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A new generalization of the Natural-Residual function

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Contents

A new generalization of the Natural-Residual function

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Abstract. NCP-functions play an important role in nonlinear complementarity problems(NCP). In this paper, we recall some definitions and properties of NCP-functions such as generalized Fisher-Burmeister function, $\phi^p_{_{\rm FB}}(a,b) = \|(a,b)\|_p - (a+b)$, and the generalized Natural-Residual function, $\phi^p_{_{\rm NR}}(a,b) = a^p - (a-b)^p_+$. We attempt to generalize Natural-Residual function as a new NCP-function: $\tilde{\phi}^p_{_{\rm NR}}(a,b) = (a+b)^p - |a-b|^p$.

Keywords. NCP, Fisher-Burmeister, Natural-Residual, complementarity.

1 Introduction

Consider the quadratic program

$$\begin{aligned} \min f(x) &= c^T x + \frac{1}{2} x^T Q x, \\ \text{s.t. } Ax &\geq b, \\ x &\geq 0, \end{aligned} \tag{1}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. If x is a locally optimal solution of the program (1), then there exists a vector $y \in \mathbb{R}^m$ such that the pair (x, y) satisfies the Karush-Kuhn-Tucker (KKT) conditions

$$u = c + Qx - A^T y \geq 0, \ x \geq 0, \ x^T u = 0, \tag{2}$$

$$v = -b + Ax \geq 0, \ y \geq 0, \ y^T v = 0. \tag{3}$$

Hence, the KKT conditions in (2)(3) corresponds to a linear complementarity problem

$$\bar{x} \geq 0, \ F(\bar{x}) \geq 0, \langle \bar{x}, F(\bar{x}) \rangle = 0,$$

where $F(\bar{x}) = M\bar{x} + q$, $M = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$, $q = \begin{bmatrix} c \\ -b \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Now, we consider another optimal problem

$$\begin{aligned} \min & f(x) \\ \text{s.t. } & Ax = b, \\ & g(x) \leq 0, \end{aligned}$$

where f is nonlinear. Note that the Lagrangian is $L(x, \lambda, \mu) = f(x) + \langle \lambda, Ax - b \rangle + \langle \mu, g(x) \rangle$, thus its KKT conditions are

$$\nabla_x L(x, \lambda, \mu) = \nabla f(x) + A^T \lambda + \nabla g(x) \cdot \mu = 0,$$

$$Ax = b, \tag{4}$$

$$g(x) \leq 0, \tag{5}$$

$$\mu \geq 0, \tag{6}$$

$$g(x) \cdot \mu = 0. \tag{7}$$

Thus,

$$\nabla g(x) \cdot \mu = -\nabla f(x) - A^T \lambda.$$

If $(\nabla g(x))^T \nabla g(x)$ is nonsingular, then from (7) we have

$$\mu = -[(\nabla g(x))^T \nabla g(x)]^{-1} \cdot \nabla g(x)^T \cdot (\nabla f(x) + A^T \lambda).$$

Hence, the KKT conditions in (4), (5), (6), and (7) are corresponding to equations $Ax = b$ and a nonlinear complementarity problem

$$\mu \geq 0, \ -g(x) \geq 0, \ \langle \mu, -g(x) \rangle = 0.$$

Therefore, the nonlinear complementarity problem is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \ F(x) \geq 0, \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \cdots, F_n)^T$ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

We assume that F is continuously differentiable throughout this paper. The nonlinear complementarity problem has attracted much attention due to its various applications in operations research, economics, and engineering, see [18, 26, 27] and references therein. There have been many methods proposed for solving the nonlinear complementarity problem. Among which, one of the most popular and powerful approaches that has been studied intensively recently is to reformulate the nonlinear complementarity problem as a system of nonlinear equations [21]. For constructing a merit function, a class of functions, called NCP-functions and defined below, plays a significant role.

Definition 1.1. *A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if it satisfies*

$$\phi(a, b) = 0 \Leftrightarrow a \geq 0, \ b \geq 0, \ ab = 0.$$

The nonlinear complementarity problem can be reformulated as a system of nonsmooth equations

$$\Phi(x) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix} = 0.$$

Therefore, the function $\Psi : \mathbb{R}^n \to \mathbb{R}_+$ defined as below is a merit function for the nonlinear complementarity problem

$$\Psi(x) = \|\Phi(x)\|^2 = \sum_{i=1}^{n} \psi(x_i, F_i(x))$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}_+$ is the square of ϕ. Such a function that can constitute an equivalent unconstrained minimization problem for the nonlinear complementarity problem is called a merit function. Consequently, the nonlinear complementarity problem is equivalent to an unconstrained minimization problem[13, 14]:

$$\min_{x\in\mathbb{R}^n} \|\Phi(x)\|^2.$$

In other words, a merit function is a function whose global minimum are coincident with the solutions of the original nonlinear complementarity problem.

Many NCP-functions and merit functions have been explored and proposed in many literature, see [16] for a survey. Among them, the Fischer-Burmeister (FB) function and the Natural-Residual (NR) function are two famous and effective NCP-functions. The FB function $\phi_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{FB}}(a, b) = \sqrt{a^2 + b^2} - (a + b), \ \forall (a, b) \in \mathbb{R}^2 \tag{8}$$

and the NR function $\phi_{\text{NR}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{NR}}(a, b) = a - (a - b)_+ = \min\{a, b\}, \ \forall (a, b) \in \mathbb{R}^2. \tag{9}$$

Recently, the generalized Fischer-Burmeister function ϕ^p_{FB} which includes the Fischer-Burmeister as a special case was considered in [1, 2, 3, 7, 24]. Indeed, the function ϕ^p_{FB} is a natural extension of the ϕ_{FB} function, in which the 2-norm in ϕ_{FB} is replaced by general p-norm. In other words, $\phi^p_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\phi^p_{\text{FB}}(a, b) = \|(a, b)\|_p - (a + b), \tag{10}$$

where $p > 1$ and $\|(a, b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. The detailed geometric view of ϕ^p_{FB} is depicted in [24]. Corresponding to ϕ^p_{FB}, there is a merit function $\psi^p_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$\psi^p_{\text{FB}}(a, b) = \frac{1}{2} \left|\phi^p_{\text{FB}}(a, b)\right|^2. \tag{11}$$

To the contrast, what does "generalized natural-residual function" look like? In [6], Chen *et al.* give an answer to the long-standing open question. More specifically, the generalized natural-residual function, denoted by ϕ^p_{NR}, is defined by

$$\phi^p_{\text{NR}}(a, b) = a^p - (a - b)^p_+ \tag{12}$$

with $p > 1$ being a positive odd integer. As remarked in [6], the main idea to create it relies on "discrete generalization", not the "continuous generalization". Note that when $p = 1$, $\phi^p_{\rm NR}$ is reduced to the natural residual function $\phi_{\rm NR}$.

Unlike the surface of $\phi^p_{\rm FB}$, the surface of $\phi^p_{\rm NR}$ is not symmetric which may cause some difficulties in further analysis in designing solution methods. To this end, Chang *et al.* [12] try to symmetrize the function $\phi^p_{\rm NR}$. The first-type symmetrization of $\phi^p_{\rm NR}$, denoted by $\phi^p_{\rm S-NR}$ is proposed as

$$\phi^p_{\rm S-NR}(a,b) = \begin{cases} a^p - (a-b)^p & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ b^p - (b-a)^p & \text{if} \quad a < b, \end{cases} \tag{13}$$

where $p > 1$ being a positive odd integer. It is shown in [12] that $\phi^p_{\rm S-NR}$ is an NCP-function with symmetric surface, but it is not differentiable. Therefore, it is natural to ask whether there exists another symmetrization function that has not only symmetric surface but also is differentiable. Fortunately, Chang *et al.* [12] also figure out the second symmetrization of $\phi^p_{\rm NR}$, denoted by $\psi^p_{\rm S-NR}$, which is proposed as

$$\psi^p_{\rm S-NR}(a,b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ a^p b^p - (b-a)^p a^p & \text{if} \quad a < b, \end{cases} \tag{14}$$

where $p > 1$ being a positive odd integer.

