§3 Convergence Results With Strong Convexity

In the first part of this section, we prove the well-definedness of the sequence $\{x_k\}$ generated by Algorithm 1 under some weak monotonicity assumptions on the operator T. Then we give the convergence results obtained under an additional continuous assumption. The following existence result is due to Shih and Tan [23].

Proposition 3.1 [23, Theorem 2]. *Let C be a nonempty closed convex subset of X and let F be a (l, w)-u.s.c. and monotone set-valued operator from C into X^* such that $F(x)$ is weakly compact and convex for each $x \in C$. If C is either compact or*

$$\lim_{\substack{\|z\|\to\infty \\ z\in C}} \inf_{y\in F(z)} \frac{\langle y, z - x_0\rangle}{\|z\|} = \infty,$$

for some $x_0 \in C$*, then there exists a solution to the problem* $VI(F,C)$.

Using this, we can establish an existence result for strongly monotone operators.

Proposition 3.2. *Let* C *be a nonempty closed convex subset of* X *and let* F *be a* (l,w)*-u.s.c. and strongly monotone set-valued operator from* C *into* X^* *such that* $F(x)$ *is weakly compact and convex for each* $x \in C$*, then there exists a unique solution to the problem* $VI(F,C)$.

Proof : Let $z, x \in C$ and $y_1 \in F(z)$, $y_2 \in F(x)$. Since F is strongly monotone, there exists $a > 0$ such that

$$\begin{aligned} \langle y_1, z-x\rangle &\geq \langle y_2, z-x\rangle + a\|z-x\|^2 \\ &\geq -\|y_2\|\|z-x\| + a\|z-x\|^2 \\ &= (a\|z-x\| - \|y_2\|)\|z-x\|. \end{aligned}$$

Thus,

$$\begin{aligned} \inf_{y_1\in F(z)} \frac{\langle y_1, z-x\rangle}{\|z\|} &\geq (a\|z-x\| - \|y_2\|)\frac{\|z-x\|}{\|z\|} \\ &\to \infty, \qquad as\ \|z\| \longrightarrow \infty. \end{aligned}$$

By Proposition 3.1, $VI(F,C)$ has a solution, and it is trivial that the solution is unique, since F is strongly monotone. □

In [25], Yao obtained the existence theorem for pseudomonotone operator as follows.

Proposition 3.3 [25, Corollary 2.1]. *Let* C *be a nonempty closed convex subset of* X *and let* F *be a u.s.c. pseudomonotone set-valued operator from* C *into* X^* *such that* $F(x)$ *is weakly compact and convex for each* $x \in C$*. If* C *is either compact or*

$$\liminf_{\substack{\|z\|\to\infty \\ z\in C}} \inf_{y\in F(z)} \langle y, z-x_0\rangle > 0,$$

for some $x_0 \in C$*, then there exists a solution to the problem* $VI(F,C)$.

Now we make some basic assumptions on our Algorithm 1.

Assumption 1.

(i) T is (l,w)-u.s.c. and weakly montone with constant L on C.

(ii) M is differentiable and strongly convex with constant $\alpha = \frac{b}{2}$, and M' is (l,w)-continuous.

(iii) The sequence $\{\varepsilon_k\}$ satisfies the following condition :

$$\varepsilon_k > 0\,,\ \sum_{k=0}^{\infty} \varepsilon_k = +\infty\,,\ \sum_{k=0}^{\infty} \varepsilon_k^2 < +\infty.$$

The following result analyzes the convergence of the Algorithm 1 involving pseudo-Dunn property.

