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A quadruple set-valued equidistribution over permutations

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Abstract

In this paper we give a detailed constructive proof of an equidistribution between two quadruples of set-valued statistics

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal}) \sim (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})$$

over the set of permutations, where $\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal}$ stand for the statistics sorting index, cycle set, left to right maximal place set, left to right maximal letter set and $\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip}$ stand for the statistics inversion, left to right maximal place set, right to left minimum letter set, right to left minimum place set respectively. Our main result will be proved by way of a bijection $F : \mathfrak{S}_n \to \mathfrak{S}_n$, which is a composition of four mappings.

Key words: Permutations, sorting index, Cycle, Lmap, Lmal, inversion, Rmil, Rmip

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Chapter 1

Introduction

The study of statistics of combinatorial structures is one of the core themes in algebraic and enumerative combinatorics. The reason is that one of the fundamental goal of combinatorics is to understand combinatorial structures, and a result of statistic not only reveals an inner relation but also helps us understand the combinatorial structures more.

As the set of permutations is the most fundamental combinatorial structure, there have been numerous important work on the statistics of permutations. The study of statistics of permutations can trace back to Euler (or even further). Many mathematicians made important contributions in this topic. To name a few we have Euler [9], McMahon [19], Stanley [22], Foata [10, 11, 12], Foata and Schützenberger [14, 15, 16], Foata and Han [13], Humpherey [17], Carlitz [3] and many others.

One may think that this field *permuations statistics* seems old and maybe there is nothing new to say. This is very misleading – in fact, the study of permutation statistics is still very active and new important results appear occasionally. For example, the generating function on $(\mathsf{maj}, \mathsf{exc})$ is not known until 2007 [21], the concepts of sorting index [20], set valued statistic [13], folding phenomenon [7, 6] are all relatively new. A recent highlight is the connection between polytopal geometry and permutation statistics [1]. Also, there are still many of unsolved problems [2, 5].

In this paper, we will give a detailed constructive proof of an equidistribution between two quadruples of set-valued statistics

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal}) \sim (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})$$

over the set of permtuations. The definitions and notations will be given shortly.

1.1 Notation of a permutation

A *permutation* on $[n] := \{1, 2, \ldots, n\}$ is a bijection on $[n]$. We denote by $\mathfrak{S}_n$ the set of all permutations on $[n]$. The 2-*line notation* of a permutation $\sigma \in \mathfrak{S}_n$ is

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix},$$

in which $\sigma(i) = \sigma_i$. By deleting the upper row we obtain its *one-line notation* $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$. A *l-cycle* $C = (c_1, c_2, \ldots, c_l)$ is a permutation with $C(c_i) = c_{i+1}$ for $i = 1, \ldots, l-1$ and $C(c_l) = c_1$. It is well known that we can write a permutation $\sigma \in \mathfrak{S}_n$ as $\sigma = C_1C_2 \cdots C_k$, a product of disjoint cycles. In this case we say $C_1C_2 \cdots C_k$ the *cycle notation* of σ.

1.2 Transposition array

We need one more notation of a permutation, namely the transposition array, which plays a central role in this thesis. A 2-cycle is called a *transposition*. It is also known that $\sigma \in \mathfrak{S}_n$ can be written as

$$\sigma = (p_1, 1)(p_2, 2) \cdots (p_n, n),$$

a composition of n transpositions. We call $p_1 p_2 \ldots p_n$ the *transposition array* of σ and denote it by $TA(\sigma)$.

Let us look an example.

Example 1.2.1. *For the permutation in its two line notation*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix},$$

its one-line notation is 78246513 *and cycle notation* $(1,7)(2,8,3)(4)(5,6)$. *To represent* σ *into transposition array we do the following:*

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & \mathbf{8} & 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix} \qquad (2,8)$$

$$\Rightarrow \begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \mathbf{7} & 3 & 2 & 4 & 6 & 5 & 1 & 8 \end{pmatrix} \qquad (1,7)(2,8)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & 7 & 8 \\ 1 & 3 & 2 & 4 & 6 & \mathbf{5} & 7 & 8 \end{pmatrix} \qquad (5,6)(1,7)(2,8)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & 6 & 7 & 8 \\ 1 & 3 & 2 & 4 & \mathbf{5} & 6 & 7 & 8 \end{pmatrix} \qquad (5,5)(5,6)(1,7)(2,8)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & \mathbf{4} & 5 & 6 & 7 & 8 \\ 1 & 3 & 2 & \mathbf{4} & 5 & 6 & 7 & 8 \end{pmatrix} \qquad (4,4)(5,5)(5,6)(1,7)(2,8)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & \mathbf{2} & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \qquad (2,3)(4,4)(5,5)(5,6)(1,7)(2,8)$$

$$\Rightarrow \begin{pmatrix} 1 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \qquad (2,2)(2,3)(4,4)(5,5)(5,6)(1,7)(2,8)$$

$$\Rightarrow \begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \mathbf{1} & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \qquad (1,1)(2,2)(2,3)(4,4)(5,5)(5,6)(1,7)(2,8)$$

Hence

$$TA(\sigma) = 12245512.$$

Note that only the following two actions are performed on the digits of the 2−line notation when we constuct the transposition array:

(i) exchange a with a smaller digit b, in which b is to the right of a.

