for some $\zeta > 0$. Applying Lemma 3.9 to $f(x) = \|x-x^*\|^\alpha$ and $\xi = 0$, we conclude that $\{x_k\}$ strongly converges to x^* . $\square$

§4 A Modified Algorithm Without Strong Convexity

In this section, we modify Algorithm 1 in order to weakness of the auxiliary function M, without strong monotonicity. Indeed, we make assumption as follows.

Assumption 2.

(i) T is (l, w)-u.s.c. and weakly monotone with constant L on C.

(ii) M is convex and differentiable and satisfies the following :

(a) $\langle M'(x), x\rangle \geq c\|x\|^2,\ c \geq 1.$

(b) $M(x) \leq \alpha\|x\|^2,\ \alpha < 1.$

(c) $\|M'(x)\| \leq \beta\|x\|,\ 0 < \beta < 1.$

Remark. If $M(x) = \frac{1}{2}\|x\|^2$, then $M'(x) = J(x) = \partial\left(\frac{1}{2}\|x\|^2\right)$ satisfies Assumption 2, where $J(x)$ is usually called the duality mapping.

Algorithm 2 to VI(T,C) :

Step 1: Take any $x_0 \in C$ and $y_0 \in T(x_0)$.

Step 2: Knowing (x_k, y_k) and ε_k, compute $x_{k+1} \in C$ such that

$$\varepsilon_k\langle y_k, z - x_{k+1}\rangle + B(x_{k+1}, x_k, z) \geq 0,\ \forall\, z \in C,$$

and

$$\langle y_k, x_k - x_{k+1}\rangle < 0. \tag{13}$$

Step 3: Take any $y_{k+1} \in T(x_{k+1})$ and return to Step 2 , until $\|x_{k+1} - x_k\|$ is below some threshold.

The followig lemma will be used to prove the well-definedness of the sequence $\{x_k\}$ generated by our Algorithm 2.

Lemma 4.1. *Let $x \in C$, T and M satisfy Assumption 2, and $\beta + 2\varepsilon L < 1$, then the operator $F(y) = \varepsilon T(y) + M'(y) - M'(x)$ is strongly monotone with constant $-\varepsilon L + \frac{1-\beta}{2}$ on C.*

Proof : For all (y_1, y_1^*) and $(y_2, y_2^*) \in G(F)$, we have some $z_1^* \in T(y_1)$ and $z_2^* \in T(y_2)$. since T is weakly monotone with constant L,

$$\langle z_1^* - z_2^*, y_1 - y_2 \rangle \geq -L\|y_1 - y_2\|^2.$$

By Assumption 2, there exist $c \geq 1$ and $\beta > 0$ such that

$$\begin{aligned}
\langle y_1^* - y_2^*, y_1 - y_2 \rangle &= \varepsilon\langle (z_1^* - z_2^*) + M'(y_1) - M'(y_2), y_1 - y_2 \rangle \\
&= \varepsilon\langle z_1^* - z_2^*, y_1 - y_2 \rangle + \langle M'(y_1) - M'(y_2), y_1 - y_2 \rangle \\
&= \varepsilon\langle z_1^* - z_2^*, y_1 - y_2 \rangle + \langle M'(y_1), y_1 - y_2 \rangle - \langle M'(y_2), y_1 - y_2 \rangle \\
&= \varepsilon\langle z_1^* - z_2^*, y_1 - y_2 \rangle + \langle M'(y_1), y_1 \rangle - \langle M'(y_1), y_2 \rangle \\
&\quad - \langle M'(y_2), y_1 \rangle + \langle M'(y_2), y_1 \rangle \\
&\geq \varepsilon(-L)\|y_1 - y_2\|^2 + c\|y_1\|^2 - \beta\|y_1\|\|y_2\| - \beta\|y_1\|\|y_2\| + c\|y_2\|^2 \\
&\geq \varepsilon(-L)\|y_1 - y_2\|^2 + \|y_1\|^2 - 2\beta\|y_1\|\|y_2\| + \|y_2\|^2, \\
&\geq \varepsilon(-L)\|y_1 - y_2\|^2 + \|y_1\|^2 + \|y_2\|^2 - \beta(\|y_1\|^2 + \|y_2\|^2), \\
&= \varepsilon(-L)\|y_1 - y_2\|^2 + (1 - \beta)(\|y_1\|^2 + \|y_2\|^2), \\
&= \varepsilon(-L)\|y_1 - y_2\|^2 + \frac{1-\beta}{2}(\|y_1 - y_2\|^2 + \|y_1 + y_2\|^2), \\
&\geq (-\varepsilon L + \frac{1-\beta}{2})\|y_1 - y_2\|^2.
\end{aligned}$$

Since $\beta + 2\varepsilon L < 1$, F is strongly monotone. □

Remarks.