Theorem 3.4. *Suppose that the problem* $(P1)$ *has a solution* x^*. *Under Assumption 1, for each step k, there exists a unique solution* x_{k+1} *generated by Algorithm 1 to the auxiliary problem* $(P2)$. *Moreover, if* T *has the pseudo-Dunn property with constant* t *on* C, *and if*

$$\forall\, k \in N,\ \varepsilon_{k+1} \leq \varepsilon_k \ \text{ and } \ \alpha < \varepsilon_k < \tfrac{2b}{t+\beta},\ \text{ where } \alpha > 0, \beta > 0,$$

then the sequence $\{x_k\}$ *is bounded. In addition, if* M' *is Lipschitz continuous with constant* m, *and if* T *is* (w,s)*-u.s.c. on* C, *then every weakly cluster point of the sequence* $\{x_k\}$ *is a solution of the problem* $(P1)$.

Proof : Under Assumption 1, the operator F defined by

$$F(y) = \varepsilon T(y) + M'(y) - M'(x)$$

is (l,w)-u.s.c. and strongly monotone with constant $b - \varepsilon L$ on C , by Lemma 2.3. According to Proposition 3.2, we get that there exists a unique solution to the auxiliary problem $(P2)$.

Consider the function Λ_{x^*} defined by

$$\Lambda_{x^*}(x,\varepsilon) := \Phi_{x^*}(x) + \Omega_{x^*}(x,\varepsilon),\ \forall\, x \in C,$$

where

$$\begin{aligned} \Phi_{x^*}(x) &:= M(x^*) - M(x) - \langle M'(x), x^* - x\rangle, \\ \Omega_{x^*}(x,\varepsilon) &:= \varepsilon\langle y^*, x - x^*\rangle. \end{aligned}$$

Since M' is strongly monotone with constant b, by Proposition 2.2, M is strongly convex with constant $\alpha = \frac{b}{2}$, and hence

$$\Phi_{x^*}(x_k) \geq \frac{b}{2}\|x_k - x^*\|^2 \geq 0.$$

Since x^* is a solution of the problem $(P1)$, $\Omega_{x^*}(x,\varepsilon)$ is nonnegative, hence $\Lambda_{x^*}(x,\varepsilon) \geq 0$. Now we study the variation of Λ_{x^*} for Algorithm 1 :

$$\Gamma_k = \Lambda_{x^*}(x_{k+1},\varepsilon_k) - \Lambda_{x^*}(x_k,\varepsilon_k) = s_1 + s_2 + s_3,$$

where

$$\begin{aligned} s_1 &:= M(x_k) - M(x_{k+1}) - \langle M'(x_k), x_k - x_{k+1}\rangle, \\ s_2 &:= \langle M'(x_k) - M'(x_{k+1}), x^* - x_{k+1}\rangle, \\ s_3 &:= \varepsilon_{k+1}\langle y^*, x_{k+1} - x^*\rangle - \varepsilon_k\langle y^*, x_k - x^*\rangle. \end{aligned}$$

Since M' is strongly monotone,

$$s_1 \leq -\frac{b}{2}\|x_{k+1} - x_k\|^2.$$

Using (3) with $z = x^*$, $\varepsilon_k\langle y_k, x^* - x_{k+1}\rangle + B(x_{k+1}, x_k, x^*) \geq 0$, and therefore

$$s_2 \leq \varepsilon_k\langle y_k, x^* - x_{k+1}\rangle = \varepsilon_k\langle y_k, x^* - x_k\rangle + \varepsilon_k\langle y_k, x_k - x_{k+1}\rangle.$$

Using (1) with $z = x_k$, we have some $y^* \in T(x^*)$ such that

$$\langle y^*, x_k - x^*\rangle \geq 0.$$

Since T has the pseudo-Dunn property with constant t and for some $y_k \in T(x_k)$, we have

$$\langle y_k, x_k - x^*\rangle \geq \frac{1}{t}\|y_k - y^*\|^2.$$

Thus,

$$s_2 \leq -\frac{\varepsilon_k}{t}\|y_k - y^*\|^2 + \varepsilon_k\langle y_k, x_k - x_{k+1}\rangle.$$