(ii) exchange a with a bigger digit b, in which b is to the left of a

1.3 Words

We need two more sets in this thesis. Let

$$\mathsf{CC}_n := \{a_1a_2\ldots a_n : 0 \le a_i \le i-1\}$$

and

$$\mathsf{RCC}_n := \{a_1a_2\ldots a_n : 0 \le a_i \le n-i\}.$$

In other words, the first place of an element $w \in \mathsf{CC}_n$ is 0, the second place can be $0, 1$, while the third place can be $0, 1$ or 2, etc. And the elements of RCC_n is the reversal of the elements of CC_n. It is clear that

$$|\mathsf{CC}_n| = |\mathsf{RCC}_n| = n!.$$

Example 1.3.1. *we have*

$$\mathsf{CC}_3 = \{000, 001, 002, 010, 011, 012\}$$

and

$$\mathsf{RCC}_3 = \{000, 100, 200, 010, 110, 210\}.$$

1.4 Statistics and Set valued statistics

Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n = C_1C_2 \cdots C_k \in \mathfrak{S}_n$ with the transposition array $TA(\sigma) = p_1p_2 \ldots p_n$. We define its *inversion* stastics by

$$\mathsf{inv}(\sigma) := |\{(i, j) \in [n] \times [n] : i < j \text{ and } \sigma_i > \sigma_j\}|,$$

its *sorting index* by

$$\mathsf{sort}(\sigma) := \sum_{i=1}^{n} (i - p_i),$$

its *cycle* (set-valued) index by

$$\mathsf{Cyc}(\sigma) := \{i \in [n] : i \text{ is the minimal number of } C_j, \text{ for some } j = 1, 2, \ldots, k\}.$$

We also define the following four set-valued indices:

$$\begin{aligned}
\mathsf{Lmap}(\sigma) &:= \{i \in [n] : \sigma_i > \sigma_j, \text{ for } j = 1, 2, \ldots, i - 1\},\\
\mathsf{Lmal}(\sigma) &:= \{\sigma_i \in [n] : \sigma_i > \sigma_j, \text{ for } j = 1, 2, \ldots, i - 1\},\\
\mathsf{Rmip}(\sigma) &:= \{i \in [n] : \sigma_i < \sigma_j, \text{ for } j = i + 1, i + 2, \ldots, n\},\\
\mathsf{Rmil}(\sigma) &:= \{\sigma_i \in [n] : \sigma_i < \sigma j, \text{ for } j = i + 1, i + 2, \ldots, n\}.
\end{aligned}$$

Example 1.4.1. *For the permutation* $\sigma = 78246513 = (1, 7)(2, 8, 3)(4)(5, 6)$ *with its transposition array* $TA(\sigma) = 12245512$*, its inversion index is* $\mathsf{inv}(\sigma) = 20$ *as there are* 20 *pairs of* (i, j) *with* $\sigma_i > \sigma_j$*, namely* $(1, 2), (1, 3), \ldots, (6, 8)$*. The sortin index can be calculated by*

$$\begin{aligned}
\mathsf{sort}(\sigma) &= (1 - 1) + (2 - 2) + (3 - 2) + (4 - 4) + (5 - 5) + (6 - 5) + (7 - 1) + (8 - 2)\\
&= 14.
\end{aligned}$$

The cycle index is $\mathsf{Cyc}(\sigma) = \{1, 2, 4, 5\}$*, and* $\mathsf{Lmap}(\sigma) = \{1, 2\}$,$\mathsf{Lmal}(\sigma) = \{7, 8\}$, $\mathsf{Rmip}(\sigma) = \{7, 8\}$ *and* $\mathsf{Rmil}(\sigma) = \{1, 3\}$.

1.5 Foata-Han and Chen-Gong-Guo's results

For $\sigma \in \mathfrak{S}_n$, its *Lehmer code* [18], defined by Lehmer, is

$$Leh(\sigma) = a_1 a_2 \ldots a_n,$$

where $a_i = |\{j : 1 \le j \le i, \sigma_j \le \sigma_i\}|$, for $1 \le i \le n$. From Lehmer code Foata and Han defined the A-code [13] of σ by

$$\text{A-code}(\sigma) = Leh(\sigma^{-1}).$$

Foata and Han also defined the B-code of a $\sigma \in \mathfrak{S}_n$. Let $\sigma = C_1 C_2 \cdots C_k$ be the cycle notation, the its B-code is

$$\text{B-code}(\sigma) = b_1 b_2 \ldots b_n,$$

where b_i is determined in the following way. Suppose $i \in C_j$, for some $j \in \{1, \ldots, k\}$, Write C_j as $(c_{ji_1}, \ldots, c_{ji_l}, i)$, then

$$b_i := \max\{1 \le m \le l : c_{ji_m} < i\}.$$

Also if $c_{ji_m} < i$ for $1 \le m \le l$, then $b_i := i$. In other word, b_i is to search, to the left of i, the first digit not greater than i.