(i) If T is monotone, then F is strongly monotone with constant $\frac{1-\beta}{2}$.

(ii) If T is strongly monotone with constant a, then F is strongly monotone with constant $\varepsilon a + \frac{1-\beta}{2}$.

(iii) If T has the Dunn property, then F is strongly monotone with constant $\frac{1-\beta}{2}$.

To illustrate the uniqueness of solutions to Algorithm 2, we have the following property.

Lemma 4.2. *Suppose that the problem* $(P1)$ *has a solution and the sequence* $\{x_k\}$ *generated from Algorithm 2 is well-defined. Let* T *and* M *satisfy Assumption 2 and* M' *be* (l, w)*-continuous, then there exists a unique solution to the variational inequality* $(P2)$.

Proof : Under the previous assumptions, the operator F defined by

$$F(y) = \varepsilon T(y) + M'(y) - M'(x)$$

is (l, w)-u.s.c. and strongly monotone on C, by Lemma 2.3 and Lemma 4.1. According to Proposition 3.2, we get that there exists a unique solution to the problem $(P2)$. □

The following result deals with the convergence of the Algorithm 2 involving pseudomonotone operator.

Theorem 4.3. *Suppose that the problem* $(P1)$ *has a solution* $x^* \in C$*. Under Assumption 2, if* T *is pseudomonotone on* C*, then the sequence* $\{x_k\}$ *generated from Algorithm 2 is bounded. In addition, if* M' *is Lipschitz continuous with constant* m*, and if* T *is* (w, s)*-u.s.c. on* C*, then every weakly cluster point of the sequence* $\{x_k\}$ *is a solution of the problem* $(P1)$.

Proof : Consider the Bregman distance

$$\phi_{x^*}(x) := M(x^*) - M(x) - \langle M'(x), x^* - x\rangle, \forall\, x \in C.$$

Since M is convex,

$$\phi_{x^*}(x_k) = M(x^*) - M(x_k) - \langle M'(x_k), x^* - x_k\rangle \geq 0.$$

Now we study the variation of ϕ_{x^*} for Algorithm 2 : $\Delta_k = \phi_{x^*}(x_{k+1}) - \phi_{x^*}(x_k)$ for each k. So we have

$$\begin{aligned} \Delta_k &= M(x_k) - M(x_{k+1}) - \langle M'(x_k), x_k - x_{k+1}\rangle + \langle M'(x_k) - M'(x_{k+1}), x^* - x_{k+1}\rangle \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 := M(x_k) - M(x_{k+1}) - \langle M'(x_k), x_k - x_{k+1}\rangle \leq 0 \tag{14}$$

and

$$S_2 := \langle M'(x_k) - M'(x_{k+1}), x^* - x_{k+1}\rangle.$$

As before, by the pseudomonotonicity of T, for $y_k \in T(x_k)$, we have

$$S_2 \leq \varepsilon_k \langle y_k, x_k - x_{k+1} \rangle < 0, \tag{15}$$

by using (13). Therefore, combining (14) and (15) to get the variation $\Delta_k < 0$, which implies that Δ_k is negative. So the sequence $\{\phi_{x^*}(x_k)\}$ is strictly decreasing and bounded below by 0. Thus, $\{\phi_{x^*}(x_k)\}$ is convergent, so it is bounded (i.e. there exists c such that $\phi_{x^*}(x_{n_k}) < c$). We want to prove $\{x_k\}$ is bounded. If $\{x_k\}$ is unbounded, then there exists $\{x_{n_k}\}$ with $\lim_{k\to\infty} \|x_{n_k}\| = +\infty$. It follows that

