3 Existence of a minimizer of (1.4)

From the above discussion, (1.4) is equivalent to finding $Q_0 \subset \Omega \times \mathbf{R}$ such that

$$E(u_{Q_0}, Q_0) = \min_{Q \subset \Omega \times \mathbf{R}} E(u_Q, Q),$$

where u_Q is the solution of (2.1) described in § 2. Recall that the vertices A and B of Q defined in § 2 satisfy the following property

$$x_A < x_B,\ y_A \le y_B,\ (x_A - x_B)^2 + (y_A - y_B)^2 = 4r^2. \tag{3.1}$$

Also, it is easy to check that

$$x_C = x_B - (y_A - y_B),\ y_C = y_B + (x_A - x_B). \tag{3.2}$$

Note that $Q \subset \Omega \times \mathbf{R}$ means $A, B, C \in \Omega \times \mathbf{R}$, or equivalently,

$$x_A, x_B, x_C \in (0, 1). \tag{3.3}$$

Hence we have

$$E(u_Q, Q) = \int_0^1 \sqrt{1 + u_Q'(x)^2}dx - Gy_p^Q := g(x_A, y_A, x_B, y_B) \tag{3.4}$$

for $(x_A, y_A, x_B, y_B) \in \mathbf{R}^4$ such that (3.1) and (3.3) hold. We would like to get an expression for the function g defined in (3.4). For convenience, for $(x_A, y_A, x_B, y_B) \in \mathbf{R}^4$, we shall denote $g(x_A, y_A, x_B, y_B)$ by $E_i(x_A, y_A, x_B, y_B)$ if Q is in the **Case i**, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$.

Recall $y_p^Q = (y_A + y_C)/2 = (y_A + y_B + x_A - x_B)/2$. For different Q, the minimum energy can be computed as follows.

$$E_1(x_A, y_A, x_B, y_B) = 1 - \frac{G}{2}(y_A + y_B + x_A - x_B). \tag{3.5}$$

$$\begin{aligned} E_2(x_A, y_A, x_B, y_B) &= \int_0^{x_B} \sqrt{1 + (\frac{y_B}{x_B})^2}dx + \int_{x_B}^1 \sqrt{1 + (\frac{y_B}{1 - x_B})^2}dx \\ &\quad -\frac{G}{2}(y_A + y_B + x_A - x_B) \\ &= x_B\sqrt{1 + (\frac{y_B}{x_B})^2} + (1 - x_B)\sqrt{1 + (\frac{y_B}{1 - x_B})^2} \\ &\quad -\frac{G}{2}(y_A + y_B + x_A - x_B). \end{aligned} \tag{3.6}$$

$$E_3(x_A, y_A, x_B, y_B) = \int_0^{x_A} \sqrt{1 + (\frac{y_A}{x_A})^2}dx + \int_{x_A}^{x_B} \sqrt{1 + (\frac{y_B - y_A}{x_B - x_A})^2}dx$$

$$
\begin{aligned}
& +\int_{x_B}^{1}\sqrt{1+(\frac{y_B}{1-x_B})^2}dx-\frac{G}{2}(y_A+y_B+x_A-x_B) \\
&= x_A\sqrt{1+(\frac{y_A}{x_A})^2}+(x_B-x_A)\sqrt{1+(\frac{y_B-y_A}{x_B-x_A})^2} \\
& +(1-x_B)\sqrt{1+(\frac{y_B}{1-x_B})^2}-\frac{G}{2}(y_A+y_B+x_A-x_B). \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
E_4(x_A,y_A,x_B,y_B) &= \int_0^{x_B}\sqrt{1+(\frac{y_B}{x_B})^2}dx+\int_{x_B}^{x_C}\sqrt{1+(\frac{y_C-y_B}{x_B-x_C})^2}dx \\
& +\int_{x_C}^{1}\sqrt{1+(\frac{y_C}{1-x_C})^2}dx-\frac{G}{2}(y_A+y_B+x_A-x_B) \\
&= x_B\sqrt{1+(\frac{y_B}{x_B})^2}+(y_B-y_A)\sqrt{1+(\frac{x_B-x_A}{y_B-y_A})^2} \\
& +(1-x_C)\sqrt{1+(\frac{y_C}{1-x_C})^2}-\frac{G}{2}(y_A+y_B+x_A-x_B). \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
E_5(x_A,y_A,x_B,y_B) &= \int_0^{x_A}\sqrt{1+(\frac{y_A}{x_A})^2}dx+\int_{x_A}^{x_B}\sqrt{1+(\frac{y_B-y_A}{x_B-x_A})^2}dx \\
& +\int_{x_B}^{x_C}\sqrt{1+(\frac{y_C-y_B}{x_C-x_B})^2}dx+\int_{x_C}^{1}\sqrt{1+(\frac{y_C}{1-x_C})^2}dx \\
& -\frac{G}{2}(y_A+y_B+x_A-x_B) \\
&= x_A\sqrt{1+(\frac{y_A}{x_A})^2}+(x_B-x_A)\sqrt{1+(\frac{y_B-y_A}{x_B-x_A})^2} \\
& +(y_B-y_A)\sqrt{1+(\frac{x_B-x_A}{y_B-y_A})^2}+(1-x_C)\sqrt{1+(\frac{y_C}{1-x_C})^2} \\
& -\frac{G}{2}(y_A+y_B+x_A-x_B). \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
E_6(x_A,y_A,x_B,y_B) &= \int_0^{x_A}\sqrt{1+(\frac{y_A}{x_A})^2}dx+(x_B-x_A) \\
& +\int_{x_B}^{1}\sqrt{1+(\frac{y_B}{1-x_B})^2}dx-\frac{G}{2}(2y_A+x_A-x_B) \\
&= x_A\sqrt{1+(\frac{y_A}{x_A})^2}+(x_B-x_A)+(1-x_B)\sqrt{1+(\frac{y_B}{1-x_B})^2} \\
& -\frac{G}{2}(2y_A+x_A-x_B). \qquad (3.10)
\end{aligned}
$$