Since $\varepsilon_{k+1} \leq \varepsilon_k$ for each k, we obtain

$$s_2 + s_3 \leq -\frac{\varepsilon_k}{t}\|y_k - y^*\|^2 + \varepsilon_k\langle y_k - y^*, x_k - x_{k+1}\rangle.$$

Therefore,

$$\begin{aligned} \Gamma_k \ \leq \ & -\frac{b}{2}\|x_{k+1} - x_k\|^2 - \frac{\varepsilon_k}{t}\|y_k - y^*\|^2 \\ & +\varepsilon_k\langle y_k - y^*, x_k - x_{k+1}\rangle. \end{aligned}$$

Since

$$\varepsilon_k\|y_k - y^*\| \cdot \|x_{k+1} - x_k\| \leq \frac{\varepsilon_k^2}{2\lambda}\|y_k - y^*\|^2 + \frac{\lambda}{2}\|x_{k+1} - x_k\|^2,$$

where $\lambda > 0$, we get

$$\Gamma_k \leq -(\frac{b}{2} - \frac{\lambda}{2})\|x_{k+1} - x_k\|^2 - \varepsilon_k(\frac{1}{t} - \frac{\varepsilon_k}{2\lambda})\|y_k - y^*\|^2.$$

Thus, this condition $\alpha < \varepsilon_k < \frac{2b}{t+\beta}$, where $\alpha > 0$, $\beta > 0$ yields

$$\Gamma_k \leq -(\frac{b}{2} - \frac{\lambda}{2})\|x_{k+1} - x_k\|^2 - \frac{\alpha\beta}{t(t+\beta)}\|y_k - y^*\|^2,$$

and for $\lambda < b$, Γ_k is negative unless $x_{k+1} = x_k$ and $y_k = y^*$. Then, according to (3), $\{x_k\}$ is a solution to the problem $(P1)$. Note that the sequence $\{\Lambda_{x^*}(x_k\,,\, \varepsilon_k)\}$ is strictly decreasing, and since it is positive, it must be convergent and the difference between two consecutive terms tends to zero. Therefore, $\|x_{k+1} - x_k\|$ and $\|y_k - y^*\|$ are convergent to zero. Moreover, since the sequence $\{\Lambda_{x^*}(x_k\,,\, \varepsilon_k)\}$ converges, it is bounded, and so is $\{x_k\}$.

Let $\bar{x}$ be a weakly cluster point of the sequence $\{x_k\}$, and let $\{x_{k_i}\}$ be a subsequence weakly converging to $\bar{x}$. By (3), since M' is Lipschitz continuous with constant m and $\varepsilon_k > \alpha$, we have some $y_k \in T(x_k)$,

$$\langle y_k, z - x_{k+1}\rangle \geq -\frac{m}{\alpha}\|x_{k+1} - x_k\| \cdot \|z - x_{k+1}\|, \forall z \in C.$$

Since $\|x_{k_i+1} - x_{k_i}\|$ converges to zero and y_{k_i} strongly converges to y^*, taking the limit for the subsequence $\{k_i\}$ in the last inequality yields

$$\langle y^*, z - \bar{x}\rangle \geq 0, \forall z \in C. \tag{4}$$

Moreover, $\langle \bar{y}, x_{k_i} - \bar{x}\rangle$ converges to 0, for each $\bar{y} \in T(\bar{x})$. If $0 \in T(\bar{x})$, then $\bar{x}$ is clearly a solution of the problem $(P1)$. Otherwise, we let $\bar{y} \in T(\bar{x})$ with $\bar{y} \neq 0$, and let

$$w_{k_i} = x_{k_i} - \frac{\langle \bar{y}, x_{k_i} - \bar{x}\rangle}{\|\bar{y}\|^2}\bar{y}.$$

