1: Statistics fix on hyperoctahedral groups B_n

$n \setminus k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	5	2	1					
3	29	15	3	1				
4	233	116	30	4	1			
5	2329	1165	290	50	5	1		
6	27949	13974	3495	580	75	6	1	
7	391285	195643	48909	8155	1015	105	7	1

We are going to prove a few identities involving the sequence $f_{n,k}$. By picking out all fixed points and standardizing the remaining letters, the following result follows immediately.

Proposition 2.1. *Let n, k be integers with $0 \leq k \leq n$. Then we have*

$$f_{n,k} = \binom{n}{k} f_{n-k,0}. \tag{4}$$

Proof. Let $\sigma \in B_n$ have k fixed points. The number of ways to choose k fixed points from n is $\binom{n}{k}$. By removing all fixed points of σ and standardizing the remaining letters, we obtain a derangement in $f_{n-k,0}$. □

The derangement numbers satisfy the harder recurrence $d_n = nd_{n-1} + (-1)^n$. T. Benjamin and Joel Ornstein [BO17] used the cycle notation to present a simple combinatorial proof, and we use the similar idea to prove the recurrence of the derangement numbers $f_{n,0}$ as follows.

Theorem 2.2. *The sequence $\langle f_{n,0}\rangle_n$ satisfies the following recursion:*

$$f_{0,0} = f_{1,0} = 1; \qquad f_{n,0} = (2n-1)f_{n-1,0} + 2(n-1)f_{n-2,0}, \quad \forall\ n \geq 2. \tag{5}$$

Proof. Let $\sigma \in F_{n,0}$ be written in the cycle notation. We consider how to generate σ from elements in $F_{n-1,0}$ and $F_{n-2,0}$:

1. In $F_{n-1,0}$, we can insert a cycle $(\overline{n})$ and get $f_{n-1,0}$ elements in $F_{n,0}$. Or we can put n or $\overline{n}$ behind $n-1$ letters within any cycle, so we can get $(n-1)\cdot 2\cdot f_{n-1,0}$ elements in $F_{n,0}$.

2. Elements in $F_{n-1,1}$ can be inserted n or $\overline{n}$ into the cycle (j), where $j \in \{1, 2, \ldots, n-1\}$, and we have $n-1$ choice of fixed point, by removing the only fixed point and standardizing, we have element in $F_{n-2,0}$, so we can get $(n-1)\cdot 2\cdot f_{n-2,0}$ elements in $F_{n,0}$.

For $n \geq 2$, we have

$$\begin{aligned} f_{n,0} &= f_{n-1,0} + (n-1)\cdot 2\cdot f_{n-1,0} + (n-1)\cdot 2\cdot f_{n-2,0} \\ &= (2n-1)f_{n-1,0} + 2(n-1)f_{n-2,0} \end{aligned}$$

□

The previous result is parallel to the recursive formula satisfied by d_n like Equation (1). Next we prove the result for $f_{n,0}$ which is analogous to Equation (2).

Theorem 2.3. *For $n \geq 1$, we have $f_{n,0} = 2 \cdot f_{n,1} + (-1)^n$.*

Proof. We provide two proofs here.

First proof. From the Equation (4) and (5), we have $f_{0,0} = 1$, $f_{0,1} = 0$, and

$$\begin{aligned} f_{n,0} - 2 \cdot f_{n,1} &= f_{n,0} - 2n \cdot f_{n-1,0} \\ &= (2n-1) \cdot f_{n-1,0} + 2(n-1) \cdot f_{n-1,0} - 2n \cdot f_{n-2,0} \\ &= -f_{n-1,0} + 2(n-1) \cdot f_{n-2,0} \\ &= -f_{n-1,0} + 2 \cdot f_{n-1,1} \\ &= (-1) \cdot (f_{n-1,0} - 2 \cdot f_{n-1,1}) \\ &= \cdots = (-1)^n \cdot (f_{0,0} - 2 \cdot f_{0,1}) \\ &= (-1)^n \cdot (1 - 0) \\ &= (-1)^n \end{aligned} \tag{6}$$

Second proof. We give a combinatorial proof here. It is obvious that the identity holds for $n = 1$. The statement will then follow recursively by the identity

$$f_{n,0} - 2 \cdot f_{n,1} = 2 \cdot f_{n-1,1} - f_{n-1,0}, \qquad \forall\, n \geq 2. \tag{7}$$

We now give a combinatorial proof for Equation (7). Let $2 \cdot F_{n,1}$ denote the set

$$2 \cdot F_{n,1} := \big\{(\varepsilon, \sigma) \colon \varepsilon \in \{+, -\}, \sigma \in F_{n,1}\big\},$$

and $2 \cdot H_{n,1}$ be the subset of $2 \cdot F_{n,1}$ consisting of the elements (ε, σ) with $\sigma(n) \neq n$. We now construct a map θ from $2 \cdot H_{n,1}$ into $F_{n,0}$. For $(\varepsilon, \sigma) \in 2 \cdot H_{n,1}$ and ℓ being the fixed point of σ, let $\theta(\varepsilon, \sigma)$ be the signed permutation on $[n]$ with 2 colors given by

$$\big(\theta(\varepsilon, \sigma)\big)(k) := \begin{cases} \sigma(n), & \text{if } k = \ell; \\ \varepsilon \cdot \ell, & \text{if } k = n; \\ \sigma(k), & \text{otherwise} \end{cases} \qquad 1 \leq k \leq n.$$