Let

$$
\mathcal{O} = \{(x_A,y_A,x_B,y_B)\in\mathbf{R}^4 \mid x_A,x_B,x_C\in(0,1), x_A<x_B, y_A<y_B\}, \qquad (3.11)
$$

$$\widehat{\mathcal{O}} = \{(x_A, y_A, x_B, y_B) \in \mathbf{R}^4 \mid x_A, x_B, x_C \in (0,1), x_A < x_B, y_A \le y_B\}, \quad (3.12)$$

and $h(x_A, y_A, x_B, y_B) = (x_A - x_B)^2 + (y_A - y_B)^2 - 4r^2$ for all $(x_A, y_A, x_B, y_B) \in \mathbf{R}^4$. We define two surfaces

$$\mathcal{S} = \{(x_A, y_A, x_B, y_B) \in \widehat{\mathcal{O}} \mid h(x_A, y_A, x_B, y_B) = 0\} \quad (3.13)$$

and

$$\overline{\mathcal{S}} = \{(x_A, y_A, x_B, y_B) \in \overline{\mathcal{O}} \mid h(x_A, y_A, x_B, y_B) = 0\}. \quad (3.14)$$

Then (1.4) is equivalent to finding

$$\min_{(x_A,\, y_A,\, x_B,\, y_B) \in \mathcal{S}} g(x_A, y_A, x_B, y_B), \quad (3.15)$$

i.e., we transform (1.4) into a minimization problem in $\mathbf{R}^4$ over $\mathcal{S} \subset \mathbf{R}^4$.

To study this minimization problem, we first extend g continuously on $x_A = x_B$, $x_A = 0$, $x_B = 1$, and $x_C = 1$ respectively as follows. We extend g on $x_A = x_B$ by

$$g(x_A, y_A, x_A, y_B) = 1 - \frac{G}{2}(y_A + y_B) \quad (3.16)$$

if $y_B \le 0$ and by

$$\begin{aligned} g(x_A, y_A, x_A, y_B) &= x_A\sqrt{1 + (\frac{y_B}{x_A})^2} + y_B - y_A \\ &\quad + (1 - x_A - y_B + y_A)\sqrt{1 + (\frac{y_B}{1 - x_A - y_B + y_A})^2} \\ &\quad - \frac{G}{2}(y_A + y_B) \end{aligned} \quad (3.17)$$

if $y_B > 0$ and $x_A > 0$. We extend g at $(0, y_A, 0, y_B)$ by

$$g(0, y_A, 0, y_B) = 2y_B - y_A + (1 - y_B + y_A)\sqrt{1 + (\frac{y_B}{1 - y_B + y_A})^2} - \frac{G}{2}(y_A + y_B) \quad (3.18)$$

if $y_B > 0$. We extend g at $(0, y_A, x_B, y_B)$ with $x_B > 0$ by

$$g(0, y_A, x_B, y_B) = 1 - \frac{G}{2}(y_A + y_B - x_B) \quad (3.19)$$

if $y_B \le 0$. If $y_B > 0$, then we extend g at $(0, y_A, x_B, y_B)$ with $x_B > 0$ by

$$g(0, y_A, x_B, y_B) = x_B\sqrt{1 + (\frac{y_B}{x_B})^2} + (1 - x_B)\sqrt{1 + (\frac{y_B}{1 - x_B})^2} - \frac{G}{2}(y_A + y_B - x_B). \quad (3.20)$$

if $m_{\overline{BO}} < m_{\overline{AB}}$ and $m_{\overline{BC}} < m_{\overline{BD}}$; by

$$\begin{aligned} g(0, y_A, x_B, y_B) &= y_A + x_B\sqrt{1 + (\frac{y_B - y_A}{x_B})^2} + (1 - x_B)\sqrt{1 + (\frac{y_B}{1 - x_B})^2} \\ &\quad -\frac{G}{2}(y_A + y_B - x_B), \end{aligned} \tag{3.21}$$