The idea of "discrete generalization" looks simple, but it is novel and important. In fact, we also apply such idea to construct more NCP-functions. For example, we apply it to the Fischer-Burmeister function to obtain $\phi^p_{\rm D-FB} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi^p_{\rm D-FB}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p \tag{15}$$

where $p > 1$ being a positive odd integer.

One can see that the second symmetrization of $\phi^p_{\rm NR}$ is differentiable and symmetric simultaneously. Now, we have a new idea comes from

$$\phi_{\rm NR}(a,b) = \min\{a,b\} = a - (a-b)_+ = \frac{(a+b) - |a-b|}{2}.$$

Hence, we can define a hole new NCP-function $\phi^{\tilde{p}}_{\rm NR}(a,b) = (a+b)^p - |a-b|^p$ with $p > 1$ being a positive odd integer. In fact, $\phi^{\tilde{p}}_{\rm NR}(a,b)$ is a differentiable NCP function, and hence a symmetric surface.

2 Fischer-Burmeister Functions

In this section, we focus on Fischer-Burmeister (FB) function and its generalizations as defined in (9), (11) and (12). We first recall some basic definitions and then we discuss the differentiability.

Definition 2.1. *Let* $\phi : \mathbb{R}^2 \to \mathbb{R}$

(a) ϕ *is sub-additive if* $\phi(w + w') \leq \phi(w) + \phi(w') \ \ \forall\, w, w' \in \mathbb{R}^2$.

(b) ϕ *is positive homogeneous if* $\phi(\alpha \cdot w) = \alpha \cdot \phi(w) \ \ \forall\, w \in \mathbb{R}^2$ *and* $\alpha \geq 0$.

(c) ϕ *is convex if* $\phi(\alpha w + (1-\alpha) w') \leq \alpha\phi(w) + (1-\alpha)\phi(w') \ \ \forall\, w, w' \in \mathbb{R}^2$ *and* $0 \leq \alpha \leq 1$.

Definition 2.2. *Let* $\phi : \mathbb{R}^n \to \mathbb{R}^m$ *is locally Lipschitz continuous (also called strictly continuous) at* $x \in \mathbb{R}^n$ *if there exist scalars* $\kappa > 0$ *and* $\delta > 0$ *such that*

$$\|\phi(y) - \phi(z)\| \leq \kappa \|y - z\|$$

for all $y, z \in \mathbb{R}^n$ *with* $\|y - x\| \leq \delta$ *and* $\|z - x\| \leq \delta$.

Proposition 2.1. *[7, 11] Let* $\phi^p_{\text{FB}}(a, b)$ *be defined as in (11), then the following hold.*

(a) ϕ^p_{FB} *is a NCP-function.*

(b) ϕ^p_{FB} *is sub-additive.*

(c) ϕ^p_{FB} *is positive homogeneous of degree p.*

(d) ϕ^p_{FB} *is convex.*

(e) ϕ^p_{FB} *is Lipschitz continuous with* $\kappa_1 = \sqrt{2} + 2^{(1/p-1/2)}$ *when* $1 < p < 2$, *and with* $\kappa_2 = 1 + \sqrt{2}$ *when* $p \geq 2$.

Proposition 2.2. *[7, 11] Let* $\psi^p_{\text{FB}} = \frac{1}{2}|\phi^p_{\text{FB}}|^2$ *and* ϕ^p_{FB} *be defined as in (11). Then, the following hold.*

(a) ψ^p_{FB} *is an NCP-function.*

(b) $\psi^p_{\text{FB}}(a, b) \geq 0$ *for all* $(a, b) \in \mathbb{R}^2$.

(c) ψ^p_{FB} *is continuously differentiable everywhere.*

(d) $\nabla_a \psi^p_{\text{FB}}(a, b) \cdot \nabla_b \psi^p_{\text{FB}}(a, b) \geq 0$ *for all* $(a, b) \in \mathbb{R}^2$. *The equality holds if and only if* $\phi^p_{\text{FB}}(a, b) = 0$.

(e) $\nabla_a \psi^p_{\text{FB}}(a,b) = 0 \Longleftrightarrow \nabla_b \psi^p_{\text{FB}}(a,b) = 0 \Longleftrightarrow \phi^p_{\text{FB}}(a,b) = 0.$

Proposition 2.3. *[8] Let ϕ^p_{FB} be defined as in (11). Then, the generalized gradient $\partial\phi^p_{\text{FB}}(a,b)$ of ϕ^p_{FB} at a point (a,b) is equal to the set of all (v_a, v_b) such that*

$$(v_a, v_b) = \begin{cases} \left(\dfrac{\operatorname{sgn}(a)\cdot|a|^{p-1}}{\|(a,b)\|_p^{p-1}} - 1, \dfrac{\operatorname{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}} - 1 \right) & \text{if} \quad (a,b) \neq (0,0), \\ \\ (\xi - 1, \zeta - 1) & \text{if} \quad (a,b) = (0,0), \end{cases}$$

where (ξ, ζ) is any vector satisfying $|\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1$.

Proposition 2.4. *[11] Let $\phi^p_{\text{D-FB}}(a,b) = (\sqrt{a^2+b^2})^p - (a+b)^p : \mathbb{R}^2 \to \mathbb{R}$ where p is a positive odd integer and $p > 1$, then*

(a) *$\phi^p_{\text{D-FB}}$ is a NCP-function.*

(b) *$\phi^p_{\text{D-FB}}$ is positive homogeneous of degree p.*

(c) *$\phi^p_{\text{D-FB}}$ is locally Lipschitz continuous, but not (globally) Lipschitz continuous.*

(d) *$\phi^p_{\text{D-FB}}$ is not α-Hölder continuous for any $\alpha \in (0,1]$.*

(e) *$\phi^p_{\text{D-FB}} < 0 \Leftrightarrow a > 0, b > 0$.*

(f) *$\nabla_a \phi^p_{\text{D-FB}}(a,b) \cdot \nabla_b \phi^p_{\text{D-FB}}(a,b) > 0$ on the first quadrant.*

(g) *$\nabla_a \phi^p_{\text{D-FB}}(a,b) \cdot \nabla_b \phi^p_{\text{D-FB}}(a,b) = 0$ provided that $\phi^p_{\text{D-FB}}(a,b) = 0$.*

Proposition 2.5. *[11] Let $\phi^p_{\text{D-FB}}$ be defined as in (12) where p being a positive odd integer. Then, the followings hold.*

(a) *For $p > 1$, $\phi^p_{\text{D-FB}}$ is continuously differentiable with*

$$\nabla \phi^p_{\text{D-FB}}(a,b) = p \begin{bmatrix} a(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \\ b(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \end{bmatrix}.$$

(b) *For $p > 3$, $\phi^p_{\text{D-FB}}$ is twice continuously differentiable with $\nabla^2 \phi^p_{\text{D-FB}}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,*

and for $(a,b) \neq (0,0)$,

$$\nabla^2 \phi^p_{\text{D-FB}}(a,b) = \begin{bmatrix} \dfrac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a^2} & \dfrac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a \partial b} \\ \\ \dfrac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b \partial a} & \dfrac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b^2} \end{bmatrix}$$

where

$$\begin{aligned}
\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a^2} &= p\left\{[(p-1)a^2+b^2](\sqrt{a^2+b^2})^{p-4}-(p-1)(a+b)^{p-2}\right\},\\
\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a \partial b} &= p[(p-2)ab(\sqrt{a^2+b^2})^{p-4}-(p-1)(a+b)^{p-2}] = \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b \partial a},\\
\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b^2} &= p\left\{[a^2+(p-1)b^2](\sqrt{a^2+b^2})^{p-4}-(p-1)(a+b)^{p-2}\right\}.
\end{aligned}$$

3 Natural Residual Functions

In this section, we would also talk about some properties and differentiability of Natural-Residual (NR) function and its generalizations as defined in (10), (13), (14) and (15).