By way of the bijections $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ and $\mathbf{i} : \mathfrak{S}_n \to \mathfrak{S}_n$, where

$$\phi := (\text{B-code})^{-1} \circ \text{A-code}$$

and

$$\mathbf{i}(\sigma) = \sigma^{-1},$$

Foata and Han [13] proved the following theorem.

Theorem 1.5.1. *([13]) We have*

$$(\mathsf{Cyc}, \mathsf{Lmap}) \sim (\mathsf{Lmap}, \mathsf{Rmil}),$$

that is, two pairs of triple statistics $(\mathsf{Cyc}, \mathsf{Lmap})$ *and* $(\mathsf{Lmap}, \mathsf{Rmil})$ *are equidistributed over* $\mathfrak{S}_n$.

It is extend by Chen, Gong and Guo [4] to the following.

Theorem 1.5.2. *([4]) We have*

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}) \sim (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}),$$

that is, two pairs of triple statistics $(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap})$ *and* $(\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil})$ *are equidistributed over* $\mathfrak{S}_n$.

Their strategy of proof can be depicted by the following diagram:

$$\begin{tikzcd} \mathfrak{S}_n \arrow[r, "\phi^{-1}"] & \mathfrak{S}_n \arrow[r, "\mathbf{i}"] & \mathfrak{S}_n \\ \begin{pmatrix} \mathsf{sort} \\ \mathsf{Cyc} \\ \mathsf{Lmap} \end{pmatrix} & \begin{pmatrix} \mathsf{inv} \\ \mathsf{Rmil} \\ \mathsf{Lmap} \end{pmatrix} & \begin{pmatrix} \mathsf{inv} \\ \mathsf{Lmap} \\ \mathsf{Rmil} \end{pmatrix} \end{tikzcd}$$

1.6 Our main result

The main result of this paper is to extend the above result even further into an equidistribution of a pair of quadruple set-valued statistics.

Theorem 1.6.1 (Main result I). *We have*

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal}) \sim (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})$$

are equidistributed over $\mathfrak{S}_n$

Example 1.6.2. *We give a full table for* $n = 2, 3$ *and* 4 *to show that this result is really nontrivial.*

σ	$\mathsf{sort}(\sigma)$	$\mathsf{Cyc}(\sigma)$	$\mathsf{Lmap}(\sigma)$	$\mathsf{Lmal}(\sigma)$	$\mathsf{inv}(\sigma)$	$\mathsf{Lmap}(\sigma)$	$\mathsf{Rmil}(\sigma)$	$\mathsf{Rmip}(\sigma)$
12	0	1, 2	1, 2	1, 2	0	1, 2	1, 2	1, 2
21	1	1	1	2	1	1	1	2

σ	$\mathsf{sort}(\sigma)$	$\mathsf{Cyc}(\sigma)$	$\mathsf{Lmap}(\sigma)$	$\mathsf{Lmal}(\sigma)$	$\mathsf{inv}(\sigma)$	$\mathsf{Lmap}(\sigma)$	$\mathsf{Rmil}(\sigma)$	$\mathsf{Rmip}(\sigma)$
123	0	1, 2, 3	1, 2, 3	1, 2, 3	0	1, 2, 3	1, 2, 3	1, 2, 3
132	1	1, 2	1, 2	1, 3	1	1, 2	1, 2	1, 3
213	1	1, 3	1, 3	2, 3	1	1, 3	1, 3	2, 3
231	2	1	1, 2	2, 3	2	1, 2	1	3
312	3	1	1	3	2	1	1, 2	2, 3
321	2	1, 2	1	3	3	1	1	3

σ	$\mathsf{sort}(\sigma)$	$\mathsf{Cyc}(\sigma)$	$\mathsf{Lmap}(\sigma)$	$\mathsf{Lmal}(\sigma)$	$\mathsf{inv}(\sigma)$	$\mathsf{Lmap}(\sigma)$	$\mathsf{Rmil}(\sigma)$	$\mathsf{Rmip}(\sigma)$
1234	0	1,2,3,4	1,2,3,4	1,2,3,4	0	1,2,3,4	1,2,3,4	1,2,3,4
1243	1	1,2,3	1,2,3	1,2,4	1	1,2,3	1,2,3	1,2,4
1324	1	1,2,4	1,2,4	1,3,4	1	1,2,4	1,2,4	1,3,4
1342	2	1,2	1,2,3	1,3,4	2	1,2,3	1,2	1,4
1423	3	1,2	1,2	1,4	2	1,2	1,2,3	1,3,4
1432	2	1,2,3	1,2	1,4	3	1,2	1,2	1,4
2134	1	1,3,4	1,3,4	2,3,4	1	1,3,4	1,3,4	2,3,4
2143	2	1,3	1,3	2,4	2	1,3	1,3	2,4
2314	2	1,4	1,2,4	2,3,4	**2**	**1,2,4**	**1,4**	**3,4**
2341	3	1	1,2,3	2,3,4	3	1,2,3	1	4
2413	4	1	1,2	2,4	3	1,2	1,3	3,4
2431	3	1,3	1,2	2,4	4	1,2	1	4
3124	3	1,4	1,4	3,4	2	1,4	1,2,4	2,3,4
3142	4	1	1,3	3,4	3	1,3	1,2	2,4
3214	**2**	**1,2,4**	**1,4**	**3,4**	3	1,4	1,4	3,4
3241	3	1,2	1,3	3,4	4	1,3	1	4
3412	4	1,2	1,2	3,4	4	1,2	1,2	3,4
3421	5	1	1,2	3,4	5	1,2	1	4
4123	6	1	1	4	3	1	1,2,3	2,3,4
4132	4	1,3	1	4	4	1	1,2	2,4
4213	5	1,2	1	4	4	1	1,3	3,4
4231	3	1,2,3	1	4	5	1	1	4
4312	5	1	1	4	5	1	1,2	3,4
4321	4	1,2	1	4	6	1	1	4