if $m_{\overline{BO}} \geq m_{\overline{AB}}$ and $m_{\overline{BC}} < m_{\overline{BD}}$; by

$$\begin{aligned} g(0, y_A, x_B, y_B) &= x_B\sqrt{1 + (\frac{y_B}{x_B})^2} + (y_B - y_A)\sqrt{1 + (\frac{x_B}{y_B - y_A})^2} \\ &\quad +(1 - x_B - y_B + y_A)\sqrt{1 + (\frac{y_B - x_B}{1 - x_B - y_B + y_A})^2} \\ &\quad -\frac{G}{2}(y_A + y_B - x_B), \end{aligned} \tag{3.22}$$

if $m_{\overline{BO}} < m_{\overline{AB}}$ and $m_{\overline{BC}} \geq m_{\overline{BD}}$; by

$$\begin{aligned} g(0, y_A, x_B, y_B) &= x_B\sqrt{1 + (\frac{y_B - y_A}{x_B})^2} + (y_B - y_A)\sqrt{1 + (\frac{x_B}{y_B - y_A})^2} \\ &\quad +(1 - x_B - y_B + y_A)\sqrt{1 + (\frac{y_B - x_B}{1 - x_B - y_B + y_A})^2} \\ &\quad -\frac{G}{2}(y_A + y_B - x_B) + y_A, \end{aligned} \tag{3.23}$$

if $m_{\overline{BO}} \geq m_{\overline{AB}}$ and $m_{\overline{BC}} \geq m_{\overline{BD}}$; and by

$$g(0, y_A, x_B, y_B) = y_B + x_B + (1 - x_B)\sqrt{1 + (\frac{y_B}{1 - x_B})^2} - \frac{G}{2}(2y_B - x_B), \tag{3.24}$$

if $y_A = y_B$.

If $x_B = 1$, then $y_B = y_A$. (Otherwise, $x_C = x_B + y_B - y_A > 1$, a contradiction.) In this case, we extend g at $(x_A, y_A, 1, y_B)$ by

$$g(x_A, y_A, 1, y_B) = 1 - \frac{G}{2}(2y_A + x_A - 1) \tag{3.25}$$

if $y_B \leq 0$ and by

$$g(x_A, y_A, 1, y_B) = x_A\sqrt{1 + (\frac{y_A}{x_A})^2} + 1 - x_A + y_B - \frac{G}{2}(2y_A + x_A - 1) \tag{3.26}$$

if $y_B > 0$.

For $x_C = 1$, we extend g at (x_A, y_A, x_B, y_B) by (3.5) if $y_B \leq 0$. If $y_B > 0$, we extend

g at (x_A, y_A, x_B, y_B) with $x_C = 1$ by (3.6), if $m_{\overline{BO}} < m_{\overline{AB}}$ and $m_{\overline{BC}} < m_{\overline{BD}}$; by (3.7), if $m_{\overline{BO}} \geq m_{\overline{AB}}$ and $m_{\overline{BC}} < m_{\overline{BD}}$; by

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= x_B\sqrt{1+(\frac{y_B}{x_B})^2} + (y_B - y_A)\sqrt{1+(\frac{x_B - x_A}{y_B - y_A})^2} \\ &\quad + y_B + x_A - x_B - \frac{G}{2}(y_A + y_B + x_A - x_B), \end{aligned} \tag{3.27}$$

if $m_{\overline{BO}} < m_{\overline{AB}}$ and $m_{\overline{BC}} \geq m_{\overline{BD}}$; by

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= x_A\sqrt{1+(\frac{y_A}{x_A})^2} + (x_B - x_A)\sqrt{1+(\frac{y_B - y_A}{x_B - x_A})^2} \\ &\quad + (y_B - y_A)\sqrt{1+(\frac{x_B - x_A}{y_B - y_A})^2} \\ &\quad + y_B + x_A - x_B - \frac{G}{2}(y_A + y_B + x_A - x_B), \end{aligned} \tag{3.28}$$

if $m_{\overline{BO}} \geq m_{\overline{AB}}$ and $m_{\overline{BC}} \geq m_{\overline{BD}}$; and by

$$g(x_A, y_A, x_B, y_B) = x_A\sqrt{1+(\frac{y_B}{x_A})^2} + x_B - x_A + y_B - \frac{G}{2}(2y_B + x_A - x_B), \tag{3.29}$$

if $y_A = y_B$. Note that $x_B = x_C = 1$ in the last case.

Hence g is well-defined on $\overline{\mathcal{O}}$ and the following theorem can be derived easily.

Theorem 3.1 *The energy function g defined above is continuous on*

$$\overline{\mathcal{O}} = \{(x_A, y_A, x_B, y_B) \in \mathbf{R}^4 \mid x_A, x_B, x_C \in [0,1], x_A \leq x_B,\ y_A \leq y_B\}.$$

Furthermore, we have the following result.