Proposition 3.1. *[6] Let $\phi^p_{\text{NR}}(a,b)$ be defined as in (13) with $p>1$ being a positive odd integer. Then, the following hold.*

(a) ϕ^p_{NR} *is a NCP-function.*

(b) $\phi^p_{\text{NR}}(a,b) > 0 \iff a>0,\ b>0.$

(c) ϕ^p_{NR} *is positive homogeneous of degree p.*

(d) ϕ^p_{NR} *is locally Lipschitz continuous, but not (globally) Lipschitz continuous.*

(e) ϕ^p_{NR} *is not α-Hölder continuous for any $\alpha \in (0,1]$.*

(f) $$\nabla_a \phi^p_{\text{NR}}(a,b)\cdot \nabla_b \phi^p_{\text{NR}}(a,b) \begin{cases} >0 & \text{on } \{(a,b) \mid a>b>0 \text{ or } a>b>2a\},\\ =0 & \text{on } \{(a,b) \mid a\le b \text{ or } a>b=2a \text{ or } a>b=0\},\\ <0 & \text{otherwise.}\end{cases}$$

(g) $\nabla_a \phi^p_{\text{NR}}(a,b)\cdot \nabla_b \phi^p_{\text{NR}}(a,b) = 0$ *provided that* $\phi^p_{\text{NR}}(a,b)=0$.

Proposition 3.2. *[6] Let ϕ^p_{NR} be defined as in (13) with $p>1$ being a positive odd integer, and let $p=2k+1$ where $k\in\mathbb{N}$. Then, the following hold.*

(a) *An alternative expression of ϕ^p_{NR} is*

$$\phi^p_{\text{NR}}(a,b) = a^{2k+1} - \frac{1}{2}\left((a-b)^{2k+1}+(a-b)^{2k}|a-b|\right).$$

(b) *The function ϕ^p_{NR} is continuously differentiable with*

$$\nabla \phi^p_{\text{NR}}(a,b) = p\begin{bmatrix} a^{p-1}-(a-b)^{p-2}(a-b)_+ \\ (a-b)^{p-2}(a-b)_+ \end{bmatrix}.$$

(c) *The function* ϕ^p_{NR} *is twice continuously differentiable with*

$$\nabla^2 \phi^p_{\text{NR}}(a,b) = p(p-1)\begin{bmatrix} a^{p-2}-(a-b)^{p-3}(a-b)_+ & (a-b)^{p-3}(a-b)_+ \\ (a-b)^{p-3}(a-b)_+ & -(a-b)^{p-3}(a-b)_+ \end{bmatrix}.$$

Proposition 3.3. *[11] Let* $\phi^p_{\text{S-NR}}$ *be defined as in (14) with* $p > 1$ *being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{\text{S-NR}}(a,b) > 0 \iff a > 0,\ b > 0.$

(b) $\phi^p_{\text{S-NR}}$ *is positive homogeneous of degree* p.

(c) $\phi^p_{\text{S-NR}}$ *is not Lipschitz continuous.*

(d) $\phi^p_{\text{S-NR}}$ *is not* α*-Hölder continuous for any* $\alpha \in (0,1]$.

(e) $\nabla_a \phi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \phi^p_{\text{S-NR}}(a,b) > 0$ *on* $\{(a,b)\,|\,a > b > 0\} \bigcup \{(a,b)\,|\,b > a > 0\}$.

(f) $\nabla_a \phi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \phi^p_{\text{S-NR}}(a,b) = 0$ *provided that* $\phi^p_{\text{S-NR}}(a,b) = 0$ *and* $(a,b) \neq (0,0)$.

Proposition 3.4. *[12] Let* $\phi^p_{\text{S-NR}}$ *be defined in (14) with* $p > 1$ *being a positive odd integer. Then,* $\phi^p_{\text{S-NR}}$ *is an NCP-function and is positive only on the first quadrant*

$$\Omega = \{(a,b)\,|\,a > 0,\ b > 0\}.$$

Proposition 3.5. *[12] Let* $\psi^p_{\text{S-NR}}$ *be defined in (15) with* $p > 1$ *being a positive odd integer. Then,* $\psi^p_{\text{S-NR}}$ *is an NCP-function and is positive on the set*

$$\Omega' = \{(a,b)\,|\,ab \neq 0\} \cup \{(a,b)\,|\,a < b = 0\} \cup \{(a,b)\,|\,0 = a > b\}.$$

Now, we elaborate more about the above three functions as below.

(i) For p being an even integer, all of above are not NCP-functions. A counterexample is given as below.

$$\begin{aligned} &\phi^2_{\text{NR}}(-1,-2) = (-1)^2 - (-2+1)^2_+ = 0. \\ &\phi^2_{\text{S-NR}}(-1,-2) = (-1)^2 - (-1+2)^2 = 0. \\ &\psi^2_{\text{S-NR}}(-1,-2) = (-1)^2(-2)^2 - (-1+2)^2(-2)^2 = 0. \end{aligned}$$

(ii) The above three functions are neither convex nor concave function. To see this, taking $p = 3$ and using the following argument verify the assertion.

$$\begin{aligned}
1 = \phi^3_{_{\rm NR}}(1,1) &< \frac{1}{2}\phi^3_{_{\rm NR}}(0,1) + \frac{1}{2}\ \phi^3_{_{\rm NR}}(2,1) = \frac{0}{2} + \frac{7}{2} = \frac{7}{2}.\\
1 = \phi^3_{_{\rm NR}}(1,1) &> \frac{1}{2}\phi^3_{_{\rm NR}}(1,-1) + \frac{1}{2}\ \phi^3_{_{\rm NR}}(1,3) = -\frac{7}{2} + \frac{1}{2} = -3.\\
1 = \phi^3_{_{\rm S-NR}}(1,1) &< \frac{1}{2}\phi^3_{_{\rm S-NR}}(0,0) + \frac{1}{2}\phi^3_{_{\rm S-NR}}(2,2) = \frac{0}{2} + \frac{8}{2} = 4.\\
1 = \phi^3_{_{\rm S-NR}}(1,1) &> \frac{1}{2}\phi^3_{_{\rm S-NR}}(2,0) + \frac{1}{2}\phi^3_{_{\rm S-NR}}(0,2) = \frac{0}{2} + \frac{0}{2} = 0.\\
1 = \psi^3_{_{\rm S-NR}}(1,1) &< \frac{1}{2}\psi^3_{_{\rm S-NR}}(0,0) + \frac{1}{2}\psi^3_{_{\rm S-NR}}(2,2) = \frac{0}{2} + \frac{64}{2} = 32.\\
1 = \psi^3_{_{\rm S-NR}}(1,1) &> \frac{1}{2}\psi^3_{_{\rm S-NR}}(2,0) + \frac{1}{2}\psi^3_{_{\rm S-NR}}(0,2) = \frac{0}{2} + \frac{0}{2} = 0.
\end{aligned}$$