An exemplary example is in boldface. We can see that

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal})(3214) = (2, \{1,2,4\}, \{1,4\}, \{3,4\}),$$

which corresponds to

$$(\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})(2314) = (2, \{1,2,4\}, \{1,4\}, \{3,4\}).$$

To prove our theorem it suffices to find a bijection $F : \mathfrak{S}_n \to \mathfrak{S}_n$ such that

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal})(\sigma) = (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})(F(\sigma)).$$

As the table shows, F should send 3214 to 2314, 1234 to 1234, etc. Note that F is not an involution, as 2314 will be sent to 3124 (rather than 3214). During the investigation we found ourselves very much want to know how these statistics vary and what really happens among these permutations. It turns out that we will give a very close investigation and our main result will be proved by the bijection $F : \mathfrak{S}_n \to \mathfrak{S}_n$, which is a composition of four mappings A, R, C, and B.

The rest of the thesis is organized as follows. In the next section we introduce the four bijections needed for the proof. In Section 3 we prove our main result. Discussion and concluding remarks are put in Section 4.

Chapter 2

Four bijections

As we said in the introduction, we will find a bijection $F : \mathfrak{S}_n \to \mathfrak{S}_n$ composed of four bijections. In fact, we will see that

$$F = B \circ C \circ R \circ A.$$

In this chapter we will define these bijections A, R, C, B and prove that they are well-defined.

2.1 The bijection A

Firstly we look at the mapping A. Recall that $\mathsf{CC}_n := \{a_1 a_2 \ldots a_n : 0 \leq a_i \leq i-1\}$. Let $A : \mathfrak{S}_n \longrightarrow \mathsf{CC}_n$ be defined as follows. For $\sigma \in \mathfrak{S}_n$ with $TA(\sigma) = p_1 p_2 \ldots p_n$, define

$$A(\sigma) := (1-p_1)(2-p_2)\ldots(n-p_n).$$

Example 2.1.1. $A : \mathfrak{S}_3 \longrightarrow \mathsf{CC}_3$

σ	$TA(\sigma)$	$A(\sigma)$
123	123	000
132	122	001
213	113	010
231	112	011
312	111	012
321	121	002

It is clear that from the definition of $TA(\sigma)$ we have $1 \leq p_i \leq i$, so $i-1 \geq i-p_i \geq 0$ and $A(\sigma) \in \mathsf{CC}_n$, hence A is well-defined.

Now we prove the mapping A is an injection.

Lemma 2.1.2. *The mapping A is injective.*

Proof. Let $\sigma, \sigma' \in \mathfrak{S}_n$ and

$$A(\sigma) = a_1 a_2 \ldots a_n \neq a'_1 a'_2 \ldots a'_n = A(\sigma').$$

From the definition of A we know $A(\sigma) \neq A(\sigma')$ implies $TA(\sigma) = p_1 p_2 \ldots p_n \neq p'_1 p'_2 \ldots p'_n = TA(\sigma')$. Now we define

$$k = \max\{i \in [n] : p_i \neq p'_i\}.$$

We will prove that in the one-line notations of σ and σ', their positions of k are different. Hence $\sigma \neq \sigma'$ and the mapping A is injective.

First note that in the one-line notation of

$$\tau := (p_{k+1}, k+1)(p_{k+2}, k+2) \cdots (p_n, n) = (p'_{k+1}, k+1)(p'_{k+2}, k+2) \cdots (p'_n, n),$$

the positions of $1, 2, \ldots, n$ are of the same.

However,

$$\omega := (p_k, k)(p_{k+1}, k+1)(p_{k+2}, k+2) \cdots (p_n, n)$$

is to switch p_k and k in τ, and

$$\omega' := (p'_k, k)(p'_{k+1}, k+1)(p'_{k+2}, k+2) \cdots (p'_n, n)$$

is to switch p'_k and k in τ. Since $p_k \neq p'_k$, the positions of k in ω and ω' are different.

Note that if we compose by $(p_{k-1}, k-1)$, $(p_{k-2}, k-2)$, $\ldots$, $(p_1, 1)$ successively to ω, the position of k in the one-line notation will not change. Similary when we compose $(p'_{k-1}, k-1)$, $(p'_{k-2}, k-2)$, $\ldots$, $(p'_1, 1)$ successively to the left of ω', the position of k in its one-line notation will not change either. Hence the positions of k in σ and σ' are different and we are done. $\square$

Also, the mapping A is a surjection.