For example, if $\sigma = \overline{2}\,4\,3\,\overline{5}\,1$ with the fixed point 3, then $\theta(-,\sigma) = \overline{2}\,4\,1\,\overline{5}\,\overline{3}$. Clearly θ is an injective map into $F_{n,0}$. The derangements $\pi \in F_{n,0}$ which is not in the image of θ must assume one of the two following forms:

- $\pi(n) = |n|$. By deleting the trailing $|n|$ from the word form of π, we obtain a derangement in $F_{n-1,0}$.

- $\ell := |\pi(n)| < n$, and $\pi(\ell) = n$. In this case, these derangements can be mapped bijectively to $2 \cdot F_{n-1,1}$ by setting $\pi \mapsto (\varepsilon, \pi')$, where $\varepsilon := \mathsf{sgn}(\pi(n))$, and

$$\pi'(k) := \begin{cases} \pi(k), & \text{if } k \neq \ell; \\ \ell, & \text{if } k = \ell; \end{cases} \qquad 1 \leq k \leq n-1$$

Besides, the elements in $2 \cdot F_{n,1}$ which are not mapped by θ are those (ε, σ) such that $\sigma(n) = n$; by deleting the trailing n, it is easy to see that the number of those elements are $2 \cdot f_{n-1,0}$. By interpreting θ being a bijection between subsets of $F_{n,0}$ and $2 \cdot F_{n,1}$, we see that

$$f_{n,0} - \big(f_{n-1,0} + 2 \cdot f_{n-1,1}\big) = 2 \cdot f_{n,1} - 2 \cdot f_{n-1,0}$$

which is clearly equivalent to Equation (7), and the theorem is now proved. $\square$

From the second proof of Theorem 2.3, we construct an almost bijection θ_n from $2F_{n,1}$ to $F_{n,0}$. Let $(\varepsilon, \sigma) \in 2F_{n,1}$, where $\varepsilon \in \{+,-\}$ and $\sigma \in F_{n,1}$. The action of θ_n on (ε, σ) depends on the fixed point σ as follows.

- $\mathsf{FIX}(\sigma) = \{j\}$, $j < n$. We simply define

$$\big(\theta_n(\varepsilon, \sigma)\big)(k) := \begin{cases} \sigma(n), & \text{if } k = j, \\ \sigma(k), & \text{if } k \neq j, n, \\ \varepsilon \cdot j, & \text{if } k = n, \end{cases} \qquad 1 \leq k \leq n.$$

$F_{n,0}$		$2 \cdot F_{n,1}\ (\pm, F_{n,1})$
$\sigma(n) = \pm\ell$ or $\sigma(\ell) = n$	$\leftrightarrow$	$FIX(\sigma) = \{n\},\ \theta(+,\sigma)$
$\sigma(n) = \overline{n}$	$\leftrightarrow$	$FIX(\sigma) = \{n\},\ \theta(-,\sigma)$
$\theta(2 \cdot H_{n,1})$	$\leftrightarrow$	$FIX(\sigma) \neq \{n\},\ 2 \cdot H_{n,1}$

Figure 1: Bijection between sets $F_{n,0}$ and $2 \cdot F_{n,1}$

For example, if $\sigma = \overline{2}\,4\,3\,\overline{5}\,1$ with the fixed point $j = 3$, by exchanging j with $\sigma(n)$, then we have $\theta_5(+,\sigma) = \overline{2}\,4\,1\,\overline{5}\,3$, and $\theta_5(-,\sigma) = \overline{2}\,4\,1\,\overline{5}\,\overline{3}$. Clearly θ is an injective map into $F_0^{(n)}$.

- $\mathsf{FIX}(\sigma) = \{n\}$. The sign ε will be significant in defining θ_n as shown below.

$$(a)\ \big(\theta_n(-,\sigma)\big)(k) := \begin{cases} \sigma(k), & \text{if } k \neq n, \\ -n, & \text{if } k = n, \end{cases} \qquad 1 \leq k \leq n.$$

(b) When $\varepsilon = +$, we first purge the fixed point n from σ to obtain the derangement $\sigma' = \sigma_1\sigma_2\ldots\sigma_{n-1}$, then we look for the inverse image $\theta_{n-1}^{-1}(\sigma') = (\varepsilon', \pi)$, where $\pi \in F_{n-1,1}$ (whenever it is possible). Let j be the fixed point of π. Then we set

$$\big(\theta_n(+,\sigma)\big)(k) := \begin{cases} \pi(k), & \text{if } k \neq j, n, \\ n, & \text{if } k = j, \\ \varepsilon' \cdot j, & \text{if } k = n, \end{cases} \qquad 1 \leq k \leq n.$$

