2 Preliminary

In this section, we first show the existence of a minimizer of (1.5) for a fixed square Q.

Lemma 2.1 *If $u \in K_Q$, then*

$$\int_0^1 \frac{u'(x)(v'-u')(x)}{\sqrt{1+u'(x)^2}}dx \geq 0 \quad \textit{for all} \quad v \in K_Q \tag{2.1}$$

if and only if

$$\int_0^1 \frac{v'(x)(v'-u')(x)}{\sqrt{1+v'(x)^2}}dx \geq 0 \quad \textit{for all} \quad v \in K_Q. \tag{2.2}$$

Proof: Suppose that (2.1) holds. Let $v \in K_Q$. Since $X/\sqrt{1+X^2}$ is monotone increasing for all $X \in \mathbf{R}$,

$$\left(\frac{v'(x)}{\sqrt{1+v'(x)^2}} - \frac{u'(x)}{\sqrt{1+u'(x)^2}} \right) [v'(x) - u'(x)] \geq 0 \quad \text{for all} \quad x \in [0,1].$$

We obtain

$$\int_0^1 \left(\frac{v'(x)}{\sqrt{1+v'(x)^2}} - \frac{u'(x)}{\sqrt{1+u'(x)^2}} \right) [v'(x) - u'(x)] dx \geq 0$$

and hence

$$\int_0^1 \frac{v'(x)(v'-u')(x)}{\sqrt{1+v'(x)^2}}dx \geq \int_0^1 \frac{u'(x)(v'-u')(x)}{\sqrt{1+u'(x)^2}}dx \geq 0.$$

To prove the converse, suppose that (2.2) holds. Since K_Q is convex and u, $v \in K_Q$, $u + \lambda(v-u) \in K_Q$ for all $\lambda \in (0,1]$. Applying (2.2), we obtain

$$\int_0^1 \lambda \frac{[u' + \lambda(v'-u')](x)(v'-u')(x)}{\sqrt{1+[u'+\lambda(v'-u')]^2(x)}}dx \geq 0 \quad \text{for all} \quad \lambda \in (0,1]$$

and so

$$\int_0^1 \frac{u'(x)(v'-u')(x)}{\sqrt{1+[u'+\lambda(v'-u')]^2(x)}}dx + \lambda \int_0^1 \frac{[(v'-u')(x)]^2}{\sqrt{1+[u'+\lambda(v'-u')]^2(x)}}dx \geq 0.$$

Letting $\lambda \to 0$, (2.1) follows. Therefore we complete the proof. $\square$

Note that since $A(\zeta)$ defined by

$$< A(\zeta), v >:= \int_0^1 \frac{\zeta'(x)v'(x)}{\sqrt{1+\zeta'(x)^2}} dx$$

is a strictly monotone operator on $C^{0,1}([0,1])$, the uniqueness of solution of (2.1) follows from the standard theory of variational inequalities (cf. [5]).

Theorem 2.2 *Suppose that u is the solution of (2.1). Then u is a minimizer of* (1.5).

Proof: Given $v \in K_Q$ and let u be the solution of (2.1). By the mean value theorem (see [7], p.265) we obtain

$$E(v) - E(u) = GDE(u + t(v-u); v-u) = \int_0^1 \frac{[u' + t(v'-u')](x)(v'-u')(x)}{\sqrt{1 + [u' + t(v'-u')]^2(x)}} dx$$

for some $t \in (0,1)$. Here $GDE(x;h)$ is the Gâteaux derivative of E at x in the direction h. Putting $w = u + t(v-u)$, it follows from (2.2) that

$$E(v) - E(u) = \frac{1}{t} \int_0^1 \frac{w'(x)(w'-u')(x)}{\sqrt{1+w'(x)^2}} dx \geq 0,$$

since $w \in K_Q$. Hence u is a minimizer of E on K_Q. $\square$

The uniqueness of the minimizer of (1.5) follows from the uniqueness of solution of the variational inequality as follows.