Then,

$$\langle \bar{y}, w_{k_i} - \bar{x}\rangle = 0, \tag{5}$$

and $\|w_{k_i} - x_{k_i}\|$ converges to 0. Thus, w_{k_i} weakly converges to $\bar{x}$. On the other hand, since T is (w, s)-u.s.c., for $u_{k_i} \in T(w_{k_i})$, u_{k_i} strongly converges to y^*, due to $\|y_{k_i} - y^*\| \longrightarrow 0$ and $\|u_{k_i} - y_{k_i}\| \longrightarrow 0$. By (5) and the pseudo-Dunn property of T, we get

$$\langle u_{k_i}, w_{k_i} - \bar{x}\rangle \geq \frac{1}{t}\|u_{k_i} - \bar{y}\|^2.$$

Therefore, by taking the limit, we conclude that u_{k_i} strongly converges to $\bar{y}$. Then, $y^* = \bar{y} \in T(\bar{x})$ and hence $\bar{x}$ is a solution to the problem $(P1)$. □

Corollary 3.5. *Under all the conditions of Theorem 3.4, if M' is continuous from X equipped with the weak topology to X^* equipped with the weak topology, then the sequence $\{x_k\}$ weakly converges to a unique solution of the problem $(P1)$.*

Proof : Following the proof of Theorem 3.4, we need only to prove the uniqueness of $\bar{x}$. Assume that the sequence $\{x_k\}$ has two weakly cluster points $\bar{x}$ and $\hat{x}$. Then, both cluster points can be used as x^* to defined the function Λ_{x^*}. Consider the subsequence $\{k_i\}$ and $\{l_j\}$ such that x_{k_i} and x_{l_j} weakly converge to $\bar{x}$ and $\hat{x}$, respectively. Then, by Lemma 1.4 (i) and the fact $\bar{y} = \hat{y}$, we get

$$\hat{\Lambda}_{x^*}(x_{k_i}, \varepsilon_{k_i}) = \bar{\Lambda}_{x^*}(x_{k_i}, \varepsilon_{k_i}) + R_{x^*}(x_{k_i}),$$

where

$$R_{x^*}(x_{k_i}) = M(\hat{x}) - M(\bar{x}) - \langle M'(x_{k_i}), \hat{x} - \bar{x}\rangle.$$

Note that $\bar{\Lambda}_{x^*}(x_{k_i}, \varepsilon_{k_i})$ and $\hat{\Lambda}_{x^*}(x_{k_i}, \varepsilon_{k_i})$ tend to some $\bar{l}$ and $\hat{l}$, respectively. Since M' is (w, w)-continuous, $R_{x^*}(x_{k_i})$ converges to $\hat{l} - \bar{l}$. Since M is strongly convex with constant $\frac{b}{2}$, by Proposition 2.2, we obtain

$$\hat{l} \geq \bar{l} + \frac{b}{2}\|\hat{x} - \bar{x}\|^2.$$

By interchangeing the role of $\bar{x}$, the same calculations yield

$$\bar{l} \geq \hat{l} + \frac{b}{2}\|\hat{x} - \bar{x}\|^2.$$

Then

$$0 \leq \frac{b}{2}\|\hat{x} - \bar{x}\|^2 \leq \min\{\bar{l} - \hat{l}, \hat{l} - \bar{l}\} \leq 0.$$

Hence, $\bar{x} = \hat{x}$. □

We note that, if T is strongly monotone with constant a and Lipschitz continuous with constant μ on C, then it has the Dunn property with constant $\frac{\mu^2}{a}$. Without monotonicity, we shall deal with the pseudomonotone case as follows.