Theorem 3.2 *The function g is C^1 on $\mathcal{O}$.*

Proof: Let

$$\begin{aligned} \mathcal{O}_1 &= \{(x_A, y_A, x_B, y_B) \in \mathcal{O} \mid y_B < 0\}, \\ \mathcal{O}_2 &= \{(x_A, y_A, x_B, y_B) \in \mathcal{O} \mid y_B > 0, \frac{y_B}{x_B} < \frac{y_B - y_A}{x_B - x_A}, \frac{y_B}{1 - x_B} < \frac{x_B - x_A}{y_B - y_A}\}, \\ \mathcal{O}_3 &= \{(x_A, y_A, x_B, y_B) \in \mathcal{O} \mid y_B > 0, \frac{y_B}{x_B} > \frac{y_B - y_A}{x_B - x_A}, \frac{y_B}{1 - x_B} < \frac{x_B - x_A}{y_B - y_A}\}, \\ \mathcal{O}_4 &= \{(x_A, y_A, x_B, y_B) \in \mathcal{O} \mid y_B > 0, \frac{y_B}{x_B} < \frac{y_B - y_A}{x_B - x_A}, \frac{y_B}{1 - x_B} > \frac{x_B - x_A}{y_B - y_A}\}, \\ \mathcal{O}_5 &= \{(x_A, y_A, x_B, y_B) \in \mathcal{O} \mid y_B > 0, \frac{y_B}{x_B} > \frac{y_B - y_A}{x_B - x_A}, \frac{y_B}{1 - x_B} > \frac{x_B - x_A}{y_B - y_A}\}. \end{aligned}$$

We claim that

$$\mathcal{O} = (\bigcup_{i=1}^{5} \mathcal{O}_i) \cup \bigcup_{j=2}^{4} (\partial\mathcal{O}_1 \cap \partial\mathcal{O}_j) \cup \bigcup_{k=3}^{4} (\partial\mathcal{O}_2 \cap \partial\mathcal{O}_k) \cup \bigcup_{l=3}^{4} (\partial\mathcal{O}_l \cap \partial\mathcal{O}_5) \cup (\bigcap_{m=2}^{5} \partial\mathcal{O}_m).$$

If $(x_A, y_A, x_B, y_B) \in \partial\mathcal{O}_1 \cap \partial\mathcal{O}_5$, then $y_A = y_B = y_C = 0$ and so $x_A = x_B$ which contradicts to $A \neq B$. Hence $\partial\mathcal{O}_1 \cap \partial\mathcal{O}_5$ is empty. Note that

$$\begin{aligned} \bigcap_{m=2}^{5} \partial\mathcal{O}_m = \{(x_A, y_A, x_B, y_B) \in \mathcal{O} \mid y_B > 0, (y_B/x_B) = (y_B - y_A)/(x_B - x_A), \\ (y_B/1 - x_B) = (x_B - x_A)/(y_B - y_A)\}. \end{aligned}$$

First, we consider $\partial g/\partial x_A$ in each $\mathcal{O}_i$. By computation we have

$$\frac{\partial E_1}{\partial x_A} = -\frac{G}{2}, \tag{3.30}$$

$$\frac{\partial E_2}{\partial x_A} = -\frac{G}{2}, \tag{3.31}$$

$$\frac{\partial E_3}{\partial x_A} = \frac{1}{\sqrt{1 + (y_A/x_A)^2}} - \frac{1}{\sqrt{1 + [(y_B - y_A)/(x_B - x_A)]^2}} - \frac{G}{2}, \tag{3.32}$$

$$\frac{\partial E_4}{\partial x_A} = \frac{-(x_B - x_A)/(y_B - y_A)}{\sqrt{1 + [(x_B - x_A)/(y_B - y_A)]^2}} + \frac{(y_C/1 - x_C)}{\sqrt{1 + (y_C/1 - x_C)^2}} - \frac{G}{2}, \tag{3.33}$$

$$\begin{aligned} \frac{\partial E_5}{\partial x_A} = &\frac{1}{\sqrt{1 + (y_A/x_A)^2}} - \frac{1}{\sqrt{1 + [(y_B - y_A)/(x_B - x_A)]^2}} \\ &+ \frac{(x_A - x_B)/(y_B - y_A)}{\sqrt{1 + [(x_B - x_A)/(y_B - y_A)]^2}} + \frac{y_C/1 - x_C}{\sqrt{1 + (y_C/1 - x_C)^2}} - \frac{G}{2}. \end{aligned} \tag{3.34}$$

It is clear that $\partial g/\partial x_A$ is continuous on $\mathcal{O}_i$, $i = 1, 2, \ldots, 5$. Hence it is enough to show that $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_i$ for each i.

If we move the square Q from $\mathcal{O}_2$ to $\mathcal{O}_1$, then it is clear that $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_1 \cap \partial\mathcal{O}_2$.