Proposition 3.6. *[12] Let $\phi^p_{_{\rm S-NR}}$ be defined as in (14) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{_{\rm S-NR}}$ is*

$$\phi^p_{_{\rm S-NR}}(a,b) = \begin{cases} \phi^p_{_{\rm NR}}(a,b) & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ \phi^p_{_{\rm NR}}(b,a) & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\phi^p_{_{\rm S-NR}}$ is not differentiable. However, $\phi^p_{_{\rm S-NR}}$ is continuously differentiable on the set $\Omega := \{(a,b) \,|\, a \neq b\}$ with*

$$\nabla\phi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\,[\,a^{p-1} - (a-b)^{p-1},\ (a-b)^{p-1}\,]^T & \text{if} \quad a > b, \\ p\,[\,(b-a)^{p-1},\ b^{p-1} - (b-a)^{p-1}\,]^T & \text{if} \quad a < b. \end{cases}$$

In a more compact form,

$$\nabla\phi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\,[\,\phi^{p-1}_{_{\rm NR}}(a,b),\ (a-b)^{p-1}\,]^T & \text{if} \quad a > b, \\ p\,[\,(b-a)^{p-1},\ \phi^{p-1}_{_{\rm NR}}(b,a)\,]^T & \text{if} \quad a < b. \end{cases}$$

(c) *The function $\phi^p_{_{\rm S-NR}}$ is twice continuously differentiable on the set $\Omega = \{(a,b) \,|\, a \neq b\}$ with*

$$\nabla^2\phi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} a^{p-2} - (a-b)^{p-2} & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if} \quad a > b, \\ \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & b^{p-2} - (b-a)^{p-2} \end{bmatrix} & \text{if} \quad a < b. \end{cases}$$

In a more compact form,

$$\nabla^2 \phi^p_{\text{S-NR}}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} \phi^{p-2}_{\text{NR}}(a,b) & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if} \quad a > b, \\ \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & \phi^{p-2}_{\text{NR}}(b,a) \end{bmatrix} & \text{if} \quad a < b. \end{cases}$$

Proposition 3.7. *[11] Let $\psi^p_{\text{S-NR}}$ be defined as in (15) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) $\psi^p_{\text{S-NR}}(a,b) \geq 0$ *for all* $(a,b) \in \mathbb{R}^2$.

(b) $\psi^p_{\text{S-NR}}$ *is positive homogeneous of degree* $2p$.

(c) $\psi^p_{\text{S-NR}}$ *is locally Lipschitz continuous, but not Lipschitz continuous.*

(d) $\psi^p_{\text{S-NR}}$ *is not* α*-Hölder continuous for any* $\alpha \in (0,1]$.

(e) $\nabla_a \psi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \psi^p_{\text{S-NR}}(a,b) > 0$ *on the first quadrant* $\mathbb{R}^2_{++}$.

(f) $\psi^p_{\text{S-NR}}(a,b) = 0 \iff \nabla \psi^p_{\text{S-NR}}(a,b) = 0$. *In particular, we have* $\nabla_a \psi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \psi^p_{\text{S-NR}}(a,b) = 0$ *provided that* $\psi^p_{\text{S-NR}}(a,b) = 0$.

Proposition 3.8. *[12] Let $\psi^p_{\text{S-NR}}$ be defined as in (15) with $p > 1$ being a positive odd integer. Then, $\psi^p_{\text{S-NR}}$ is an NCP-function and is positive on the set*

$$\Omega' = \{(a,b) \,|\, ab \neq 0\} \cup \{(a,b) \,|\, a < b = 0\} \cup \{(a,b) \,|\, 0 = a > b\}.$$

Proposition 3.9. *[12] Let $\psi^p_{\text{S-NR}}$ be defined as in (15) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of* $\phi^p_{\text{S-NR}}$ *is*

$$\psi^p_{\text{S-NR}}(a,b) = \begin{cases} \phi^p_{\text{NR}}(a,b)b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ \phi^p_{\text{NR}}(b,a)a^p & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\psi^p_{_{\rm S-NR}}$ is continuously differentiable with*

$$\nabla\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\,[\,a^{p-1}b^p - (a-b)^{p-1}b^p,\ a^pb^{p-1} - (a-b)^pb^{p-1} + (a-b)^{p-1}b^p\,]^T & \text{if} \quad a>b, \\ p\,[\,a^{p-1}b^p, a^pb^{p-1}\,]^T = pa^{2p-1}[1\ ,1\,]^T & \text{if} \quad a=b, \\ p\,[\,a^{p-1}b^p - (b-a)^pa^{p-1} + (b-a)^{p-1}a^p,\ a^pb^{p-1} - (b-a)^{p-1}a^p\,]^T & \text{if} \quad a<b. \end{cases}$$

In a more compact form,

$$\nabla\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\,[\,\phi^{p-1}_{_{\rm NR}}(a,b)b^p,\ \phi^p_{_{\rm NR}}(a,b)b^{p-1} + (a-b)^{p-1}b^p\,]^T & \text{if} \quad a>b, \\ p\,[\,a^{2p-1}, a^{2p-1}\,]^T & \text{if} \quad a=b, \\ p\,[\,\phi^p_{_{\rm NR}}(b,a)a^{p-1} + (b-a)^{p-1}a^p,\ \phi^{p-1}_{_{\rm NR}}(b,a)a^p\,]^T & \text{if} \quad a<b, \end{cases}$$

(c) *The function $\psi^p_{_{\rm S-NR}}$ is twice continuously differentiable with*

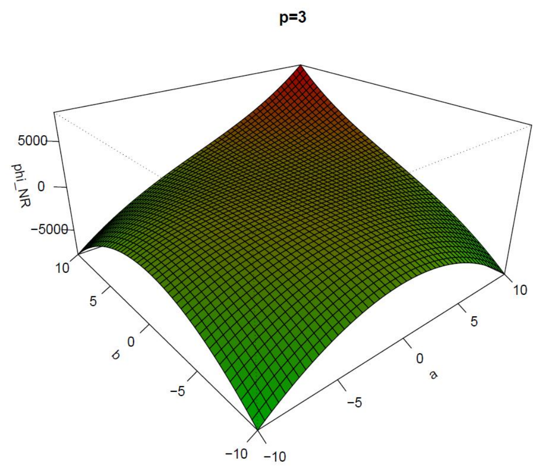
$$\nabla^2\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\begin{bmatrix} (p-1)[a^{p-2}-(a-b)^{p-2}]b^p & \begin{matrix}(p-1)(a-b)^{p-2}b^p \\ +p[a^{p-1}-(a-b)^{p-1}]b^{p-1}\end{matrix} \\ \begin{matrix}(p-1)(a-b)^{p-2}b^p \\ +p[a^{p-1}-(a-b)^{p-1}]b^{p-1}\end{matrix} & \begin{matrix}(p-1)[a^p-(a-b)^p]b^{p-2} \\ +2p(a-b)^{p-1}b^{p-1} \\ -(p-1)(a-b)^{p-2}b^p\end{matrix} \end{bmatrix} & \text{if} \quad a>b, \\ p\begin{bmatrix} (p-1)a^{p-2}b^p & pa^{p-1}b^{p-1} \\ pa^{p-1}b^{p-1} & (p-1)a^pb^{p-2} \end{bmatrix} & \text{if} \quad a=b, \\ p\begin{bmatrix} \begin{matrix}(p-1)[b^p-(b-a)^p]a^{p-2} \\ +2p(b-a)^{p-1}a^{p-1} \\ -(p-1)(b-a)^{p-2}a^p\end{matrix} & \begin{matrix}(p-1)(b-a)^{p-2}a^p \\ +p[b^{p-1}-(b-a)^{p-1}]a^{p-1}\end{matrix} \\ \begin{matrix}(p-1)(b-a)^{p-2}a^p \\ +p[b^{p-1}-(b-a)^{p-1}]a^{p-1}\end{matrix} & (p-1)[b^{p-2}-(b-a)^{p-2}]a^p \end{bmatrix} & \text{if} \quad a<b. \end{cases}$$