For example, if $\sigma = 2\,\overline{3}\,\overline{1}\,4$ with the fixed point 4, then $\theta_4(-,\sigma) = 2\,\overline{3}\,\overline{1}\,\overline{4}$. And by deleting the fixed point 4, we have $\theta_4(+,\sigma)$ from the word form of σ, we obtain a derangement $\pi = 2\,\bar{3}\,\bar{1} \in F_{3,0}$, then we know $\theta_3^{-1}(\pi) = (-, 1\,\bar{3}\,2)$ in $F_{3,1}$, so we can construct the map $\theta_4(+, 1\,\bar{3}\,2\,4) = 4\,\bar{3}\,2\,\bar{1}$.

The correspondences $F_{2,0} \leftrightarrow F_{2,1}$ and $F_{3,0} \leftrightarrow F_{3,1}$ are shown below for the cases $n = 2$ and $n = 3$, where an element $\sigma \in B_2$ and an element $\pi \in B_3$ is represented by the sequence $\sigma = \sigma_1\sigma_2$ and $\pi = \pi_1\pi_2\pi_3$, respectively:

$F_{2,0}$	$\leftrightarrow$	$(\pm, F_{2,1})$
$\bar{2}\,1$		$(+, 1\bar{2})$
$\bar{2}\,\bar{1}$		$(-, 1\bar{2})$
$2\,\bar{1}$		$(+, \bar{1}2)$
$\bar{1}\,\bar{2}$		$(-, \bar{1}2)$
$2\,1$		**

$F_{3,0}$	$\leftrightarrow$	$(\pm, F_{3,1})$	$F_{3,0}$	$\leftrightarrow$	$(\pm, F_{3,1})$	$F_{3,0}$	$\leftrightarrow$	$(\pm, F_{3,1})$
$\bar{3}\,\bar{2}\,1$		$(+, 1\,\bar{2}\,\bar{3})$	$2\,\bar{3}\,1$		$(+, 1\,\bar{3}\,2)$	$3\,\bar{2}\,\bar{1}$		$(+, \bar{2}\,\bar{1}\,3)$
$\bar{3}\,2\,\bar{1}$		$(-, 1\,\bar{2}\,\bar{3})$	$2\,\bar{3}\,\bar{1}$		$(-, 1\,\bar{3}\,2)$	$\bar{2}\,\bar{1}\,\bar{3}$		$(-, \bar{2}\,\bar{1}\,3)$
$\bar{1}\,\bar{3}\,2$		$(+, \bar{1}\,2\,\bar{3})$	$\bar{2}\,\bar{3}\,1$		$(+, 1\,\bar{3}\,\bar{2})$	$3\,1\,2$		$(+, 3\,2\,1)$
$\bar{1}\,\bar{3}\,\bar{2}$		$(-, \bar{1}\,2\,\bar{3})$	$\bar{2}\,\bar{3}\,\bar{1}$		$(-, 1\,\bar{3}\,\bar{2})$	$3\,1\,\bar{2}$		$(-, 3\,2\,1)$
$\bar{1}\,3\,\bar{2}$		$(+, \bar{1}\,\bar{2}\,3)$	***		$(+, 2\,1\,3)$	$3\,\bar{1}\,2$		$(+, 3\,2\,\bar{1})$
$\bar{1}\,\bar{2}\,\bar{3}$		$(-, \bar{1}\,\bar{2}\,3)$	$2\,1\,\bar{3}$		$(-, 2\,1\,3)$	$3\,\bar{1}\,\bar{2}$		$(-, 3\,2\,\bar{1})$
$2\,3\,1$		$(+, 1\,3\,2)$	$\bar{1}\,3\,2$		$(+, 2\,\bar{1}\,3)$	$\bar{3}\,1\,2$		$(+, \bar{3}\,2\,1)$
$2\,3\,\bar{1}$		$(-, 1\,3\,2)$	$2\,\bar{1}\,\bar{3}$		$(-, 2\,\bar{1}\,3)$	$\bar{3}\,1\,\bar{2}$		$(-, \bar{3}\,2\,1)$
$\bar{2}\,3\,1$		$(+, 1\,3\,\bar{2})$	$3\,\bar{2}\,1$		$(+, \bar{2}\,1\,3)$	$\bar{3}\,\bar{1}\,2$		$(+, \bar{3}\,2\,\bar{1})$
$\bar{2}\,3\,\bar{1}$		$(-, 1\,3\,\bar{2})$	$\bar{2}\,1\,\bar{3}$		$(-, \bar{2}\,1\,3)$	$\bar{3}\,\bar{1}\,\bar{2}$		$(-, \bar{3}\,2\,\bar{1})$

Elements of B_n can be thought as permutations whose letters are painted by either of two colors, next, we consider the elements in $\mathfrak{S}_n^{(r)}$ which is painted by r

colored in $\mathcal{S}_n$. Define

$$F_{n,k}^{(r)} := \{\sigma \in \mathfrak{S}_n^{(r)} \colon \mathsf{fix}(\sigma) = k\}, \qquad f_{n,k}^{(r)} := |F_{n,k}^{(r)}|, \qquad k = 0, 1, \ldots, n.$$

Elements in $F_{n,0}^{(r)}$ are also known as the *derangements* in $\mathfrak{S}_n^{(r)}$. By the same reasoning as in Propsition 2.1, we see that $f_{n,k}^{(r)} = \binom{n}{k} f^{(r)} n - k, 0$.