Lemma 2.1.3. *The mapping A is surjective.*

Proof. For $a_1a_2\dots a_n \in \mathsf{CC}_n$, let

$$\tau = (1-a_1)(2-a_2)\dots(n-a_n) \equiv p_1p_2\dots p_n$$

with $1 \le p_i \le i$. Let $\sigma = (p_1, 1)(p_2, 2)\cdots(p_n, n)$ and we have $A(\sigma) = a_1a_2\dots a_n$. Hence the mapping A is onto. Here we use the fact that if $1 \le p_i \le i$ doesn't hold, then $TA(\sigma) \ne p_1\dots p_n$. ☐

Proposition 2.1.4. *The mapping A is a bijection.*

Proof. From the Lemmas 2.1.2 and 2.1.3, the mapping A is a bijection. ☐

2.2 The bijection R

The bijection R is easier. Let $R : \mathsf{CC}_n \longrightarrow \mathsf{RCC}_n$ defined by

$$R(a_1a_2\dots a_n) = a_na_{n-1}\dots a_1 = b_1b_2\dots b_n.$$

That is, we also set $b_i = a_{n-i+1}$.

Example 2.2.1. $R : \mathsf{CC}_3 \longrightarrow \mathsf{RCC}_3$

σ	$R(\sigma)$
000	000
001	100
010	010
011	110
012	210
002	200

For $a_1a_2\dots a_n \in \mathsf{CC}_n$, $0 \le a_i \le i-1$, $0 \le b_i = a_{n-i+1} \le n-i$, hence $R(a_1a_2\dots a_n) \in \mathsf{RCC}_n$ and R is well-defined.

Proposition 2.2.2. *The mapping R is a bijection.*

We first prove R is injective. Let $a_1a_2\dots a_n$ and $a'_1a'_2\dots a'_n \in \mathsf{CC}_n$ with $a_1a_2\dots a_n \ne a'_1a'_2\dots a'_n$. There must exist $k \in [n]$ such that $a_k \ne a'_k$, hence

$$R(a_1a_2\dots a_n) = a_na_{n-1}\dots a_1 \ne a'_na'_{n-1}\dots a'_1 = R(a'_1a'_2\dots a'_n).$$

This means R is an injection.

Now the surjectivity. For $a_1a_2\dots a_n \in \mathsf{RCC}_n$, we have $0 \le a_i \le n-i$. Hence $a_na_{n-1}\dots a_1 \in \mathsf{CC}_n$ and $R(a_na_{n-1}\dots a_1) = a_1a_2\dots a_n$, which means R is a surjection. Therefore R is a bijection. $\square$

2.3 The bijection C

The bijection C is the most complicated among four. Let $C : \mathsf{RCC}_n \longrightarrow \mathsf{CC}_n$ be defined by

$$C := C^1 \circ C^2 \circ \cdots C^n,$$

a composition of n mappings, where C^i is defined by

$$C^i(a_1a_2\dots a_n) = a_1a_2\dots a_{i-1}\,\underbrace{a_{i+1}a_{i+2}\dots a_{i+a_i-1}a_{i+a_i}a_i}\,a_{i+a_i+1}\dots a_n.$$

Note the effect of C^i on the underbraced subword. In other words, C^i moves the i-th position letter 'a_i' to the right with the distance a_i.

Example 2.3.1. $C : \mathsf{RCC}_3 \longrightarrow \mathsf{CC}_3$

σ	$C^3(\sigma) \equiv \sigma'$	$C^2(\sigma') \equiv \sigma''$	$C^1(\sigma'') = C(\sigma)$
000	000	000	000
100	100	100	010
010	010	001	001
110	110	101	011
210	210	201	012
200	200	200	002

Lemma 2.3.2. *The mapping C is well defined.*

Proof. For $a_1a_2\dots a_n \in \mathsf{RCC}_n$ we have $a_i \le n-i$ and there are $n-i$ digits to the right of a_i. Hence the action of C^i is at least performable.

Now let $j \in [n]$. By the definition of C^i we know that $C^{j+1} \circ \cdots \circ C^n$ will not affect the position of a_j, and not until C^j act on this word does the position of a_j move to the right with a_j positions. Moreover, those letters moving from the right of a_j to the left of a_j will not change the relative order with a_j when we perform $C^1 \circ C^2 \circ \cdots \circ C^{j-1}$.

Since there are at least a_j letters to the left of a_j, hence the final position of the letter a_j will be larger than the number a_j. That is, in $b_1 b_2 \dots b_n := C(a_1 a_2 \dots a_n)$ we will have $b_i < i$, for all $i \in [n]$. Therefore,

$$b_1 b_2 \dots b_n \in \mathsf{CC}_n$$

and C is well-defined. $\square$

To prove C is a bijection we will show that there indeed exists an inverse function D. Define $D : \mathsf{CC}_n \longrightarrow \mathsf{RCC}_n$ by

$$D := D^n \circ D^{n-1} \circ \dots \circ D^1,$$

where D^i is defined by

$$D^i(a_1 a_2 \dots a_n) = a_1 a_2 \dots a_{i-a_i-1} \underbrace{a_i a_{i-a_i} a_{i-a_i+1} \dots a_{i-1}}_{} a_{i+1} a_{i+2} \dots a_n.$$

In other words, D^i moves the i-th letter a_i to the left of distant a_i.