Lemma 3.6. *If T is strongly pseudomonotone with constant e and Lipschitz continuous with constant μ on C, then it has the pseudo-Dunn property with constant $\frac{\mu^2}{e}$.*

Proof : Given any (x_1, y_1) and $(x_2, y_2) \in G(T)$, since T is strongly pseudomonotone with constant e,

$$\langle y_1, x_2 - x_1\rangle \geq 0 \Longrightarrow \langle y_2, x_2 - x_1\rangle \geq e\|x_2 - x_1\|^2.$$

Since T is Lipschitz continuous with constant μ,

$$\|y_2 - y_1\| \leq \mu\|x_2 - x_1\|,$$

which implies

$$\|x_2 - x_1\|^2 \geq \frac{1}{\mu^2}\|y_2 - y_1\|^2.$$

So we have

$$\langle y_2, x_2 - x_1\rangle \geq e \cdot \frac{1}{\mu^2}\|y_2 - y_1\|^2.$$

We complete the proof. □

We next analyze the convergence of Algorithm 1 invloving strongly monotone operator with Lipschitz property.

Theorem 3.7. *Suppose that the problem* $(P1)$ *has a solution* $x^* \in C$. *Under Assumption 1, for each step* k, *there exists a unique solution* x_{k+1} *generated by Algorithm 1 to the auxiliary problem* $(P2)$. *If* T *is strongly pseudomonotone with constant* e *on* C *(*x^* *is then unique) and Lipschitz continuous with constant* μ *on* C, *and if*

$\forall\, k \in N,\ \alpha < \varepsilon_k < \frac{2eb}{\mu^2+\beta},$ *where* $\alpha > 0,\ \beta > 0$,

then the sequence $\{x_k\}$ *strongly converges to* x^*. *Moreover, if* M' *is Lipschitz continuous with constant* m *on* C *and* $t = \frac{m}{e\alpha} + \frac{\mu}{e} < 1$, *then the error estimation of solution* x^* *is*

$$\|x_{k+1} - x^*\| \leq \frac{t}{1-t}\|x_k - x^*\|.$$

Proof : By using Theorem 3.4 and Lemma 3.6, $\{x_k\}$ weakly converges to x^*. Furthermore, $\{x_k\}$ is strongly convergent. We illustrate as follows : As before, we consider $\Phi_{x^*}(x)$, but ignore $\Omega_{x^*}(x, \varepsilon)$. Note that $\Phi_{x^*}(x) \geq 0$, and define $\Delta_k = \Phi_{x^*}(x_{k+1}) - \Phi_{x^*}(x_k) = s_1 + s_2$, where the definitions of s_1 and s_2 are same as Theorem 3.4. Thus,

$$s_1 \ \leq\ -\frac{b}{2}\|x_{k+1} - x_k\|^2,$$

and

$$\begin{aligned} s_2 &\leq \varepsilon_k\langle y_k, x^* - x_{k+1}\rangle \\ &= \varepsilon_k\langle y_k - y_{k+1}, x^* - x_{k+1}\rangle + \varepsilon_k\langle y_{k+1}, x^* - x_{k+1}\rangle. \end{aligned}$$

By (1) with $z = x_{k+1}$, and the strong pseudomonotonicity of T, we get

$$s_2 \ \leq\ -e\varepsilon_k\|x_{k+1} - x^*\|^2 + \varepsilon_k\langle y_k - y_{k+1}, x^* - x_{k+1}\rangle.$$

Since T is Lipschitz continuous with constant μ, it follows that

$$\Delta_k \ \leq\ -\frac{b}{2}\|x_{k+1} - x_k\|^2 - e\varepsilon_k\|x_{k+1} - x^*\|^2 + \varepsilon_k\mu\|x_{k+1} - x_k\| \cdot \|x_{k+1} - x^*\|.$$

By the inequality:

$$\begin{aligned}\varepsilon_k\mu\|x_{k+1}-x_k\|\cdot\|x_{k+1}-x^*\| &\leq \frac{\lambda}{2}\|x_{k+1}-x_k\|^2\\ &\quad+\frac{\varepsilon_k^2\mu^2}{2\lambda}\|x_{k+1}-x^*\|^2,\end{aligned}$$

with $\lambda = b$, we have

$$\Delta_k \leq \varepsilon_k^2(\frac{-e}{\varepsilon_k}+\frac{\mu^2}{2b})\|x_{k+1}-x^*\|^2 \leq \frac{-\alpha\beta}{2b}\|x_{k+1}-x^*\|^2 \leq 0.$$

So the sequence $\{\Phi_{x^*}(x_k)\}$ is strictly decreasing (unless $x_{k+1} = x^*$) and bounded below by zero. Thus, $\{\Phi_{x^*}(x_k)\}$ converges. This yields Δ_k tends to 0 and hence $\{x_k\}$ strongly converges to x^*.