4 Main Results

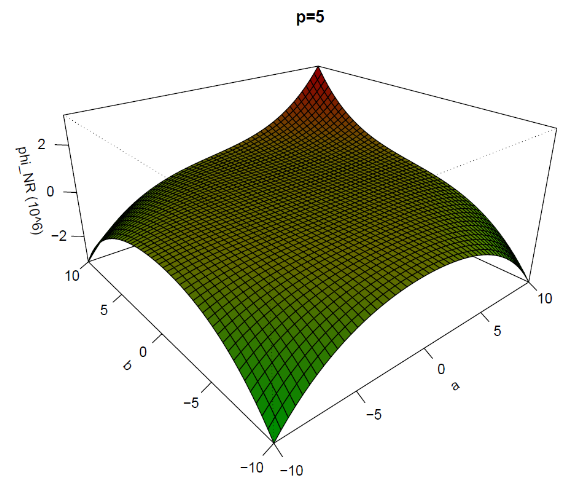
In this section, we focus on our new generalization of the Natural-Residual (NR) function

$$\tilde{\phi}^p_{_{\rm NR}}(a,b) = (a+b)^p - |a-b|^p \tag{16}$$

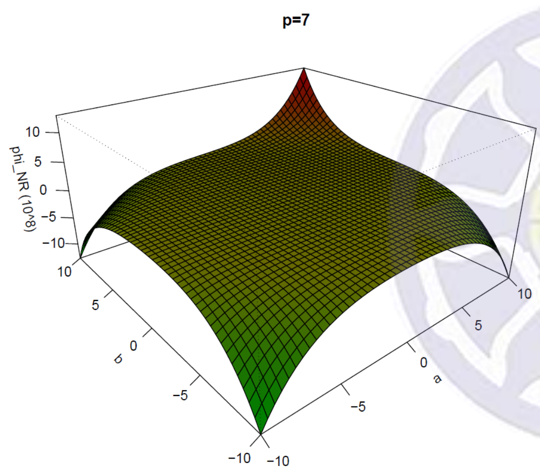
and check some basic definitions and then we discuss the differentiability.



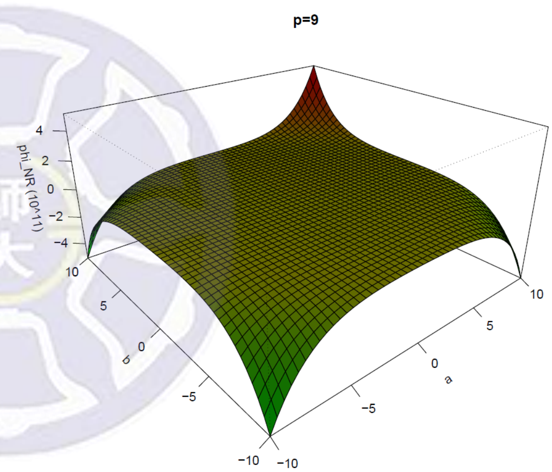
(a) $p = 3$



(b) $p = 5$



(c) $p = 7$



(d) $p = 9$

Figure 1: The surfaces of $\tilde{\phi}^p_{\text{NR}}(a, b) = (a + b)^p - |a - b|^p$ with $p = 3, 5, 7, 9$.

The surfaces of $\tilde{\phi}^p_{\text{NR}}(a, b) = (a + b)^p - |a - b|^p$ with $p = 3, 5, 7, 9$ are show in Figure 1. We can easily see that the surfaces of $\tilde{\phi}^p_{\text{NR}}(a, b)$ are symmetric to a=b.

Theorem 4.1. *Let $\tilde{\phi}^p_{\text{NR}}(a, b) = (a + b)^p - |a - b|^p$ with $p > 1$ being a positive odd integer. Then, $\tilde{\phi}^p_{\text{NR}}$ is an NCP-function.*

Proof. For any $a, b \in \mathbb{R}$, we note

$$\begin{aligned}\tilde{\phi}^p_{\text{NR}}(a,b) = 0 &\Longleftrightarrow (a+b)^p - |a-b|^p = 0 \\ &\Longleftrightarrow (a+b) - |a-b| = 0 \\ &\Longleftrightarrow 2\min\{a,b\} = 0 \\ &\Longleftrightarrow a \geq 0,\ b \geq 0\ ,\ ab = 0.\end{aligned}$$

Hence, $\tilde{\phi}^p_{\text{NR}}$ is an NCP-function. □

Lemma 4.1. *Let $p > 1$, then*

(a) *the function $f(t) = |t|^p$ is differentiable and $f'(t) = \text{sgn}(t)p|t|^{p-1}$.*

(b) *the function $f(t) = t^p|t|$ is differentiable and $f'(t) = (p+1)t^{p-1}|t|$.*

Proof. The proofs are straightforward which are omitted here. □

Theorem 4.2. *Let $\tilde{\phi}^p_{\text{NR}}(a,b) = (a+b)^p - |a-b|^p$ with $p > 1$ being a positive odd integer. The function $\tilde{\phi}^p_{\text{NR}}$ is continuously differentiable with*

$$\nabla\tilde{\phi}^p_{\text{NR}}(a,b) = p \begin{bmatrix} p(a+b)^{p-1} - (a-b)|a-b|^{p-2} \\ p(a+b)^{p-1} + (a-b)|a-b|^{p-2} \end{bmatrix}.$$

Proof. By Lemma 4.1, one can directly calculate the partial derivative of $\tilde{\phi}^p_{\text{NR}}$

$$\begin{aligned}\frac{\partial\tilde{\phi}^p_{\text{NR}}}{\partial a} &= p(a+b)^{p-1} - \text{sgn}(a-b)p|a-b|^{p-1}, \\ &= p(a+b)^{p-1} - (a-b)|a-b|^{p-2},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\tilde{\phi}^p_{\text{NR}}}{\partial b} &= p(a+b)^{p-1} + \text{sgn}(a-b)p|a-b|^{p-1}, \\ &= p(a+b)^{p-1} + (a-b)|a-b|^{p-2},\end{aligned}$$

Note that $\frac{\partial\tilde{\phi}^p_{\text{NR}}}{\partial a}$ and $\frac{\partial\tilde{\phi}^p_{\text{NR}}}{\partial b}$ are both continuous. Hence $\tilde{\phi}^p_{\text{NR}}$ is continuously differentiable and $\nabla\tilde{\phi}^p_{\text{NR}}(a,b) = p \begin{bmatrix} p(a+b)^{p-1} - (a-b)|a-b|^{p-2} \\ p(a+b)^{p-1} + (a-b)|a-b|^{p-2} \end{bmatrix}$. □

Theorem 4.3. *Let $\tilde{\phi}^p_{\rm NR}(a,b) = (a+b)^p - |a-b|^p$ with $p > 1$ being a positive odd integer. The function $\tilde{\phi}^p_{\rm NR}$ is twice continuously differentiable with*

$$\nabla^2 \tilde{\phi}^p_{\rm NR}(a,b) = \ p(p-1) \begin{bmatrix} (a+b)^{p-2} - |a-b|^{p-2} & (a+b)^{p-2} + |a-b|^{p-2} \\ (a+b)^{p-2} + |a-b|^{p-2} & (a+b)^{p-2} - |a-b|^{p-2} \end{bmatrix}.$$

Proof. By Theorem 4.2, we can directly calculate the second partial derivative of $\tilde{\phi}^p_{\rm NR}$:

$$\begin{aligned}
\frac{\partial^2 \tilde{\phi}^p_{\rm NR}}{\partial a \partial a} &= p(p-1)(a+b)^{p-2} - p(p-1)|a-b|^{p-2}, \\
\frac{\partial^2 \tilde{\phi}^p_{\rm NR}}{\partial b \partial b} &= p(p-1)(a+b)^{p-2} - p(p-1)|a-b|^{p-2}, \\
\frac{\partial^2 \tilde{\phi}^p_{\rm NR}}{\partial a \partial b} &= p(p-1)(a+b)^{p-2} + p(p-1)|a-b|^{p-2}, \\
\frac{\partial^2 \tilde{\phi}^p_{\rm NR}}{\partial b \partial a} &= p(p-1)(a+b)^{p-2} + p(p-1)|a-b|^{p-2}.
\end{aligned}$$