An argument similar to Theorem 2.3 can be used to prove the recurrence of the derangement numbers $f_{n,0}^{(r)}$ as follows:

Theorem 2.4. *For $r \in \mathbb{N}$. The sequence $\langle f_{n,0}^{(r)} \rangle_n$ satisfies the following recursion:*

$$f_{n,0}^{(r)} = (rn-1) \cdot f_{n-1,0}^{(r)} + r \cdot (n-1) \cdot f_{n-2,0}^{(r)}, \qquad \forall\ n \geq 2. \tag{8}$$

Proof. Let $\sigma \in \mathfrak{S}_n^{(r)}$ be the cycle notation. As before, we consider to generate σ from elements in $F_{n-1,0}^{(r)}$ and $F_{n-2,0}^{(r)}$,

1. If elements in $F_{n-1,0}^{(r)}$, we can insert a cycle (n), the letter n can be painted by $(r-1)$ colors to get $(r-1) \cdot f_{n-1,0}^{(r)}$ elements in $F_{n,0}^{(r)}$. Or we can put the letter n which is painted by r colors behind the letter j, $j = 1, 2, ..., n-1$, so we can get $r \cdot (n-1) \cdot f_{n-1,0}^{(r)}$ elements in $F_{n,0}^{(r)}$.

2. If elements in $F_{n-2,0}^{(r)}$, each element can be inserted a cycle which is $(j\ \ n)$, $j = 1, 2, ..., n-1$, and the letter n can be painted by r colors. Standardize, we can get $r \cdot (n-1) \cdot f_{n-2,0}^{(r)}$ elements in $F_{n,0}^{(r)}$.

For $n \geq 2$ we have

$$\begin{aligned} f_{n,0}^{(r)} &= (r-1) \cdot f_{n-1,0}^{(r)} + r \cdot (n-1) \cdot f_{n-1,0}^{(r)} + r \cdot (n-1) \cdot f_{n-2,0}^{(r)} \\ &= (rn-1) \cdot f_{n-1,0}^{(r)} + r \cdot (n-1) \cdot f_{n-2,0}^{(r)} \end{aligned}$$

□

Theorem 2.5. *For $n \in \mathbb{N}$, we have:*

$$f_{n,0}^{(r)} = rf_{n,1}^{(r)} + (-1)^n. \tag{9}$$

Proof. The proof is analogous to that of Theorem 2.3. Using Theorem 2.4, we get

$$\begin{aligned} f_{n,0}^{(r)} - rf_{n,1}^{(r)} &= f_{n,0}^{(r)} - rnf_{n-1,0}^{(r)} \\ &= \big((rn-1)f_{n-1,0}^{(r)} + r(n-1)f_{n-2,0}^{(r)}\big) - rnf_{n-1,0}^{(r)} \\ &= -\big(f_{n-1,0}^{(r)} - r(n-1)f_{n-2,0}^{(r)}\big) \\ &= -\big(f_{n-1,0}^{(r)} - rf_{n-1,1}^{(r)}\big) \\ &= \cdots = (-1)^n\big(f_{0,0}^{(r)} - rf_{0,1}^{(r)}\big) = (-1)^n(1-0) = (-1)^n. \end{aligned} \tag{10}$$

□

We know the generating function of d_n is given by $f(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!} = \frac{\mathrm{e}^{-x}}{1-x}$. In the end of this section, we prove the analogous closed form for the generating function of $f_{n,0}^{(r)}$.

Theorem 2.6. *The generating function of $f_{n,0}^{(r)}$ is*

$$F(x) = \sum_{n\geq 0} f_{n,0}^{(r)} \frac{x^n}{n!} = \frac{\mathrm{e}^{-x}}{1-rx} \tag{11}$$

Proof. Let $a_n = f_{n,0}^{(r)}$, and we have the recurrences for the number of a_n is $a_0 = 1$ and $a_n = r \cdot n \cdot a_{n-1} + (-1)^n$ for $n > 0$. A routine computation shows that

$$\begin{aligned} \sum_{n\geq 1} a_n \frac{x^n}{n!} &= \sum_{n\geq 1} n \cdot r \cdot a_{n-1} \frac{x^n}{n!} + \sum_{n\geq 1} (-1)^n \frac{x^n}{n!}. \\ F(x) - 1 &= r \cdot x \cdot F(x) + (\mathrm{e}^{-x} - 1). \\ F(x)(1-rx) &= \mathrm{e}^{-x}. \\ F(x) &= \frac{\mathrm{e}^{-x}}{1-rx}. \end{aligned} \tag{12}$$

□

Notice that the special case $r = 1$ reduces to the generating function of the number of derangements of the ordinary permutations.

