1 Introduction

We consider a solid square allowed to move freely on an elastic wire and we would like to determine its equilibrium position. Let Q denote the closed set of points in or on the square. We shall denote by $2r$, $r > 0$, the length of the sides of the square, by G its weight, and by $P = (x_p^Q, y_p^Q)$ its barycenter. We shall always assume that

$$Q \subset \Omega \times \mathbf{R}, \tag{1.1}$$

where $\Omega = (0,1)$ and

$$r < \frac{1}{4}. \tag{1.2}$$

Furthermore, we shall assume that the wire, in its undeformed position, occupied Ω. We called a function u an admissible deformation if

$$u \in K_Q := \{v \in C^{0,1}(\overline{\Omega}) \mid v(0) = v(1) = 0 \ \text{ and } \ Q \subset S(v)\},$$

where

$$S(v) := \{(x,y) \in \Omega \times \mathbf{R} \mid y \leq v(x)\}.$$

Recall that $C^{0,1}(\overline{\Omega})$ denotes the space of Lipschitz-continuous functions on $\overline{\Omega}$.

For each square Q satisfying (1.1) and u an admissible deformation of the wire, we consider the following total energy corresponding to this configuration

$$E(u,Q) = \int_0^1 \sqrt{1 + u'(x)^2}dx - Gy_p^Q, \tag{1.3}$$

where u' denotes the derivative of u with respect to x. Note that the integral term in (1.3) is the arc length of u and the second term on the right hand side of (1.3) is the potential energy.

Having chosen r such that (1.2) holds, we would like to minimize (1.3) over the set of all admissible pairs (u,Q), i.e., $u \in K_Q$ and $Q \subset \Omega \times \mathbf{R}$. So the problem we would like to study can be stated as follows. Find $Q_0 \subset \Omega \times \mathbf{R}$ and $u_0 \in K_{Q_0}$ such that

$$E(u_0, Q_0) = \min_{u \in K_Q,\, Q \subset \Omega \times \mathbf{R}} E(u,Q). \tag{1.4}$$

The main idea to study this variational problem is to transform (1.4) into a minimization problem in $\mathbf{R}^4$.

Suppose that the position of the square Q is fixed in $\Omega \times \mathbf{R}$. Denote by

$$I_Q = \{x \in (0,1) \mid (x,y) \in Q \ \text{ for some } \ y \in \mathbf{R}\}.$$

For $x \in I_Q$, set

$$\psi_Q(x) = \sup \{y \mid (x, y) \in Q\}.$$

Then the function ψ_Q is the function describing the lower border of Q, if we direct the y direction downward. For example, ψ_Q equals constant if Q has its sides parallel to the axes, a hat function otherwise. It is easy to see that

$$K_Q = \{u \in C^{0,1}(\overline{\Omega}) \mid \ u(0) = u(1) = 0 \ \text{ and } u(x) \geq \psi_Q \ \text{ on } I_Q\}.$$

For Q being fixed with barycenter $P = (x_p^Q, y_p^Q)$, we first search for $u = u_Q$ the solution of

$$E(u_Q, Q) = \min_{v \in K_Q} \{ \int_0^1 \sqrt{1 + v'(x)^2} dx - G y_p^Q \},$$

or equivalently, since G and y_p^Q are fixed,

$$\min_{v \in K_Q} \int_0^1 \sqrt{1 + v'(x)^2} dx. \tag{1.5}$$

Then the problem (1.4) is equivalent to finding $Q_0 \subset \Omega \times \mathbf{R}$ such that

$$E(u_{Q_0}, Q_0) = \min_{Q \subset \Omega \times \mathbf{R}} E(u_Q, Q). \tag{1.6}$$

Since, for a fixed Q, u_Q and y_p^Q are uniquely determined by the positions of any two of its vertices, we can transform (1.4) into a minimization problem in $\mathbf{R}^4$.

The motivation of this study is from a series of works by Chipot et al (cf. [1, 2, 3]). In these papers, they considered, instead of (1.3), the following total energy

$$E(u, Q) = \frac{1}{2} \int_0^1 [u'(x)]^2 dx - G y_p^Q. \tag{1.7}$$

Our method is quite similar to their method by using a compactness argument. But, instead of restricting the problem in a compact subset of $(0, 1)$, we extend our problem to be defined in $[0, 1]$ continuously. Also, a minimizer exists for all $G > 0$ for the case with total energy (1.7). But, in our case (with total energy (1.3)), there is a minimizer if and only if $G < 2$. Moreover, the problem (1.6) admits infinitely many solutions for $G = \sqrt{2}$. Yet there are only finitely many solutions for any $G > 0$ in the case with total energy (1.7).

This paper is organized as follows. We first study the existence of the minimizer of (1.5) for a fixed square in §2. Then the existence of a minimizer of (1.6) is given in §3. In §4, we compute the minimizers.