Example 2.3.3. $D : \mathsf{CC}_3 \longrightarrow \mathsf{RCC}_3$

σ	$D^1(\sigma) \equiv \sigma'$	$D^2(\sigma') \equiv \sigma''$	$D^3(\sigma'') = D(\sigma)$
000	000	000	000
010	010	100	100
001	001	001	010
011	011	101	110
012	012	102	210
002	002	002	200

Lemma 2.3.4. *The mapping D is well defined.*

Proof. For $a_1 a_2 \dots a_n \in \mathsf{CC}_n$ we have $a_i \le i - 1$ and there are $i - 1$ letters to the left of a_i, hence the action of D^i make senses.

Let $j \in [n]$. From the defition of D^i we know $D^{j-1} \circ \dots \circ D^1$ will not affect the position of a_j. And not until D^j applies on the word does the letter a_j move to the left with distance a_j. Moreover, those letters moves from the left of a_j to the right of a_j will not change their relative positions with a_j when acted by $D^n \circ D^{n-1} \circ \dots \circ D^{j+1}$.

Since there are at least a_j letters to the right of a_j, hence the final position of the letter a_j will be less than the number $n - a_j$. In other words, in $b_1b_2 \dots b_n := D(a_1a_2 \dots a_n)$ we have $b_i \leq n - i$, for all $i \in [n]$. Hence

$$b_1b_2 \dots b_n \in \mathsf{RCC}_n.$$

This proves that D is well-defined. $\square$

From the definition one may wonder that there are some symmetry between C and D, and they are inverse to each other. Set $\tau = a_1a_2 \dots a_n \in \mathsf{RCC}_n$ and suppose $D \circ C(\tau) = b_1b_2 \dots b_n$. Our goal is to prove the following.

Proposition 2.3.5. *The mappings C and D are inverse to each other. Namely,*

$$D \circ C = id_{\mathsf{RCC}_n} \text{ and } C \circ D = id_{\mathsf{CC}_n}.$$

In other words, we have

$$a_1 \dots a_n = b_1 \dots b_n$$

and both C and D are bijections.

However this fact is never easy to prove. The bulk of this section is to prove this fact. The stategy of the proof is by Induction by combining the following two lemmas, one for the base case and the other for the inductive step.

Lemma 2.3.6. *We have $b_n = a_n$.*

Proof. Suppose $C(a_1a_2 \dots a_n) = \dots a_na_{i_1}a_{i_2} \dots a_{i_k}$. Note that the effect of C is to move letters to the right, and initially a_n is at the rightmost position. Hence after the action of C, the letters a_{i_1}, a_{i_2}, ..., a_{i_k} to the right of a_n must been moved before, which means

$$a_{i_1}, a_{i_2}, \dots, a_{i_k} > 0$$

as numbers.

Now we look at $D(\dots a_na_{i_1}a_{i_2} \dots a_{i_k})$. Since the first $n-k-1$ movements only move the letters to the left of a_n, hence

$$D^{n-(k+1)} \circ \dots \circ D^1(C(\tau)) = \dots a_na_{i_1}a_{i_2} \dots a_{i_k}.$$

As $a_1 \dots a_n \in \mathsf{RCC}_n$, $a_n = 0$, therefore

$$D^{n-k} \circ D^{n-(k+1)} \circ \dots \circ D^1(C(\tau)) = \dots a_na_{i_1}a_{i_2} \dots a_{i_k}.$$

From previous discussion we have $a_{i_1} > 0$, hence we have

$$D^{(n-k)+1} \circ D^{n-k} \circ \cdots \circ D^1(C(\tau)) = \begin{pmatrix} \cdots \\ a_{i_1} \end{pmatrix} a_n a_{i_2} \dots a_{i_k}.$$

What matters is that a_{i_1} moves to the left to the a_n and the actual position of a_{i_1} is not important. Repeat the process on $a_{i_2}, \dots, a_{i_k}$ result in

$$D^n \circ \cdots \circ D^1(C(\tau)) = \begin{pmatrix} \cdots \\ a_{i_1}, a_{i_2}, \dots, a_{i_k} \end{pmatrix} a_n.$$

Therefore in $D \circ C(\tau)$ we have $b_n = a_n$ and the lemma is proved. ☐

Now the second lemma for the inductive step.

Lemma 2.3.7. *Suppose we have $b_k = a_k$ for $k = m+1, \dots, n$. Then $b_m = a_m$.*

Proof. For convenients sake, we denote some permutation in the letters $\{a_{m+1}, \dots, a_n\}$ by $[a_{m+1}, \dots, a_n]$. The actural order of the letters is irrelavent.