3 Alternating permutations with maximal number of fixed points in hyperoctahedral groups

Han and Xin have extended alternating permutations with maximal number of fixed points [HX09] in $\mathfrak{S}_n$. A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$ is said to be *alternating* (respectively *reverse alternating*) if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$ (respectively $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$). In this section, we consider the subset G_n of B_n which consists of "snakes" (a.k.a. alternating) in B_n, i.e., $\sigma \in G_n$ when $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \cdots$. We know that $|G_n| = 2^n \cdot E_n$ [KPP94] , where E_n is the n^{th} Euler number $(1, 1, 1, 2, 5, 16, 61, \dots)$. Again we define

$$G_{n,k} := \{\sigma \in G_n \colon \mathsf{fix}(\sigma) = k\}, \qquad g_{n,k} := |G_{n,k}|, \qquad k = 0, 1, \dots, \lfloor \frac{n+1}{2} \rfloor.$$

Theorem 3.1. *We have*

$$g_{2n+1,n+1} = g_{2n,n} = f_{n,0}.$$

Proof. If $\sigma \in G_n$ has maximal fixed points, then exactly one of $2i - 1$ and $2i$ is fixed by σ for each $1 \le i \le \lfloor \frac{N+1}{2} \rfloor$; in particular $\sigma(2n+1) = 2n+1$ if $\sigma \in G_{2n+1,n+1}$. By shrinking $2n+1$ from σ, we obtain a permutation $\sigma' \in G_{2n,n}$. Hence the equality $g_{2n,n} = g_{2n+1,n+1}$ is already established.

Now a map $G_{2n,n} \to F_{n,0}$ is easily constructed by removing the fixed points of σ and standardizing the remaining letters, as seen in the following example:

$$1\,\overline{3}\,(10)\,4\,8\,6\,7\,\overline{2}\,9\,\overline{5} \quad \mapsto \quad \overline{3}\,(10)\,8\,\overline{2}\,\overline{5} \quad \mapsto \quad \overline{2}\,5\,4\,\overline{1}\,\overline{3}.$$

Table 2: Numbers of alternating permutations with k fixed points in B_n

$n \setminus k$	0	1	2	3	4
0	1				
1	1				
2	3	1			
3	10	5	1		
4	50	25	5		
5	312	156	39	5	
6	2400	1200	275	29	
7	21168	10584	2646	389	29

For the element $\sigma = \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4 \cdots\ \sigma_{2n}$ in G_{2n}, each pair of letters $\sigma_{2k-1}\ \sigma_{2k}, k = 1, 2, \cdots, n$ is a decent set and have only one fixed points, then the resulting permutation cannot have any fixed point.

Next, we need to establish the inverse for the map above, i.e., from $F_{n,0}$ back to $G_{2n,n}$. Firstly $2n-1$ must be a fixed point, hence n or $\overline{n}$ must correspond to $2n$ or $\overline{2n}$, respectively. If $2n$ is present, then it must be filled at the left position of its block, hence the right position of that block is a fixed point; if $\overline{2n}$ is present, then it must be filled at the right position of its block, hence the left position of that block is a fixed point; hence we know what the next (unfixed) number (and its sign) would be. In general, once we know the unfixed number of this current block, then we know it should be put at the left or right position of the next block if each block must be [big]-[small]. And for the element $\pi = \pi_1\ \pi_2\ \cdots\ \pi_{2i-1}\ \pi_{2i}\ \pi_{2i+1}\ \pi_{2i+2}\ \cdots\ \pi_{2n}$, if π_{2i-1} and π_{2i+1} are fixed points, then $\pi_{2i+1} > \pi_{2i-1} > \pi_{2i}$; if π_{2i-1} and π_{2i+2} are fixed points, then $\pi_{2i+1} > \pi_{2i+2} > \pi_{2i-1} > \pi_{2i}$; if π_{2i} and π_{2i+1} are fixed points, then

$\pi_{2i+1} > \pi_{2i}$; if π_{2i} and π_{2i+2} are fixed points, then $\pi_{2i+1} > \pi_{2i+2} > \pi_{2i}$, so that the resulting permutation π is a snake (i.e. alternating). Just keep going until the cycle is finished. Now find the rightmost unfilled block, its left position must be a fixed point, and trace the cycle, so on and so forth, so we are done. $\square$