Theorem 2.3 *If u is a minimizer of* (1.5)*, then u is the unique solution of the variational inequality (2.1).*

Proof: Given $v \in K_Q$. Since K_Q is a convex set, $u + \lambda(v-u) \in K_Q$ for all $\lambda \in [0,1]$. Since $E(u) \leq E(u + \lambda(v-u))$, we obtain

$$\int_0^1 \sqrt{1+u'(x)^2} dx \leq \int_0^1 \sqrt{1 + [u + \lambda(v-u)]'(x)^2} dx \quad \text{for all} \quad \lambda \in (0,1].$$

Then

$$\left(\int_0^1 \sqrt{1 + [u + \lambda(v-u)]'(x)^2} dx - \int_0^1 \sqrt{1+u'(x)^2} dx \right) / \lambda \geq 0 \tag{2.3}$$

for all $\lambda \in (0,1]$ and (2.1) follows by letting $\lambda \to 0$ in (2.3). $\square$

Now, we want to describe the minimizer $u = u_Q$ of (1.5) for any fixed square Q. Note that Q is perfectly defined by the positions of any two of its vertices. We shall always assume that the positive y direction is pointed downward. Let $A = (x_A, y_A)$ be the further left vertex of Q and $B = (x_B, y_B)$ be the further down vertex of Q, except when the sides of Q are parallel to the x, y-axes. In the latter case, we define A(B, resp.) to be the left (right, resp.) vertex of the lower side of Q. For convenience, we denote by $C = (x_C, y_C)$ the further right vertex of Q which is different from B and by m_Γ the slope of a segment Γ. Let $O = (0, 0)$, $D = (1, 0)$, and define the coincidence set Λ by

$$\Lambda := \{x \in I_Q | u(x) = \psi_Q(x)\}.$$

We divide our discussions into the following cases.

Case 1: $y_B \leq 0$ (see Figure 1). In this case, we have

$$u_Q = u_1 \equiv 0,$$

since $\psi_Q \leq 0$ on I_Q.

Case 2: $y_B > 0$, $m_{\overline{BO}} < m_{\overline{AB}}$, and $m_{\overline{BC}} < m_{\overline{BD}}$ (see Figure 2). In this case, the coincidence set Λ reduces to a point and we have

$$u_Q = u_2(x) := \begin{cases} (y_B/x_B)x, & x \in [0, x_B] \\ [y_B/(x_B - 1)](x - 1), & x \in [x_B, 1], \end{cases}$$

since $m_{\overline{BO}} = y_B/x_B$ and $m_{\overline{BD}} = y_B/(x_B - 1)$.

Case 3: $y_B > 0$, $m_{\overline{BO}} \geq m_{\overline{AB}}$, and $m_{\overline{BC}} < m_{\overline{BD}}$ (see Figure 3). In this case, $\Lambda = [x_A, x_B]$ and we have

$$u_Q = u_3(x) := \begin{cases} (y_A/x_A)x, & x \in [0, x_A] \\ [(y_B - y_A)/(x_B - x_A)](x - x_B) + y_B, & x \in [x_A, x_B] \\ [y_B/(x_B - 1)](x - 1), & x \in [x_B, 1], \end{cases}$$

since $m_{\overline{AO}} = y_A/x_A$, $m_{\overline{AB}} = (y_B - y_A)/(x_B - x_A)$, and $m_{\overline{BD}} = y_B/(x_B - 1)$.

Case 4: $y_B > 0$, $m_{\overline{BO}} < m_{\overline{AB}}$, and $m_{\overline{BC}} \geq m_{\overline{BD}}$ (see Figure 4). In this case, $\Lambda = [x_B, x_C]$ and we have

$$u_Q = u_4(x) := \begin{cases} (y_B/x_B)x, & x \in [0, x_B] \\ [(y_C - y_B)/(x_C - x_B)](x - x_B) + y_B, & x \in [x_B, x_C] \\ [y_C/(x_C - 1)](x - 1), & x \in [x_C, 1], \end{cases}$$

since $m_{\overline{BO}} = y_B/x_B$, $m_{\overline{BC}} = (y_C - y_B)/(x_C - x_B)$, and $m_{\overline{CD}} = y_C/(x_C - 1)$.