Since the action of C is to move letters to the right, hence

$$C^{m+1} \circ \cdots \circ C^n(\tau) = a_1 \dots a_m [a_{m+1}, \dots, a_n].$$

This means that the actions of $C^{m+1}, \dots, C^n$ will not affect $a_1, \dots, a_m$. Meanwhile in this stage we do not care about how do $C^{m+1}, \dots, C^n$ affect $a_{m+1}, \dots, a_n$ and we can regard $a_{m+1}, \dots, a_n$ as a whole.

For the action of C^m, we separate $a_{m+1}, \dots, a_n$ into two groups, one group contains those moved from the right of a_m to its left after the action of C^m, the other contains those staying to the right of a_m, as

$$C^m \circ \cdots \circ C^n(\tau) = a_1 \dots a_{m-1}(s_1, \dots, s_{a_m}) a_m (s_{a_m+1}, \dots, s_{n-m})$$

We have $\{a_{m+1}, \dots, a_n\} = \{s_1, \dots, s_{n-m}\}$, and we know, from the definition of C^m, among $a_{m+1}, \dots, a_n$ there are a_m letters moved from the right of a_m (as a letter) to the left of it.

Now we look at the action of $C^1 \circ \cdots \circ C^{m-1}$ and separate $a_1, \dots, a_{m-1}$ into two groups, one contains those moved from the left to the right of a_m after the action, the other contains the rest, as

$$C^1 \circ \cdots \circ C^n(\tau) = \begin{pmatrix} s_1, \dots, s_{a_m} \\ t_1, \dots, t_\alpha \end{pmatrix} a_m \begin{pmatrix} s_{a_m+1}, \dots, s_{n-m} \\ t'_1, \dots, t'_\beta \end{pmatrix}.$$

We have

$$\{a_1, \dots, a_{m-1}\} = \{t_1, \dots, t_\alpha, t'_1, \dots, t'_\beta\},$$

in which $t'_1, \ldots, t'_\beta$ are those moved from the left to the right of a_m, and $t_1, \ldots, t_\alpha$ are those stay to the left.

It must be emphasized that in this step what really matters is a_m. For the other letters we consider the relative order with a_m before and after the action of C. With this there are four possibilities.

In the following we will apply D on $C(\tau)$ and use induction to explain two facts:

(i) Those to the right of a_m will stay in the right of it after the action $D \circ C$.

(ii) Those to the left of a_m will stay in the left of it after the action $D \circ C$.

These two cases will be proved respectively in the following two lemmas.

Lemma 2.3.8. *Those to the right of a_m will stay in the right of it after the action $D \circ C$.*

Proof. We first investigate the result after the action of $D^{a_m+\alpha} \circ \cdots \circ D^1$ for those to the left of a_m (namely, $\{s_1, \ldots, s_{a_m}, t_1, \ldots, t_\alpha\}$) in $C(\tau)$.

Suppose that after the action of $D^{a_m+\alpha} \circ \cdots \circ D^1$ there exists $s_i \in \{s_1, \ldots, s_{a_m}\}$ staying to the left of $t_j \in \{t_1, \ldots, t_\alpha\}$. From the effect of D we know the $D^n \circ \cdots \circ D^{a_m+\alpha+1}$ will only move a_m and those letters in $C(\tau)$ to the right of a_m. This means the relative order of s_i and t_j is not changed. In fact, it is easy to see s_i will be still staying to the left of t_j in $D(C(\tau))$.

Now we let a letter s in $D(C(\tau))$ with $s \in \{s_1, \ldots, s_{n-m}\}$ and s be the left most letter in any letter in $\{s_1, \ldots, s_{n-m}\}$. We know that, in $D(C(\tau))$, to the right of s there are not only $n-m-1$ letters

$$\{s_1, \ldots, s_{n-m}\} \setminus \{s\},$$

but also the letter t_j – since t_j is to the right of s_i and s is the leftmost letter among $\{s_1, \ldots, s_{n-m}\}$, hence t_j is to the right of s. Hence we know that there are at least $n-m$ letters to the right of s. This means that in $D(C(\tau))$, s can be at one of the first m positions and this is a contradiction since the letters in $\{s_1, \ldots, s_{n-m}\}$ are to the right of a_m in τ and after the action of $D \circ C(\tau)$ they will be back to the initial positions.

Hence, after the action of $D^{a_m+\alpha} \circ \cdots \circ D^1$ letters in $\{s_1, \ldots, s_{a_m}\}$ will be to the right of any letters in $\{t_1, \ldots, t_\alpha\}$, as

$$D^{a_m+\alpha} \circ \cdots \circ D^1(C(\tau)) = [t_1, \ldots, t_\alpha][s_1, \ldots, s_{a_m}]a_m \begin{pmatrix} s_{a_m+1}, \ldots, s_{n-m} \\ t'_1, \ldots, t'_\beta \end{pmatrix}.$$

We therefore apply $D^{a_m+\alpha+1}$ on it and obtain

$$D^{a_m+\alpha} \circ \cdots \circ D^1(C(\tau)) = [t_1,\ldots,t_\alpha]a_m[s_1,\ldots,s_{a_m}] \begin{pmatrix} s_{a_m+1},\ldots,s_{n-m} \\ t_1',\ldots,t_\beta' \end{pmatrix}.$$