Example 3.2. Consider $\overline{2}\,1\,\overline{4}\,5\,\overline{3} \in F_{5,0}$. The recovery is made through the following procedure:

$$\begin{array}{cc|cc|cc|cc|cc||l}
* & * & * & * & * & * & * & * & 9 & * & 5 \mapsto (10) \\
* & * & * & * & * & * & (10) & 8 & 9 & * & \overline{4} \mapsto \overline{7} \\
* & * & * & * & 5 & \overline{7} & (10) & 8 & 9 & * & \overline{3} \mapsto \overline{6} \\
* & * & * & * & 5 & \overline{7} & (10) & 8 & 9 & \overline{6} & \cdots \\
* & * & 3 & * & 5 & \overline{7} & (10) & 8 & 9 & \overline{6} & \overline{2} \mapsto \overline{4} \\
1 & \overline{4} & 3 & * & 5 & \overline{7} & (10) & 8 & 9 & \overline{6} & 1 \mapsto 2 \\
1 & \overline{4} & 3 & 2 & 5 & \overline{7} & (10) & 8 & 9 & \overline{6} &
\end{array}$$

So, we have the bijection between the alternating permutations in B_{2n} (and also B_{2n+1}) with maximal numbers of fixed points and the derangements in B_n.

This sequel also can extend to $F_{n,k}^{(r)}$. Consider the subset $G_{n,k}^{(r)}$ of $F_{n,k}^{(r)}$ which consists alternating permutations in $F_{n,k}^{(r)}$, define $g_{n,k}^{(r)} = |G_{n,k}^{(r)}|$, we also have $g_{2n+1,n+1}^{(r)} = g_{2n,n}^{(r)} = f_{n,0}^{(r)}$, which can be verified analogously as in Theorem 3.1.

4 Strictly decreasing permutations in $\mathfrak{S}_n^{(r)}$

In this section, we discuss another subset in $\mathfrak{S}_n^{(r)}$. We regard C_r as the set $\{0, 1, \ldots, r-1\}$ with orders inherited from $\mathbb{Z}$. An element $\sigma \in \mathfrak{S}_n^{(r)}$ can be represented in the window notation: $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, where each σ_i is a pair (c_i, j_i) with $c \in C_r$ and $j \in [n]$. We use an order $\succ$ on the set $C_r \times [n]$ found in Bagno and Garber [BG06]: declaring $(c_1, j_1) \succ (c_2, j_2)$ when either $c_1 < c_2$, or $c_1 = c_2$ but $j_1 > j_2$. A permutation $\sigma \in \mathfrak{S}_n^{(r)}$ is called *strictly decreasing* if $\sigma(1) \succ \sigma(2) \succ \cdots \succ \sigma(n)$. Notice that this order is different from that in Section 3, nonetheless an obvious isomorphism exists between these orders.

Let $\mathcal{S}_{n,k}$ be the set of all r-colored strictly decreasing permutations on $[n]$ with k fixed points and the number of elements in $\mathcal{S}_{n,k}$ denoted by $s_{n,k}$. If a permutation has two or more fixed points, there must be an ascent between any pair of those fixed points, therefore it is impossible for any strictly decreasing permutation in $\mathfrak{S}_n^{(r)}$ with two or more fixed points. Henceforth we know the number of fixed points of $\sigma \in S_{n,k}$ is either 0 or 1, then a simple count gives $s_{n,0} + s_{n,1} = r^n$.

Let $\mathcal{S}_{n,0}$ be the set of all strictly decreasing derangements in $\mathfrak{S}_n^{(r)}$ and $\mathcal{S}_{n,1}$ be the set of all strictly decreasing permutations with exactly one fixed point in $\mathfrak{S}_n^{(r)}$. Now we give the enumerative results on these strictly decreasing permutations subject to the numbers of fixed points.

Theorem 4.1. *We have:*

(*i*) *For $n \geq 1$, we have $s_{n,0} = s_{n+1,1}$.*

(*ii*) *Let $t \in [n]$. The number of permutations in $s_{n,1}$ with t being the fixed point is $\binom{n-t}{t-1}(r-1)^{n-2t+1}r^{t-1}$. Therefore $s_{n,1} = \sum_{t=1}^{n} \binom{n-t}{t-1}(r-1)^{n-2t+1}r^{t-1}$.*

(*iii*) *For $n \geq 3$, we have $s_{n,0} = (r-1) \cdot s_{n-1,0} + r \cdot s_{n-2,0}$.*

(*iv*) *The generating function of $s_{n,0}$ is*

$$g(z) = \sum_{n=0}^{\infty} s_{n,0} z^n = \frac{1}{1-(r-1)z-rz^2}.$$

Proof. (*i*) Let $\sigma \in \mathcal{S}_{n+1,1}$, we can remove the only fixed point and standardize the remaining letters, we have a derangement in $\mathcal{S}_{n,0}$. On the other hand, let

$\pi \in \mathcal{S}_{n,0}$, we can insert a fixed point k right before the earliest deficiency of π, before we insert the fixed point k, we need to destandardize the letters $\pi(i)$ if $|\pi(i)| > k$, then we can get the element in $\mathcal{S}_{n+1,1}$.