Also, every second partial derivatives of $\tilde{\phi}^p_{\rm NR}$ are continuous. Then, the function $\tilde{\phi}^p_{\rm NR}$ is twice continuously differentiable. □

Theorem 4.4. *The function $\tilde{\phi}^p_{\rm NR}(a,b) = (a+b)^p - |a-b|^p$ is positive homogeneous of degree p.*

Proof. Let $\omega = (a,b) \in \mathbb{R}^2$ and $\alpha \geq 0$ be given. Then

$$\begin{aligned}
\tilde{\phi}^p_{\rm NR}(\alpha\omega) &= (\alpha a + \alpha b)^p - |\alpha a - \alpha b|^p \\
&= [\alpha(a+b)]^p - |\alpha(a-b)|^p \\
&= \alpha^p (a+b)^p - |\alpha|^p |a-b|^p \\
&= \alpha^p [(a+b)^p - |a-b|^p] \\
&= \alpha^p \tilde{\phi}^p_{\rm NR}(\omega).
\end{aligned}$$

Thus, the function $\tilde{\phi}^p_{\rm NR}$ is positive homogeneous of degree p with $p > 1$ being odd positive integer. □

Theorem 4.5. *The function $\tilde{\phi}^p_{\rm NR}(a,b) = (a+b)^p - |a-b|^p$ with positive odd integer p is locally Lipschitz continuous, but not (globally) Lipschitz continuous.*

Proof. By theorem 4.2, since $\tilde{\phi}^p_{\rm NR}(a,b) = (a+b)^p - |a-b|^p$ is continuously differentiable, which implies locally Lipschitz continuity. Consider the restriction of $\tilde{\phi}^p_{\rm NR}(a,b)$ on the line $L = \big\{(a,b) | a = b > 0\big\}$. Note that for any $a > 0$, $\tilde{\phi}^p_{\rm NR}(a,b) = (2a)^p$, it suffices to show that $f(t) = ct^p,\ c > 1$ is not Lipschitz continuous. Indeed, for any $M > 0$, choosing $t = \max\{1, M\}$ and $t' = t + 1$. Then

$$\begin{aligned}\frac{|f(t) - f(t')|}{|t - t'|} &= c(t+1)^p - ct^p \\ &= c[(t+1)^p - t^p] \\ &= c[(t+1) - t][(t+1)^{p-1} \cdot 1 + \cdots + 1 \cdot t^{p-1}] \\ &> cpt^{p-1} \\ &> M.\end{aligned}$$

Hence, it follows that f is not Lipschitz continuous. □

Remark 4.1. *In fact, we can use mean value theorem to say $\tilde{\phi}^p_{\text{NR}}(a, b)$ is not globally Lipschitz continuous for any $p \in \mathbb{R}$. Indeed, for any $M > 0$, there exists a real number $\xi \in [t, t']$ such that*

$$\frac{|f(t) - f(t')|}{|t - t'|} = f'(\xi) = cp\xi^{p-1} \geq cpt^{p-1} > M.$$

Hence, $\tilde{\phi}^p_{\text{NR}}(a, b)$ is not globally Lipschitz continuous for any $p \in \mathbb{R}$.

Finally, we note that ϕ_{NR} and ϕ^p_{FB} have the following inequality (see [3])

$$(2 - 2^{1/p})|\phi_{\text{NR}}| \leq |\phi^p_{\text{FB}}| \leq (2 + 2^{1/p})|\phi_{\text{NR}}|.$$

Consequently, we try to find a similar inequality about $\phi^p_{\text{D-FB}}$ and $\tilde{\phi}^p_{\text{NR}}$. Let ϕ^p_{D-FB}, $\tilde{\phi}^p_{\text{NR}}$ be defined in (15) and (16), and ratio(R) be defined as below

$$R = \frac{|\phi^p_{\text{D-FB}}(a, b)|}{|\tilde{\phi}^p_{\text{NR}}(a, b)|}.$$

According to the following tables ,we make the following conjecture

$$\alpha_1|\tilde{\phi}^p_{\text{NR}}| \leq |\phi^p_{\text{D-FB}}| \leq \alpha_2|\tilde{\phi}^p_{\text{NR}}|$$

where $\alpha_1 = \frac{1}{(\sqrt{2})^p}$ and $\alpha_2 = 2$.

a	b	$\lvert\phi^p_{\text{D-FB}}\rvert$	$\lvert\tilde{\phi}^p_{\text{NR}}\rvert$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
21.01	43.21	153903.69	253956.17	0.61	2	0.35	1.39	0.25
24.44	-42.92	126785.33	311953.65	0.41	2	0.35	1.59	0.05
-9.21	27.92	18867.94	44657.36	0.42	2	0.35	1.58	0.07
-23.20	46.63	128380.42	327552.67	0.39	2	0.35	1.61	0.04
6.29	47.70	46026.57	86409.14	0.53	2	0.35	1.47	0.18
-27.78	-41.78	462788.88	339254.64	1.36	2	0.35	0.64	1.01
11.20	35.84	51159.45	89152.88	0.57	2	0.35	1.43	0.22
-43.11	-1.38	168341.73	160772.39	1.05	2	0.35	0.95	0.69
-9.41	-46.57	282661.43	226715.52	1.25	2	0.35	0.75	0.89
44.46	36.39	338844.16	527971.72	0.64	2	0.35	1.36	0.29
-26.79	11.83	28466.45	60958.74	0.47	2	0.35	1.53	0.11
-32.35	-11.14	122302.10	91793.25	1.33	2	0.35	0.67	0.98
-26.86	11.05	28438.68	58405.57	0.49	2	0.35	1.51	0.13
-19.49	8.78	10992.35	23821.50	0.46	2	0.35	1.53	0.11
-32.12	45.54	170677.68	466023.83	0.37	2	0.35	1.63	0.01
0.46	-8.73	1232.79	1339.99	0.92	2	0.35	1.08	0.57
32.76	33.25	185850.15	287507.82	0.65	2	0.35	1.35	0.29
26.71	28.93	111245.35	172301.92	0.65	2	0.35	1.35	0.29
5.51	-35.73	74858.05	97729.85	0.77	2	0.35	1.23	0.41
-12.83	23.97	18719.61	48470.72	0.39	2	0.35	1.61	0.03
7.28	-9.61	1764.52	4828.82	0.37	2	0.35	1.63	0.01
-33.41	-2.40	83500.35	75742.54	1.10	2	0.35	0.90	0.75
9.71	3.41	1168.32	2008.19	0.58	2	0.35	1.42	0.23
21.31	6.20	9882.42	17361.07	0.57	2	0.35	1.43	0.22
37.65	22.75	135258.25	217092.28	0.62	2	0.35	1.38	0.27
6.38	9.19	2373.00	3750.31	0.63	2	0.35	1.37	0.28
11.67	-10.10	3672.25	10313.52	0.36	2	0.35	1.64	0.01
-22.93	19.55	27381.57	76647.08	0.36	2	0.35	1.64	0.01
24.70	9.90	22571.10	38167.16	0.59	2	0.35	1.41	0.24
-42.30	7.06	122558.32	163942.554	0.75	2	0.35	1.25	0.39