Now, by (3) with $z = x^*$,

$$\langle \varepsilon_k y_k + M'(x_{k+1}) - M'(x_k), x^* - x_{k+1}\rangle \geq 0.$$

Since T is strongly pseudomonotone with constnat e, by (1) with $z = x_{k+1}$,

$$\langle y_{k+1}, x_{k+1} - x^*\rangle \geq e\|x_{k+1}-x^*\|^2.$$

It follows that

$$\langle M'(x_{k+1}) - M'(x_k), x^* - x_{k+1}\rangle + \varepsilon_k\langle y_k - y_{k+1}, x^* - x_{k+1}\rangle \geq e\varepsilon_k\|x_{k+1}-x^*\|^2. \tag{6}$$

On the other hand, since M' is Lipschitz continuous with constant m, we have

$$\begin{aligned}&\langle M'(x_{k+1}) - M'(x_k), x^* - x_{k+1}\rangle + \varepsilon_k\langle y_k - y_{k+1}, x^* - x_{k+1}\rangle\\ &\leq \quad m\|x_{k+1}-x_k\|\cdot\|x_{k+1}-x^*\| + \varepsilon_k\mu\|x_{k+1}-x_k\|\cdot\|x_{k+1}-x^*\|\\ &= \quad (m+\varepsilon_k\mu)\|x_{k+1}-x_k\|\cdot\|x_{k+1}-x^*\|.\end{aligned} \tag{7}$$

Thus, combining (6) with (7), we obtain

$$(m+\varepsilon_k\mu)\|x_{k+1}-x_k\| \geq e\varepsilon_k\|x_{k+1}-x^*\|.$$

Hence,

$$\begin{aligned}\|x_{k+1}-x^*\| &\leq (\frac{m}{e\varepsilon_k}+\frac{\mu}{e})\|x_{k+1}-x_k\|\\ &\leq (\frac{m}{e\alpha}+\frac{\mu}{e})\|x_{k+1}-x_k\|\\ &\leq t(\|x_{k+1}-x^*\|+\|x_k-x^*\|).\end{aligned}$$

It follows that $\|x_{k+1}-x^*\| \leq \frac{t}{1-t}\|x_k-x^*\|$. □

The following two technical Lemmas will be used to the later theorem.

Lemma 3.8 [2, Lemma 5]. *Let $\{x_n\}$, $\{\mu_n\}$ and $\{\eta_n\}$ be three sequences such that*

$$\sum_{k=1}^{\infty} \mu_k < +\infty,$$

$$x_n \leq \eta_n + \sum_{k=1}^{n-1} \mu_k \sup_{l \leq k+1} x_l, \, \forall\, n = 1, 2, \cdots.$$

If the sequence $\{\eta_n\}$ is bounded above, then the sequence $\{x_n\}$ is also bunded above.

Lemma 3.9 [2, Lemma 4]. *Let f be a Lipschitz function on a Banach space X and consider the sequence $\{x_k\} \subset X$ and $\{\varepsilon_k\}$ are positive sequences such that*

$$\begin{aligned}
&\sum_{k \in N} \varepsilon_k = +\infty, \\
&\exists\, \zeta,\, \forall\, k \in N\,,\, \|x_{k+1} - x_k\| \leq \zeta \varepsilon_k, \\
&\exists\, \xi,\, \sum_{k \in N} \varepsilon_k |f(x_k) - \xi| < +\infty.
\end{aligned}$$

Then,

$$\lim_{k \to \infty} f(x_k) = \xi.$$

Under the condition that the norm of T does not increase faster than linearly with the norm of x; that is,

$$\exists\, p, q > 0 \, s.t. \, \|x^*\| \leq p\|x\| + q, \, \forall\, (x, x^*) \in G(T), \tag{$\star$}$$

the following result deals with the convergence of the Algorithm 1 involving α-strongly pseudomonotone operators.