If we pass from $\mathcal{O}_3$ to $\mathcal{O}_1$, then we have

$$y_A = y_B \to 0.$$

Then (3.32) converges toward (3.30). Hence $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_1 \cap \partial\mathcal{O}_3$. Similarly, if we move Q from $\mathcal{O}_4$ to $\mathcal{O}_1$, then (3.33) converges toward (3.30). Hence $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_1 \cap \partial\mathcal{O}_4$.

If we pass from $\mathcal{O}_3$ to $\mathcal{O}_2$, then we have

$$\frac{y_B - y_A}{x_B - x_A} \to \frac{y_A}{x_A}.$$

Then (3.32) converges toward (3.31). Hence $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_2 \cap \partial\mathcal{O}_3$.

If we move the square Q from $\mathcal{O}_4$ to $\mathcal{O}_2$, then we have

$$\frac{x_B - x_A}{y_B - y_A} \to \frac{y_C}{1 - x_C}.$$

Then (3.33) converges toward (3.31). Hence $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_2 \cap \partial\mathcal{O}_4$.

If we move the square Q from $\mathcal{O}_5$ to $\mathcal{O}_3$, then we have

$$\frac{x_B - x_A}{y_B - y_A} \to \frac{y_C}{1 - x_C}.$$

Then (3.34) converges toward (3.32). Hence $\partial g/\partial x_A$ is continuous across $\partial\mathcal{O}_3 \cap \partial\mathcal{O}_5$.

If we move the square Q from $\mathcal{O}_5$ to $\mathcal{O}_4$, then we have

$$\frac{y_B - y_A}{x_B - x_A} \to \frac{y_A}{x_A}.$$

Then (3.34) converges toward (3.33). Hence $\partial g/\partial x_A$ is continuous across $\partial D_4 \cap \partial\mathcal{O}_5$.

If we move the square Q from $\mathcal{O}_5$ to $\bigcap_{m=2}^{5} \mathcal{O}_m$, then we have

$$\frac{y_B - y_A}{x_B - x_A} \to \frac{y_A}{x_A} \text{ and } \frac{x_B - x_A}{y_B - y_A} \to \frac{y_C}{1 - x_C}.$$

Then (3.34) converges toward (3.31). Similarly, if we move Q from $\mathcal{O}_3$ ($\mathcal{O}_4$, resp.) to $\bigcap_{m=2}^{5} \partial\mathcal{O}_m$, then (3.32)((3.33), resp.) converges toward (3.31). Hence we obtain $\partial g/\partial x_A$ is continuous across $\bigcap_{m=2}^{5} \partial\mathcal{O}_m$. Therefore $\partial g/\partial x_A$ is continuous on $\mathcal{O}$.

The continuities of $\partial g/\partial x_B$, $\partial g/\partial y_A$, $\partial g/\partial y_B$ across $\partial\mathcal{O}_i$ for each i can be derived similarly. This completes the proof. $\square$

It is easy to show that for $Q \subset \Omega \times \mathbf{R}$

$$\int_0^1 |u_Q'(x)|dx = 2y_B \text{ for } y_B > 0. \tag{3.35}$$

Theorem 3.3 *If $G > 2$, then*

$$\inf_{(x_A,\, y_A,\, x_B,\, y_B) \in \mathcal{S}} g(x_A, y_A, x_B, y_B) = -\infty.$$

Proof: If $y_B > 2r$, then $y_A, y_C > 0$. Since $y_A > y_B - 2r$ and $-1 < x_A - x_B < 0$,

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= \int_0^1 \sqrt{1 + u_Q'(x)^2}dx - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &\le \int_0^1 (1 + |u_Q'(x)|)dx - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &< \int_0^1 |u_Q'(x)|dx - \frac{G}{2}(2y_B - 2r + x_A - x_B) + 1 \\ &< \int_0^1 |u_Q'(x)|dx - Gy_B + Gr + \frac{G}{2} + 1 \\ &= (2 - G)y_B + Gr + \frac{G}{2} + 1. \end{aligned}$$

Since $G > 2$, the theorem follows. $\square$

Theorem 3.4 *If $G = 2$, then $\inf_{(x_A,\, y_A,\, x_B,\, y_B)\in\mathcal{S}} g(x_A, y_A, x_B, y_B)$ exists.*

Proof: Suppose that $y_B \leq 0$. Then $g(x_A, y_A, x_B, y_B) = 1 - (y_B + y_A + x_A - x_B) > 0$. On the other hand, if we consider the region $y_B > 0$, then

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= \int_0^1 \sqrt{1 + u_Q'(x)^2}dx - (y_A + y_B + x_A - x_B) \\ &\geq \int_0^1 |u_Q'(x)|dx - 2y_B = 2y_B - 2y_B = 0. \end{aligned}$$

Hence 0 is a lower bound of g. Therefore the infimum of g on $\mathcal{S}$ exists. $\square$

However, the infimum can not be achieved in $\mathcal{S}$.