Table 1: $p = 3$

a	b	$\lvert\phi^p_{\text{D-FB}}\rvert$	$\lvert\tilde{\phi}^p_{\text{NR}}\rvert$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
12.48	-25.74	1.96E+7	8.20E+7	0.24	2	0.18	1.76	0.06
-19.26	44.03	2.47E+8	1.00E+9	0.25	2	0.18	1.75	0.07
-35.28	22.96	1.32E+8	6.70E+8	0.20	2	0.18	1.80	0.02
12.29	6.61	1.88E+6	2.41E+6	0.78	2	0.18	1.22	0.61
4.86	13.55	1.50E+6	2.06E+6	0.72	2	0.18	1.28	0.55

Table 2: $p = 5$

a	b	$\lvert\phi^p_{\mathrm{D-FB}}\rvert$	$\lvert\tilde{\phi}^p_{\mathrm{NR}}\rvert$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
42.12	-31.99	4.14E+8	2.24E+9	0.19	2	0.18	1.81	0.01
-36.82	47.44	7.80E+8	4.25E+9	0.18	2	0.18	1.82	0.01
47.63	35.60	3.25E+9	4.00E+9	0.81	2	0.18	1.19	0.64
-7.22	-18.26	1.36E+7	1.09E+7	1.25	2	0.18	0.75	1.08
24.43	31.73	4.55E+8	5.58E+8	0.82	2	0.18	1.18	0.64
-42.96	41.53	7.61E+8	4.30E+9	0.18	2	0.18	1.82	0.0001
32.06	12.64	1.30E+8	1.76E+8	0.74	2	0.18	1.26	0.56
6.86	-27.26	2.11E+7	4.98E+7	0.42	2	0.18	1.58	0.25
4.98	40.10	7.84E+7	1.33E+8	0.59	2	0.18	1.41	0.41
0.85	-13.45	760619.89	913935.23	0.83	2	0.18	1.17	0.66
27.81	35.93	8.58E+8	1.05E+9	0.82	2	0.18	1.18	0.64
-36.51	-13.03	3.86E+8	3.06E+8	1.26	2	0.18	0.74	1.09
-14.92	-21.03	7.15E+7	6.01E+7	1.19	2	0.18	0.81	1.01
27.87	-30.32	1.19E+8	6.67E+8	0.18	2	0.18	1.82	0.0007
41.14	-33.73	4.26E+8	2.35E+9	0.18	2	0.18	1.82	0.004
9.80	-12.38	980799.73	5365252.50	0.18	2	0.18	1.82	0.006
-8.58	-21.15	2.94E+7	2.36E+7	1.25	2	0.18	0.75	1.07
23.31	28.23	2.98E+8	3.64E+8	0.82	2	0.18	1.18	0.64
15.11	-40.31	1.58E+8	5.33E+8	0.30	2	0.18	1.70	0.12
31.41	-4.43	1.78E+7	4.48E+7	0.40	2	0.18	1.60	0.22
-8.98	48.17	1.90E+8	5.17E+8	0.37	2	0.18	1.63	0.19
17.26	-40.79	1.78E+8	6.67E+8	0.26	2	0.18	1.73	0.09
39.91	-17.86	1.55E+8	6.38E+8	0.24	2	0.18	1.76	0.07
7.34	-29.46	3.11E+7	7.28E+7	0.43	2	0.18	1.57	0.25
-17.84	42.59	2.01E+8	7.96E+8	0.25	2	0.18	1.75	0.08
-48.05	5.10	4.09E+8	5.70E+8	0.72	2	0.18	1.28	0.54

Table 3: $p = 5$

a	b	$\lvert\phi^p_{\mathrm{D-FB}}\rvert$	$\lvert\tilde{\phi}^p_{\mathrm{NR}}\rvert$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
-5.55	-13.90	1.22E+9	1.06E+9	1.16	2	0.088	0.84	1.07
9.27	-28.15	2.10E+10	1.04E+11	0.20	2	0.088	1.80	0.11
-20.96	17.32	1.10E+10	1.20E+11	0.09	2	0.088	1.91	0.0028
-31.65	21.97	1.26E+11	1.27E+12	0.098	2	0.088	1.90	0.01
-47.21	-19.93	7.08E+12	6.16E+12	1.15	2	0.088	0.85	1.06

Table 4: $p = 7$

a	b	$\|\phi^p_{\mathrm{D-FB}}\|$	$\|\tilde{\phi}^p_{\mathrm{NR}}\|$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
-26.60	30.34	1.74E+11	1.94E+12	0.089	2	0.088	1.91	0.0013
-14.29	-18.94	4.90E+10	4.48E+10	1.09	2	0.088	0.91	1.006
-45.24	2.26	6.62E+11	8.16E+11	0.81	2	0.088	1.19	0.72
27.62	3.84	1.74E+10	2.62E+10	0.66	2	0.088	1.34	0.58
39.03	-34.92	1.08E+12	1.21E+13	0.09	2	0.088	1.91	0.0009
-19.28	-15.27	6.43E+10	5.88E+10	1.09	2	0.088	0.91	1.004
10.96	-6.23	5.07E+10	4.45E+10	0.11	2	0.088	1.89	0.03
35.26	-13.19	1.05E+11	6.24E+11	0.17	2	0.088	1.83	0.08
-18.46	19.11	9.34E+10	1.06E+11	0.09	2	0.088	1.91	9.39
-20.53	23.86	3.06E+10	3.39E+11	0.09	2	0.088	1.91	0.0017
-45.47	-36.76	2.78E+13	2.54E+13	1.09	2	0.088	0.91	1.0035
-27.48	-43.20	9.73E+12	8.81E+12	1.10	2	0.088	0.895	1.02
33.46	44.03	1.52E+13	1.68E+13	0.91	2	0.088	1.09	0.82
-26.27	35.12	3.12E+11	3.28E+12	0.09	2	0.088	1.90	0.0066
48.48	-39.69	3.79E+12	4.14E+13	0.09	2	0.088	1.90	0.0031
-0.30	-39.08	2.86E+11	2.79E+11	1.03	2	0.088	0.97	0.94
41.34	37.66	1.75E+13	1.92E+13	0.91	2	0.088	1.09	0.82
32.74	-28.02	2.76E+11	3.06E+12	0.09	2	0.088	1.91	0.0018
-33.71	-38.00	1.06E+13	9.75E+12	1.09	2	0.088	0.91	1.001
23.48	-16.65	1.64E+10	1.68E+11	0.097	2	0.088	1.902	0.0092
34.27	-21.98	1.85E+11	1.78E+12	0.104	2	0.088	1.895	0.015
2.64	-21.76	3.37E+10	6.07E+10	0.55	2	0.088	1.45	0.47
-15.40	45.11	5.39E+11	2.95E+12	0.18	2	0.088	1.82	0.094
15.56	-31.09	6.16E+10	4.81E+11	0.19	2	0.088	1.87	0.039
-31.57	16.33	7.18E+10	5.79E+11	0.124	2	0.088	1.876	0.036
-40.66	-32.70	1.25E+13	1.14E+13	1.09	2	0.088	0.907	1.0036

Table 5: $p = 7$

a	b	$\|\phi^p_{\mathrm{D-FB}}\|$	$\|\tilde{\phi}^p_{\mathrm{NR}}\|$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
33.46	-10.67	7.97E+13	6.34E+14	0.13	2	0.044	1.87	0.08
-45.47	33.58	5.89E+15	1.205E+17	0.049	2	0.044	1.951	0.0046
20.58	-47.72	2.77E+15	3.22E+16	0.09	2	0.044	1.91	0.04
-4.72	-30.99	1.24E+14	1.003E+14	1.23	2	0.044	0.77	1.19
-22.28	41.52	1.145E+15	1.752E+16	0.065	2	0.044	1.935	0.021