$$\begin{aligned}
\phi_{x^*}(x_{n_k}) &= M(x^*) - M(x_{n_k}) - \langle M'(x_{n_k}), x^* - x_{n_k} \rangle \\
&= M(x^*) - M(x_{n_k}) - \langle M'(x_{n_k}), x^* \rangle + \langle M'(x_{n_k}), x_{n_k} \rangle \\
&\geq M(x^*) - M(x_{n_k}) - \|M'(x_{n_k})\| \|x^*\| + c\|x_{n_k}\|^2 \\
&\geq M(x^*) - M(x_{n_k}) - \beta\|x_{n_k}\| \|x^*\| + c\|x_{n_k}\|^2 \\
&\to \infty,
\end{aligned}$$

which is a contradiction. Hence $\{x_k\}$ is bounded. The remainder proof is similar to the proof of Theorem 3.4, hence every weakly cluster point of the sequence $\{x_k\}$ is a solution of the problem $(P1)$. □

The following result analyzes the convergence of the Algorithm 2 involving the pseudo-Dunn Property.

Theorem 4.4. *Under all the conditions of Theorem 4.3, if T has the pseudo-Dunn property with constant t on C, then every weakly cluster point of the sequence $\{x_k\}$ is a solution of the problem $(P1)$. Moreover, if $y^* \in T(x^*)$, then the correspondent sequence y_k strongly converges to y^* (and hence y^* is unique).*

Proof : The proof is similiar to Theorem 4.3, and we remain to show that y_k strongly converges to y^*. As before, by the pseudo-Dunn property of T, for $y_k \in T(x_k)$, $y^* \in T(x^*)$,

$$S_2 \leq -\frac{\varepsilon_k}{t}\|y_k - y^*\|^2 + \varepsilon_k \langle y_k, x_k - x_{k+1} \rangle.$$

By using (13), we have

$$S_2 < -\frac{\varepsilon_k}{t}\|y_k - y^*\|^2.$$

Hence

$$\Delta_k < -\frac{\varepsilon_k}{t}\|y_k - y^*\|^2 < 0,$$

which implies Δ_k is negative. So the sequence $\{\phi_{x^*}(x_k)\}$ is strictly decreasing (unless $y_k = y^*$) and bounded below by 0, we can get that the sequence $\{\phi_{x^*}(x_k)\}$ is convergent. Since it is convergent, it is bounded (i.e. there exists c such that $\phi_{x^*}(x_{n_k}) < c$) and the difference between two consecutive terms tends to zero. Therefore, by the same step of proof in Theorem 4.3, we can conclude that the sequence $\{x_k\}$ is bounded. Since $\|\phi_{x^*}(x_{n_k}) - \phi_{x^*}(x_{n_{k+1}})\|$ converges to zero, it follows that $\|y_k - y^*\|$ converges to zero; that is, y_k strongly converges to y^*. □

Corollary 4.5. *Under all the conditions of Theorem 4.4, if M' is a continuous from X equipped with the weak topology to X equipped with the weak topology, then the sequence $\{x_k\}$ weakly converges to a solution of the problem $(P1)$.*

Proof : The proof is similar to the proof of Corollary 3.5. □

Finally, we establish a norm convergence property in virtue of a strongly pseudomonotone operator T.

Theorem 4.6. *Suppose that the problem $(P1)$ has a solution $x^* \in C$. Under Assumption 2, if T is strongly pseudomonotone with constant e on C, then the sequence $\{x_k\}$ strongly converges to x^*.*

Proof : Using the same definitions of S_1, S_2 and Δ_k as in Theorem 4.3, we have that

$$S_1 \leq 0$$

and by the strong pseudomonotonicity of T,

$$S_2 < -e\varepsilon_k \|x_k - x^*\|^2.$$

Thus,

$$\Delta_k < -e\varepsilon_k \|x_k - x^*\|^2 < 0,$$

which implies that Δ_k is negative unless $x_k = x^*$. We can conclude that $\{\phi_{x^*}(x_k)\}$ is convergent. By using the same step of proof in Theorem 4.3, we conclude that $\{x_k\}$ is strongly convergent to x^*. □

References

[1] **Chadli, O., Chbani, Z. & Riahi, H.** (2000). *Equilibrium Problems with Generalized Monotone Bifunctions and Applications to Variational Inequalities*, J. Optim. Theory Appl. **105(2)**, 299-323.