Theorem 3.5 *If $G = 2$, then g does not attain its minimum on $\mathcal{S}$.*

Proof: Suppose that g attains its minimum, say α, at some $(x_{A_1}, y_{A_1}, x_{B_1}, y_{B_1}) \in \mathcal{S}$. For $(x_A, y_A, x_B, y_B) \in \mathcal{S}$ with $x_A = x_{A_1}$, $x_B = x_{B_1}$, $y_A = y_{A_1} + y$, and $y_B = y_{B_1} + y$ for some $y > 0$, let $h_i(y) = E_i(x_{A_1}, y_{A_1} + y, x_{B_1}, y_{B_1} + y)$ for each i. Then we have

$$\begin{aligned} h_1'(0) &= -2 < 0, \\ h_2'(0) &= \frac{y_{B_1}/x_{B_1}}{\sqrt{1 + (y_{B_1}/x_{B_1})^2}} + \frac{y_{B_1}/1 - x_{B_1}}{\sqrt{1 + (y_{B_1}/1 - x_{B_1})^2}} - 2 < 0, \\ h_3'(0) &= \frac{y_{A_1}/x_{A_1}}{\sqrt{1 + (y_{A_1}/x_{A_1})^2}} + \frac{y_{B_1}/1 - x_{B_1}}{\sqrt{1 + (y_{B_1}/1 - x_{B_1})^2}} - 2 < 0, \\ h_4'(0) &= \frac{y_{B_1}/x_{B_1}}{\sqrt{1 + (y_{B_1}/x_{B_1})^2}} \\ &\quad + \frac{(y_{B_1} + x_{A_1} - x_{B_1})/(1 - x_{B_1} - y_{B_1} + y_{A_1})}{\sqrt{1 + [(y_{B_1} + x_{A_1} - x_{B_1})/(1 - x_{B_1} - y_{B_1} + y_{A_1})]^2}} - 2 < 0, \\ h_5'(0) &= \frac{(y_{B_1} - y_{A_1})/(x_{B_1} - x_{A_1})}{\sqrt{1 + [(y_{B_1} - y_{A_1})/(x_{B_1} - x_{A_1})]^2}} \\ &\quad + \frac{(y_{B_1} + x_{A_1} - x_{B_1})/(1 - x_{B_1} - y_{B_1} + y_{A_1})}{\sqrt{1 + [(y_{B_1} + x_{A_1} - x_{B_1})/(1 - x_{B_1} - y_{B_1} + y_{A_1})]^2}} - 2 < 0, \\ h_6'(0) &= \frac{y_{A_1}/x_{A_1}}{\sqrt{1 + (y_{A_1}/x_{A_1})^2}} + \frac{y_{B_1}/1 - x_{B_1}}{\sqrt{1 + (y_{B_1}/1 - x_{B_1})^2}} - 2 < 0. \end{aligned}$$

Hence $g(x_{A_1}, y_{A_1} + \rho, x_{B_1}, y_{B_1} + \rho) < \alpha$ for any $\rho > 0$ small, a contradiction. Therefore, g does not attain its minimum on $\mathcal{S}$ if $G = 2$. The proof is complete. $\square$

Lemma 3.6 *Let*

$$D = \{(x_A, y_A, x_B, y_B) \in \overline{\mathcal{S}} \mid a \leq y_A \leq y_B \leq b\}$$

where $a \leq b$, $a, b \in \mathbf{R}$. *Then* $\inf_D g$ *is achieved in* $\mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$.

Proof: By Theorem 3.1, g is continuous on D. Since D is a compact subset of $\mathbf{R}^4$, $\inf_D g$ is achieved at some $(x_A, y_A, x_B, y_B) \in D$, i.e., $\inf_D g = \min_D g$. We claim that $\inf_D g$ is achieved in $\mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$. We consider the following cases.
(**1**) $y_B \leq 0$, $x_A = 0$. Then we have

$$g(0, y_A, x_B, y_B) = 1 - \frac{G}{2}(y_A + y_B - x_B)$$

and moving the square horizontally will not change the energy. Hence we may assume that $\inf_D g$ is achieved at $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$ in this case. Similarly, if $(x_A, y_A, x_B, y_B) \in D$ such that $x_B = 1$ or $x_C = 1$, then we may assume that $\inf_D g$ is achieved at $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$ in these cases.
(**2**) $y_B > 0$, $m_{\overline{BO}} < m_{\overline{AB}}$ and $m_{\overline{BC}} < m_{\overline{BD}}$. Fix y_B and rotate the square Q. Since u is unchanged but y_p^Q increases, it will decrease its energy. Hence $\inf_D g$ cannot be achieved in this case.
(**3**) $y_B > 0$, $m_{\overline{BO}} \geq m_{\overline{AB}}$ and $m_{\overline{BC}} < m_{\overline{BD}}$. Suppose that g attains its minimum, say α, at some $(0, y_A, x_B, y_B) \in D$. For $x > 0$ small, we have

$$\begin{aligned} H_1(x) &:= g(x, y_A, x_B + x, y_B) \\ &= x\sqrt{1 + (\frac{y_A}{x})^2} + x_B\sqrt{1 + (\frac{y_B - y_A}{x_B})^2} \\ &\quad +(1 - x_B - x)\sqrt{1 + (\frac{y_B}{1 - x_B - x})^2} - \frac{G}{2}(y_A + y_B - x_B). \end{aligned}$$