Table 6: $p = 9$

a	b	$\lvert\phi^p_{\mathrm{D-FB}}\rvert$	$\lvert\widetilde{\phi}^p_{\mathrm{NR}}\rvert$	R	α_2	α_1	$\alpha_2 - R$	$R - \alpha_1$
-32.40	23.22	2.54E+14	5.09E+15	0.049	2	0.044	1.950	0.0056
-12.22	34.70	1.22E+14	1.10E+15	0.11	2	0.044	1.889	0.067
-38.61	-37.52	8.97E+16	8.59E+16	1.044	2	0.044	0.955	1.00004
-12.37	35.24	1.39E+14	1.25E+15	0.111	2	0.044	1.888	0.067
-14.91	-33.30	1.52E+15	1.41E+15	1.08	2	0.044	0.92	1.037
-34.45	-42.95	1.04E+17	9.97E+16	1.046	2	0.044	0.953	1.0024
45.54	-37.27	8.46E+15	1.83E+17	0.046	2	0.044	1.954	0.002
-0.87	25.66	1.31E+12	2.97E+12	0.44	2	0.044	1.56	0.397
-0.76	-16.78	2.63E+11	2.27E+11	1.163	2	0.044	0.837	1.12
-28.85	-48.60	1.06E+17	1.003E+17	1.059	2	0.044	0.941	1.014
26.23	-11.65	1.32E+13	1.61E+14	0.08	2	0.044	1.92	0.037
21.07	44.384	2.04E+16	2.20E+16	0.92	2	0.044	1.08	0.88
-17.76	7.72	3.85E+11	4.54E+12	0.08	2	0.044	1.92	0.04
-44.16	-5.18	2.41E+15	1.94E+15	1.24	2	0.044	0.76	1.20
-48.33	13.55	2.096E+15	1.338E+16	0.157	2	0.044	1.843	0.11
-17.69	41.23	7.35E+14	8.56E+15	0.086	2	0.044	1.914	0.042
11.20	-48.29	1.94E+15	9.47E+15	0.205	2	0.044	1.795	0.161
20.30	-8.22	1.15E+12	1.25E+13	0.092	2	0.044	1.907	0.048
18.54	-26.29	3.69E+13	7.31E+14	0.05	2	0.044	1.95	0.006
-5.26	7.35	4.04E+8	8.09E+9	0.049	2	0.044	1.95	0.005
49.99	-35.79	1.26E+16	2.51E+17	0.049	2	0.044	1.950	0.005
20.92	-42.77	1.26E+15	1.73E+16	0.073	2	0.044	1.927	0.028
-31.41	0.11	5.86E+13	5.96E+13	0.983	2	0.044	1.016	0.939
-32.03	-8.45	3.40E+14	2.94E+14	1.15	2	0.044	0.84	1.11
-9.69	25.81	9.12E+12	8.93E+13	0.102	2	0.044	1.898	0.058
22.66	36.00	7.76E+15	8.21E+15	0.945	2	0.044	1.055	0.9003

Table 7: $p = 9$

References

[1] J.-S. Chen, *The semismooth-related properties of a merit function and a descent method for the nonlinear complementarity problem*, Journal of Global Optimization, 36(2006), 565-580.

[2] J.-S. Chen, *On some NCP-functions based on the generalized Fischer-Burmeister function*, Asia-Pacific Journal of Operational Research, 24(2007), 401-420.

[3] J.-S. Chen, H.-T. Gao and S. Pan, *A R-linearly convergent derivative-free algorithm for the NCPs based on the generalized Fischer-Burmeister merit function*, Journal of Computational and Applied Mathematics, 232(2009), 455-471.

[4] J.-S. Chen, Z.-H. Huang, and C.-Y. She, *A new class of penalized NCP-functions and its properties*, Computational Optimization and Applications, 50(2011), 49-73.

[5] J.-S. Chen, C.-H. Ko, and S.-H. Pan, *A neural network based on generalized Fischer-Burmeister function for nonlinear complementarity problems*, Information Sciences, 180(2010), 697-711.

[6] J.-S. Chen, C.-H. Ko, and X.-R. Wu, *What is the generalization of natural residual function for NCP*, to appear in Pacific Journal of Optimization, January, 2016, 19-27.

[7] J.-S. Chen and S. Pan, *A family of NCP-functions and a descent method for the nonlinear complementarity problem*, Computational Optimization and Applications, 40(2008), 389-404.

[8] J.-S. Chen, S.-H. Pan, and T.-C. Lin, *A smoothing Newton method based on the generalized Fischer-Burmeister function for MCPs*, Nonlinear Analysis: Theory, Methods and Applications, 72(2010), 3739-3758.

[9] J.-S. Chen, S.-H. Pan, and C.-Y. Yang, *Numerical comparison of two effective methods for mixed complementarity problems*, Journal of Computational and Applied Mathematics, 234(2010), 667-683.

[10] J.-S. Chen, D. F. Sun, and J. Sun, *The SC^1 property of the squared norm of the SOC Fischer-Burmeister function*, Operations Research Letters, vol. 36(2008), 385-392.

[11] C.-H. Huang, K.-J. Weng, J.-S. Chen, H.-W. Chu , and M.-Y. Li, *On four discrete-type families of NCP-functions*,to appear in Journal of Nonlinear and Convex Analysis, 2018.

[12] Y.-L. Chang, J.-S. Chen, C.-Y. Yang, *Symmetrization of generalized natural residual function for NCP*, Operations Research Letters, 43(2015), 354-358.

[13] F. Facchinei and J. Soares, *A new merit function for nonlinear complementarity problems and a related algorithm*, SIAM Journal on Optimization, 7(1997), 225-247.

[14] A. Fischer, *A special Newton-type optimization methods*, Optimization, 24(1992), 269-284.

[15] A. Fischer, *Solution of the monotone complementarity problem with locally Lipschitzian functions*, Mathematical Programming, 76(1997), 513-532.

[16] A. Galántai, *Properties and construction of NCP functions*, Computational Optimization and Applications, 52(2012), 805-824.

[17] C. Geiger and C. Kanzow, *On the resolution of monotone complementarity problems*, Computational Optimization and Applications, 5(1996), 155-173.

[18] P. T. Harker and J.-S. Pang, *Finite dimensional variational inequality and nonlinear complementarity problem: a survey of theory, algorithms and applications*, Mathematical Programming, 48(1990), 161-220.

[19] C. Kanzow, *Nonlinear complementarity as unconstrained optimization*, Journal of Optimization Theory and Applications, 88(1996), 139-155.

[20] C. Kanzow, N. Yamashita, and M. Fukushima, *New NCP-functions and their properties*, Journal of Optimization Theory and Applications, 94(1997), 115-135.

[21] O. L. Mangasarian, *Equivalence of the Complementarity Problem to a System of Nonlinear Equations*, SIAM Journal on Applied Mathematics, 31(1976), 89-92.

[22] J.-S. Pang, *Newton's Method for B-differentiable Equations*, Mathematics of Operations Research, 15(1990), 311-341.

[23] D. Sun and L. Qi, *On NCP-functions*, Computational Optimization and Applications, 13(1999), 201-220.

[24] H.-Y. Tsai and J.-S. Chen, *Geometric views of the generalized Fischer-Burmeister function and its induced merit function*, Applied Mathematics and Computation, 237(2014), 31-59.

[25] N. Yamashita and M. Fukushima, *On stationary points of the implict Lagrangian for nonlinear complementarity problems*, Journal of Optimization Theory and Applications, 84(1995), 653-663.

[26] M. C. Ferris and J-S Pang, *Engineering and economic applications of complementarity problems*, SIAM Review, 39(1997), 669-713.

[27] J-S Pang, *Complementarity problems. Handbook of Global Optimization, R Horst and P Pardalos (eds.)*, MA: Kluwer Academic Publishers, 271-338.