(*ii*) Let $\sigma \in \mathcal{S}_{n,1}$ with the only fixed point t, $t \in [n]$. The letters $\sigma(1), \ldots, \sigma(t-1)$ can be chosen from $t+1, \ldots, n$ without any color; there are $\binom{n-t}{t-1}$ possible choices. For those letters behind $\sigma(t)$, it can be colored by all r colors if its absolute value is less than t, but only by $r-1$ colors if otherwise. By the multiplication principle, the number of strictly permutations in $\mathcal{S}_{n,1}$ with t being the only fixed point is $\displaystyle\sum_{t=1}^{n} \binom{n-t}{t-1} (r-1)^{n-2t+1} r^{t-1}$.

(*iii*) By (*i*), for $n \geq 3$, we have

$$\begin{aligned} s_{n,0} + s_{n,1} = r^n &= r \cdot (s_{n-1,0} + s_{n-1,1}) \\ s_{n,0} + s_{n-1,0} &= r \cdot (s_{n-1,0} + s_{n-2,0}) \\ s_{n,0} &= (r-1) \cdot s_{n-1,0} + r \cdot s_{n-2,0} \end{aligned} \tag{13}$$

(*iv*) Let

$$g(z) = \sum_{n=0}^{\infty} s_{n,0} z^n = \sum_{n=0}^{\infty} s_{n+1,1} z^n.$$

Using the result of $s_{n,k} = r^n$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} s_{n,0} z^n + \sum_{n=0}^{\infty} s_{n,1} z^n &= \frac{1}{1-rz} \\ \sum_{n=0}^{\infty} s_{n+1,1} z^n + \sum_{n=0}^{\infty} s_{n,1} z^n &= \frac{1}{1-rz} \\ g(z) + zg(z) &= \frac{1}{1-rz} \\ g(z) &= \frac{1}{(1-rz)(1+z)} \\ &= \frac{1}{1-(r-1)z-rz^2} \end{aligned} \tag{14}$$

□

As an example of (i), consider the permutation $\pi = 2\ \overline{1}\ \overline{\overline{3}} \in \mathcal{S}_{3,0}$, we should put the fixed point 2 right before the element $\overline{1}$. Before we insert the fixed point 2, we need to add 1 to the absolute value of the letters $\pi(i)$ which satisfy $|\pi(i)| > 2$. As a result we reach the element $3\ 2\ \overline{1}\ \overline{\overline{4}} \in \mathcal{S}_{4,1}$. On the other hand, for $\rho = 3\ 2\ \overline{1}\ \overline{\overline{4}} \in \mathcal{S}_{4,1}$, by removing the fixed point 2 and standardizing the remaining letters, we have a derangement $2\ \overline{1}\ \overline{\overline{3}} \in \mathcal{S}_{3,0}$. We construct a bijection between elements with exactly one fixed point in $S_{n+1,1}$ and derangements in $S_{n,0}$.

By Theorem 4.1 (iv), we have the generating function of $s_{n,0}$, then we can get

Theorem 4.2. *The number of strictly decreasing derangements in $\mathcal{S}_{n,0}$ is*

$$s_{n,0} = \frac{1}{r+1} \cdot (r^{n+1} + (-1)^n). \tag{15}$$

Proof. In Theorem 4.1 (iv), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} s_{n,0} z^n &= \frac{1}{1-(r-1)z-rz^2} \\
&= \frac{1}{(1+z)(1-rz)} \\
&= \frac{1}{r+1}\left(\frac{1}{1+z} + \frac{r}{1-rz}\right) \\
&= \frac{1}{r+1}\left(\sum_{n=0}^{\infty}(-1)^n z^n + \sum_{n=0}^{\infty} r^{n+1} z^n\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{r+1}(r^{n+1} + (-1)^n)\right) z^n
\end{aligned} \tag{16}$$

So we have $s_{n,0} = \frac{1}{r+1} \cdot (r^{n+1} + (-1)^n)$. □

The number of derangements in $\mathfrak{S}_n$, we have the harder recursion $d_n = nd_{n-1} + (-1)^n$, and in section 2, we already prove the recursion $f_{n,r,0} = rf_{n,r,1} + (-1)^n$ in $\mathfrak{S}_n^{(r)}$. In the end of this section, we have

Corollary 4.3. *For $n \in \mathbb{N}$, we have:*

$$s_{n,0} = r \cdot s_{n,1} + (-1)^n. \tag{17}$$

Proof. By Theorem 4.1 (iv), we have

$$\sum_{n=0}^{\infty} s_{n,0} z^n = \frac{1}{(1-rz)(1+z)}.$$

And we also have

$$\sum_{n=0}^{\infty} (s_{n,0} + s_{n,1}) z^n = \sum_{n=0}^{\infty} r^n z^n = \frac{1}{(1-rz)}. \tag{18}$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} s_{n,1} z^n &= \frac{1}{(1-rz)} - \frac{1}{(1-rz)(1+z)} \\
&= \frac{1+z-1}{(1-rz)(1+z)} \\
&= \frac{z}{(1-rz)(1+z)} \\
\sum_{n=0}^{\infty} (s_{n,0} - r \cdot s_{n,1}) z^n &= \frac{1}{(1-rz)(1+z)} - \frac{rz}{(1-rz)(1+z)} \\
&= \frac{1}{1+z} \\
&= \sum_{n=0}^{\infty} (-1)^n \cdot z^n
\end{aligned} \tag{19}$$