By computation we obtain

$$H_1'(0) = -\frac{1}{\sqrt{1 + (y_B/1 - x_B)^2}} < 0. \tag{3.36}$$

Hence $g(x, y_A, x_B + x, y_B) < \alpha$ for $x > 0$ small, a contradiction. Therefore, we may assume that $\inf_D g$ is achieved in $\mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$. Similarly, if $(x_A, y_A, x_B, y_B) \in D$ such that $x_C = 1$, then we may assume that $\inf_D g$ is achieved at $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$ in this case.
(**4**) $y_B > 0$, $m_{\overline{BO}} < m_{\overline{AB}}$ and $m_{\overline{BC}} \geq m_{\overline{BD}}$. Suppose that g attains its minimum, say α, at some $(0, y_A, x_B, y_B) \in D$. Note that in this case $y_A < 0$ and hence $y_B < 2r$. For

$x > 0$ small, we have

$$\begin{aligned} H_2(x) &:= g(x, y_A, x_B + x, y_B) \\ &= (x_B + x)\sqrt{1 + (\frac{y_B}{x_B + x})^2} + (y_B - y_A)\sqrt{1 + (\frac{x_B}{y_B - y_A})^2} \\ &\quad +(1 - x_B - x - y_B + y_A)\sqrt{1 + (\frac{y_B - x_B}{1 - x_B - x - y_B + y_A})^2} \\ &\quad -\frac{G}{2}(y_A + y_B - x_B). \end{aligned}$$

By computation we obtain

$$H_2'(0) = \frac{1}{\sqrt{1 + (y_B/x_B)^2}} - \frac{1}{\sqrt{1 + [(y_B - x_B)/(1 - x_B - y_B + y_A)]^2}} < 0, \tag{3.37}$$

since

$$\frac{y_B}{x_B} - \frac{y_B - x_B}{1 - x_B - y_B + y_A} > 0. \tag{3.38}$$

Indeed, the inequality (3.38) follows from

$$\begin{aligned} & y_B(1 - x_B - y_B + y_A) - x_B(y_B - x_B) \\ &= y_B - x_B y_B - y_B^2 + y_A y_B - x_B y_B + x_B^2 \\ &= y_B - 2x_B y_B - y_B^2 + x_B^2 + y_A y_B \\ &\geq y_B - x_B^2 - y_B^2 - y_B^2 + x_B^2 + y_A y_B \\ &= y_B(1 - 2y_B + y_A) \\ &= y_B[1 - y_B - (y_B - y_A)] \\ &> y_B(1 - 4r) > 0 \;\text{ since }\; y_B < 2r. \end{aligned}$$

Hence $g(x, y_A, x_B + x, y_B) < \alpha$ for $x > 0$ small, a contradiction. Therefore, we may assume that $\inf_D g$ is achieved in $\mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$. Similarly, if $(x_A, y_A, x_B, y_B) \in D$ such that $x_C = 1$, then we may assume that $\inf_D g$ is achieved at $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{a \leq y_A \leq y_B \leq b\}$ in this case.

(**5**) $y_B > 0$, $m_{\overline{BO}} \geq m_{\overline{AB}}$ and $m_{\overline{BC}} \geq m_{\overline{BD}}$. Suppose that g attains its minimum, say α, at some $(0, y_A, x_B, y_B) \in D$. For $x > 0$ small, we have

$$\begin{aligned} H_3(x) &:= g(x, y_A, x_B + x, y_B) \\ &= x\sqrt{1 + (\frac{y_A}{x})^2} + x_B\sqrt{1 + (\frac{y_B - y_A}{x_B})^2} + (y_B - y_A)\sqrt{1 + (\frac{x_B}{y_B - y_A})^2} \\ &\quad +(1 - x_B - x - y_B + y_A)\sqrt{1 + (\frac{y_B - x_B}{1 - x_B - x - y_B + y_A})^2} - \frac{G}{2}(y_A + y_B - x_B). \end{aligned}$$

By computation we obtain

$$H_3'(0) = -\frac{1}{\sqrt{1 + [(y_B - x_B)/(1 - x_B - y_B + y_A)]^2}} < 0. \tag{3.39}$$