Theorem 3.10. *Suppose that the problem $(P1)$ has a solution $x^* \in C$. Assume T and M' satisfy Assumption 1. If T is α-strongly pseudomonotone with constant e on C, with the above property $(\star)$, then the sequence $\{x_k\}$ strongly converges to the unique solution x^*.*

Proof : Consider the Bregman distance

$$\phi_{x^*}(x) := M(x^*) - M(x) - \langle M'(x), x^* - x \rangle, \, \forall x \in C. \tag{8}$$

Since M' is strongly monotone with constant b, by Proposition 2.2, M is strongly convex with constant $\alpha = \frac{b}{2}$. Thus,

$$\begin{aligned}\phi_{x^*}(x_k) &= M(x^*) - M(x_k) - \langle M'(x_k), x^* - x_k\rangle \\ &\geq \alpha\|x^* - x_k\|^2 = \frac{b}{2}\|x^* - x_k\|^2.\end{aligned}$$

Now we study the variation of ϕ_{x^*} for Algorithm 1 : $\Delta_k = \phi_{x^*}(x_{k+1}) - \phi_{x^*}(x_k)$ for each k. Notices that

$$\begin{aligned}\Delta_k &= M(x_k) - M(x_{k+1}) - \langle M'(x_k), x_k - x_{k+1}\rangle + \langle M'(x_k) - M'(x_{k+1}), x^* - x_{k+1}\rangle \\ &= S_1 + S_2,\end{aligned}$$

where

$$\begin{aligned}S_1 &:= M(x_k) - M(x_{k+1}) - \langle M'(x_k), x_k - x_{k+1}\rangle \\ &\leq -\frac{b}{2}\|x_{k+1} - x_k\|^2, \end{aligned}\tag{9}$$

and

$$\begin{aligned}S_2 &:= \langle M'(x_k) - M'(x_{k+1}), x^* - x_{k+1}\rangle \\ &= -B(x_{k+1}, x_k, x^*).\end{aligned}$$

By (3) with $z = x^*$,

$$\varepsilon_k\langle y_k, x^* - x_{k+1}\rangle + B(x_{k+1}, x_k, x^*) \geq 0.$$

So

$$\begin{aligned}S_2 &\leq \varepsilon_k\langle y_k, x^* - x_{k+1}\rangle \\ &= \varepsilon_k\langle y_k, x^* - x_k\rangle + \varepsilon_k\langle y_k, x_k - x_{k+1}\rangle.\end{aligned}$$

By (1) with $z = x_k$, we have some $y^* \in T(x^*)$ such that

$$\langle y^*, x_k - x^*\rangle \geq 0.$$

Since T is α-strongly pseudomonotone with constant e, we get $\langle y_k, x^* - x_k\rangle \leq -e\|x_k - x^*\|^\alpha$. Thus,

$$\begin{aligned}S_2 &\leq \varepsilon_k\langle y_k, x_k - x_{k+1}\rangle - e\varepsilon_k\|x_k - x^*\|^\alpha \\ &\leq \varepsilon_k\|y_k\|\|x_{k+1} - x_k\| - e\varepsilon_k\|x_k - x^*\|^\alpha \\ &\leq \frac{(\varepsilon_k)^2}{2b}\|y_k\|^2 + \frac{b}{2}\|x_{k+1} - x_k\|^2 - e\varepsilon_k\|x_k - x^*\|^\alpha.\end{aligned}\tag{10}$$