□

5 Conclusion and Discussion

In this study, we generalize the enumeration problems on derangements and permutations with only one fixed point to colored permutations. In fact, the relationship between the derangements in $\mathfrak{S}_n^{(r)}$ and the elements with exactly only one fixed point in $\mathfrak{S}_n^{(r)}$ can be classified with the descent sets. At first we observed the phenomenon on the enumeration on derangements for alternating permutations in B_n, i.e. the numbers in the first column of Table 2 are twice of those in the second column. It could be thought the special case of the following setting.

For each subset J of $[n-1]$, define

$$F_{n,k}^{(r),J} := \{\sigma \in \mathfrak{S}_n^{(r)} \colon \mathsf{fix}(\sigma) = k \text{ and } \mathsf{DES} = J\},$$

$$f_{n,k}^{(r),J} := |F_{n,k}^{(r),J}|, \qquad k = 0, 1, \dots, n.$$

For example, the snakes in B_n have the descent sets $\{1,3,5,7,9,\dots\} \cap [n-1]$. Using the mathematical software `SAGE`, we enumerate the cardinalities $f_{n,k}^{(r),J}$ and find the following relations:

$$f_{n,0}^{(r),J} = \begin{cases} r \cdot f_{n,1}^{(r),J} + (-1)^n, & \text{if } J = [n-1]; \\ r \cdot f_{n,1}^{(r),J}, & \text{if } J \neq [n-1]. \end{cases} \tag{20}$$

The case $r = 1$, which is the ordinary permutations, has been dealt with by Foata and Han [FH08]. They found a bijection on $\mathfrak{S}_n$ that transforms the pair of statistics $(\mathsf{fix}, \mathsf{DES})$ to $(\mathsf{pix}, \mathsf{IDES})$ [DW93]. In section 4, we have already explained the situation with $f_{n,0} = r \cdot f_{n,1} + (-1)^n$, which corresponds to the case $J = [n-1]$. We hope the we can find the correct notions for pixed points and IDES sets of colored permutations that enables us to establish the identity (20) for $J \neq [n-1]$ in the future.

6 Acknowledgement

I am grateful to Professor Sen-Peng Eu gave the important ideas of permuatations with r colors and also for his helpful comments of this paper.

References

[Arn92] Vladimir I. Arnol'd, *The calculus of snakes and the combinatorics of Bernoulli, Euler and Springer numbers of Coxeter groups*, Russian Mathematical Surveys **47** (1992), no. 1, 1.

[Bag04] Eli Bagno, *Euler-Mahonian parameters on colored permutation groups*, Séminaire Lotharingien de Combinatoire **51** (2004), B51f.

[BG06] Eli Bagno and David Garber, *On the excedance number of colored permutation groups*, Sém. Lothar. Combin **53** (2006), B53f.

[BO17] Arthur T. Benjamin and Joel Ornstein, *A bijective proof of a derangement recurrence*, FIBONACCI QUARTERLY **55** (2017), no. 5, 28–29.

[DW93] Jacques Désarménien and Michelle L. Wachs, *Descent classes of permutations with a given number of fixed points*, Journal of Combinatorial Theory, Series A **64** (1993), no. 2, 311–328.

[FH05] Dominique Foata and Guo-Niu Han, *Signed words and permutations II; the Euler-Mahonian polynomials*, the electronic journal of combinatorics **11** (2005), no. 2, 22.

[FH08] ______, *Signed words and permutations, IV: Fixed and pixed points*, Israel Journal of Mathematics **163** (2008), no. 1, 217–240.

[HX09] Guo-Niu Han and Guoce Xin, *Permutations with extremal number of fixed points*, Journal of Combinatorial Theory, Series A **116** (2009), no. 2, 449–459.

[KPP94] Alexander G. Kuznetsov, Igor M. Pak, and Alexandr E. Postnikov, *Increasing trees and alternating permutations*, Russian Mathematical Surveys **49** (1994), no. 6, 79.

[Rei93] Victor Reiner, *Signed permutation statistics*, European journal of combinatorics **14** (1993), no. 6, 553–567.

[Rem83] Jeffrey B. Remmel, *A note on a recursion for the number of derangements*, European Journal of Combinatorics **4** (1983), no. 4, 371–374.

[Tuc80] Alan Tucker, *Applied combinatorics*, Wiley, 1980.

[WG16] Walter D. Wallis and John C. George, *Introduction to combinatorics*, Chapman and Hall/CRC, 2016.

[Wil84] Herbert S. Wilf, *A bijection in the theory of derangements*, Mathematics Magazine **57** (1984), no. 1, 37–40.