Hence $g(x, y_A, x_B + x, y_B) < \alpha$ for $x > 0$ small, a contradiction. Therefore, we may assume that $\inf_D g$ is achieved in $\mathcal{S} \cap \{a \le y_A \le y_B \le b\}$. Similarly, if $(x_A, y_A, x_B, y_B) \in D$ such that $x_C = 1$, then we may assume that $\inf_D g$ is achieved at $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{a \le y_A \le y_B \le b\}$ in this case.
(**6**) $y_A = y_B > 0$. Suppose that g attains its minimum, say α, at some $(0, y_A, x_B, y_B) \in D$. For $x > 0$ small, we have

$$\begin{aligned} H_4(x) &:= g(x, y_A, x_B + x, y_B) \\ &= x\sqrt{1 + (\frac{y_A}{x})^2} + x_B + (1 - x_B - x)\sqrt{1 + (\frac{y_B}{1 - x_B - x})^2} - \frac{G}{2}(2y_A - x_B). \end{aligned}$$

By computation we obtain

$$H_4'(0) = -\frac{1}{\sqrt{1 + (y_B/1 - x_B)^2}} < 0. \tag{3.40}$$

Hence $g(x, y_A, x_B + x, y_B) < \alpha$ for $x > 0$ small, a contradiction. Therefore, we may assume that $\inf_D g$ is achieved in $\mathcal{S} \cap \{a \le y_A \le y_B \le b\}$. Similarly, if $(x_A, y_A, x_B, y_B) \in D$ such that $x_B = 1$, then we may assume that $\inf_D g$ is achieved at $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{a \le y_A \le y_B \le b\}$ in this case. Hence $\inf_D g = \min_D g = \min_{\mathcal{S} \cap \{a \le y_A \le y_B \le b\}} g$. This completes the proof. □

Now, we prove the main theorem of the section.

Theorem 3.7 *If $G < 2$, then the problem (1.4) admits a minimizer in $\mathcal{S}$.*

Proof: First we claim that the infimum of g on $\mathcal{S}$ exists. If $y_B \le 0$, then

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= 1 - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &> 1 + Gr \\ &\ge 0. \end{aligned}$$

On the other hand, if we consider the region $y_B > 0$, then

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= \int_0^1 \sqrt{1 + u_Q'(x)^2} dx - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &\ge \int_0^1 |u_Q'(x)| dx - G y_B \\ &= (2 - G) y_B \\ &> 0. \end{aligned}$$

Hence 0 is a lower bound of g. Therefore, the infimum of g on $\mathcal{S}$ exists. Furthermore, since g is continuous on $\overline{\mathcal{S}}$, $\inf g$ on $\overline{\mathcal{S}}$ also exists.

By (3.15), it is enough to find $(x_{A_0}, y_{A_0}, x_{B_0}, y_{B_0}) \in \mathcal{S}$ such that

$$g(x_{A_0}, y_{A_0}, x_{B_0}, y_{B_0}) = \min_{(x_A,\, y_A,\, x_B,\, y_B) \in \mathcal{S}} g(x_A, y_A, x_B, y_B). \tag{3.41}$$

We claim that to find a minimizer in $\overline{\mathcal{S}}$ it suffices to find a minimizer in $\overline{\mathcal{S}}$ such that

$$-2r \leq y_A \leq y_B \leq M \tag{3.42}$$

for some constant $M = M(G) > 0$. Suppose that $y_A < -2r$. Then $y_B < 0$ and

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= 1 - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &> 1 + Gr \\ &= g(x_A, 0, x_B, 0). \end{aligned} \tag{3.43}$$

Hence $g(x_A, y_A, x_B, y_B)$ does not attain its minimum, if $y_A < -2r$. Therefore, we may assume that $-2r \leq y_A \leq y_B$. On the other hand, if $y_B > 2r$, then y_A, $y_C > 0$ and

$$\begin{aligned} g(x_A, y_A, x_B, y_B) &= \int_0^1 \sqrt{1 + u_Q'(x)^2}dx - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &\geq \int_0^1 \sqrt{1 + u_Q'(x)^2}dx - Gy_B \\ &\geq \int_0^1 |u_Q'(x)|dx - Gy_B \\ &= (2 - G)y_B. \end{aligned}$$

Since $G < 2$, $g(x_A, y_A, x_B, y_B) \to +\infty$ as $y_B \to \infty$. By (3.43), $\inf_{\overline{\mathcal{S}}} g \leq 1 + Gr$. There exists $M = M(G) \geq 2r$ such that to find a minimizer on $\overline{\mathcal{S}}$ it suffices to find a minimizer in $\overline{\mathcal{S}}$ such that (3.42) holds.

Let

$$\overline{\mathcal{S}}_M := \{(x_A, y_A, x_B, y_B) \in \overline{\mathcal{S}} \mid -2r \leq y_A \leq y_B \leq M\}.$$

By the above discussion and Lemma 3.6, we obtain $\inf_{\overline{\mathcal{S}}} g = \inf_{\overline{\mathcal{S}}_M} g$ and $\inf_{\overline{\mathcal{S}}} g$ is achieved at some $(x_A, y_A, x_B, y_B) \in \mathcal{S} \cap \{-2r \leq y_A \leq y_B \leq M\} \subset \mathcal{S}$. Therefore we complete the proof. □