Therefore, combining (9) and (10), we obtain

$$\Delta_k = S_1 + S_2 \leq \frac{\varepsilon_k^2}{2b}\|y_k\|^2 - e\varepsilon_k\|x_k - x^*\|^\alpha.$$

On the other hand, by the condition of assumption, we have

$$\|y_k\| \le p\|x_k\| + q \le p\|x_k - x^*\| + p\|x^*\| + q.$$

Then, by the simple inequaity :

$$(u+v)^2 \le 2(u^2+v^2),$$

we get

$$\|y_k\|^2 \le 2p^2\|x_k - x^*\|^2 + 2(p\|x^*\| + q)^2.$$

Thus,

$$\begin{aligned} \Delta_k &\le \frac{\varepsilon_k^2}{2b}\|y_k\|^2 - e\varepsilon_k\|x_k - x^*\|^\alpha \\ &\le \varepsilon_k^2(\gamma\|x_k - x^*\|^2 + \delta) - e\varepsilon_k\|x_k - x^*\|^\alpha. \end{aligned}$$

where $\gamma = \frac{p^2}{b}$ and $\delta = \frac{(p\|x^*\|+q)^2}{b}$. Now, summing up this inequaity from $k = 0$ to $n-1$, we have

$$\begin{aligned} \frac{b}{2}\|x_n - x^*\|^2 &\le \phi_{x^*}(x_n) = \sum_{k=0}^{n-1}\Delta_k + \phi_{x^*}(x_0) \\ &\le \phi_{x^*}(x_0) + \sum_{k=0}^{n-1}\varepsilon_k^2(\gamma\|x_k - x^*\|^2 + \delta) - e\sum_{k=0}^{n-1}\varepsilon_k\|x_k - x^*\|^\alpha \\ &\le \phi_{x^*}(x_0) + \sum_{k=0}^{n-1}\varepsilon_k^2(\gamma\|x_k - x^*\|^2 + \delta). \end{aligned} \tag{11}$$

Then,

$$\|x_n - x^*\|^2 \le \eta_n + \sum_{k=1}^{n-1}\mu_k\|x_k - x^*\|^2, \tag{12}$$

where

$$\begin{aligned} \eta_n &= \frac{2}{b}\phi_{x^*}(x_0) + \frac{2\gamma\varepsilon_0^2}{b}\|x_0 - x^*\|^2 + \sum_{k=0}^{n-1}\frac{2\delta\varepsilon_k^2}{b}, \\ \mu_k &= \frac{2\gamma\varepsilon_k^2}{b}, \forall\, k = 1, 2, \cdots, n-1. \end{aligned}$$

Since

$$\|x_k - x^*\|^2 \le \sup_{l \le k+1}\|x_l - x^*\|^2, \forall\, k = 1, 2, \cdots,$$

it follows from (12) that

$$\|x_n - x^*\|^2 \le \eta_n + \sum_{k=1}^{n-1}\mu_k \sup_{l \le k+1}\|x_l - x^*\|^2.$$

By Lemma 3.8, the sequence $\{\|x_n - x^*\|^2\}$ is bounded, and so is $\{x_n\}$. Moreover, from Assumption 1 (iii) and (11), we get

$$\sum_{k\in N} \varepsilon_k \|x_k - x^*\|^\alpha < +\infty.$$

Also by using (3) with $z = x_k$ and the strong monotonicity of M', we get

$$\|x_{k+1} - x_k\| \leq \frac{\varepsilon_k \|y_k\|}{b} \leq \frac{\varepsilon_k (p\|x_k\| + q)}{b} \leq \zeta\varepsilon_k, \ \forall k = 1, 2, \cdots,$$

for some $\zeta > 0$. Applying Lemma 3.9 to $f(x) = \|x - x^*\|^\alpha$ and $\xi = 0$, we conclude that $\{x_k\}$ strongly converges to x^* . □