Case 5: $y_B > 0$, $m_{\overline{BO}} \geq m_{\overline{AB}}$, and $m_{\overline{BC}} \geq m_{\overline{BD}}$ (see Figure 5). In this case, Λ is the orthogonal projection of the square Q on the x-axis, i.e., $\Lambda = [x_A, x_C]$. Then

$$u_Q = u_5(x) := \begin{cases} (y_A/x_A)x, & x \in [0, x_A] \\ [(y_B - y_A)/(x_B - x_A)](x - x_B) + y_B, & x \in [x_A, x_B] \\ [(y_C - y_B)/(x_C - x_B)](x - x_C) + y_C, & x \in [x_B, x_C] \\ [y_C/(x_C - 1)](x - 1), & x \in [x_C, 1], \end{cases}$$

since $m_{\overline{AO}} = y_A/x_A$, $m_{\overline{AB}} = (y_B - y_A)/(x_B - x_A)$, $m_{\overline{BC}} = (y_C - y_B)/(x_C - x_B)$, and $m_{\overline{CD}} = y_C/(x_C - 1)$.

Case 6: $y_A = y_B > 0$ (see Figure 6). In this case, $\Lambda = [x_A, x_B]$ and

$$u_Q = u_6(x) := \begin{cases} (y_A/x_A)x, & x \in [0, x_A] \\ y_B, & x \in [x_A, x_B] \\ [y_B/(x_B - 1)](x - 1), & x \in [x_B, 1], \end{cases}$$

since $m_{\overline{AO}} = y_A/x_A$, $m_{\overline{BD}} = y_B/(x_B - 1)$. Note that **Case 6** is a special case of **Case 3**.

For any given square Q, Q must be one of the cases above. By using Theorem 2.2, it is easy to check that u_Q is the unique minimizer of (1.5). For example, for Q in **Case 3** for any $v \in K_Q$ we compute

$$\begin{aligned} \int_0^1 \frac{u_3'(v' - u_3')}{\sqrt{1 + u_3'^2}} dx &= \frac{m_{\overline{AO}}}{\sqrt{1 + m_{\overline{AO}}^2}}[v(x_A) - u_3(x_A)] + \frac{m_{\overline{AB}}}{\sqrt{1 + m_{\overline{AB}}^2}}[v(x_B) - u_3(x_B)] \\ &\quad - \frac{m_{\overline{AB}}}{\sqrt{1 + m_{\overline{AB}}^2}}[v(x_A) - u_3(x_A)] - \frac{m_{\overline{BD}}}{\sqrt{1 + m_{\overline{BD}}^2}}[v(x_B) - u_3(x_B)] \\ &= \frac{m_{\overline{AB}}}{\sqrt{1 + m_{\overline{AB}}^2}}[v(x_B) - u_3(x_B)] - \frac{m_{\overline{BD}}}{\sqrt{1 + m_{\overline{BD}}^2}}[v(x_B) - u_3(x_B)] \\ &\quad + \left(\frac{m_{\overline{AO}}}{\sqrt{1 + m_{\overline{AO}}^2}} - \frac{m_{\overline{AB}}}{\sqrt{1 + m_{\overline{AB}}^2}}\right)[v(x_A) - u_3(x_A)] \geq 0, \end{aligned}$$

since $m_{\overline{AO}} \geq m_{\overline{AB}} > 0$, $m_{\overline{BD}} < 0$, and $X/\sqrt{1 + X^2}$ is monotone increasing for all $X \in \mathbf{R}$. The other cases can be derived similarly and we omit it.