Finally we consider the letters $s_{a_m+1},\ldots,s_{n-m}$. Again, suppose that after the action of D there exists $s_i \in \{s_{a_m+1},\ldots,s_{a_{n-m}}\}$ to the left of a_m. Similarly to above, we let $s \in \{s_1,\ldots,s_{n-m}\}$ to be the letter in $D(C(\tau))$ and be the leftmost letter among $\{s_1, \ldots, s_{n-m}\}$. We know that in $D(C(\tau))$ to the right of s there are not only $n-m-1$ letters $\{s_1,\ldots,s_{n-m}\} \setminus \{s\}$ but also a_m – since a_m is to the right of s_i and s is the left most letter among $\{s_1,\ldots,s_{n-m}\}$ and therefore a_m is to the right of s.

Hence there are at least $n-m$ letters to the right of s, which means that in $D(C(\tau))$ the letter s can only occupy one of the first m positions, a contradiction from the fact that since letters in $\{s_1,\ldots,s_{n-m}\}$ are to the right of a_m in τ and after the action of $D \circ C$ they should be back to the places.

Therefore we reach the conclusion that after the action of D, letters in

$$\{s_{a_m+1},\ldots,s_{a_{n-m}}\}$$

will stay to the right of a_m. □

Lemma 2.3.9. *Those to the left of a_m will stay in the left of it after the action $D \circ C$.*

Proof. We suppose $t_1',\ldots,t_\beta'$ are to the left of a_m in $C(\tau)$. In the following we focus on t_1'. The discussion of $t_2',\ldots,t_\beta'$ is similar.

Suppose in $C(\tau)$, among elements $\{s_{a_m+1},\ldots,s_{a_{n-m}}\}$ there are j_1 letters to the left of t_1', say

$$s_{a_m+1}, s_{a_m+2}, \ldots, s_{a_m+j_1}.$$

From the discussion of the previous lemma we know that after the action of D the letters $s_{a_m+1}, s_{a_m+2},\ldots,s_{a_m+j_1}$ are still to the right of a_m. We look at the instance right before D acts on t_1', which will be

$$D^{a_m+\alpha+1+j_1} \circ \cdots \circ D^1(C(\tau)) = [t_1,\ldots,t_\alpha]a_m \begin{pmatrix} s_1,\ldots,s_{a_m} \\ s_{a_m+1},\ldots,s_{a_m+j_1} \end{pmatrix} t_1' \begin{pmatrix} s_{a_m+j_1+1},\ldots,s_{n-m} \\ t_2',\ldots,t_\beta' \end{pmatrix}.$$

Since we suppose that in $C(\tau)$ the letters $s_{a_m+1}, s_{a_m+2},\ldots,s_{a_m+j_1}$ are to the left of t_1', hence this shows the instant that the movements of $s_{a_m+1}, s_{a_m+2},\ldots,$ $s_{a_m+j_1}$ via the action of D are already done.

Now we act by D to move $t_1^{'}$. We need to see how the effect of C on $t_1^{'}$ in $C(\tau)$. Note that $t_1^{'}$ is to the left of a_m in τ, hence $t_1^{'}$ moves later than $a_m, s_1, \ldots, s_{n-m}$ when we perform $C(\tau)$. We infer that when C acts on τ, when acting on $t_1^{'}$, its right-hand side element $a_m, s_1, \ldots, s_{a_m}, s_{a_m+1}, \ldots, s_{a_m+j_1}$ will move to the left hand side of $t_1^{'}$. This means as a number

$$t_1^{'} \geq 1 + a_m + j_1.$$

Hence when applied by D, $t_1^{'}$ will move to the left of a_m, namely

$$D^{a_m+\alpha+1+j_1+1} \circ \cdots \circ D^1(C(\tau)) = \left(\begin{smallmatrix} t_1, \ldots, t_\alpha \\ t_1^{'} \end{smallmatrix} \right) a_m \left(\begin{smallmatrix} s_1, \ldots, s_{a_m} \\ s_{a_m+1}, \ldots, s_{a_m+j_1} \end{smallmatrix} \right) \left(\begin{smallmatrix} s_{a_m+j_1+1}, \ldots, s_{n-m} \\ t_2^{'}, \ldots, t_\beta^{'} \end{smallmatrix} \right).$$

Now we know $t_1^{'}$ is moved to the left of a_m. For $t_2^{'}, \ldots$ the arguments are similar and we reach to the result that in $D \circ C(\tau)$, letters $t_1^{'}, \ldots, t_\beta^{'}$ will move to the left of a_m. $\square$

Proof of Lemma 2.3.7 The proof of Lemma 2.3.7 is done by combining the above two lemmas, namely

$$D(C(\tau)) = \begin{pmatrix} t_1, \ldots, t_\alpha \\ t_1^{'}, \ldots, t_\beta^{'} \end{pmatrix} a_m \begin{pmatrix} s_1, \ldots, s_{a_m} \\ s_{a_m+1}, \ldots, s_{n-m} \end{pmatrix}.$$

and we have $a_m = b_m$. $\square$

Finally we can reach our goal.

Proof of Proposition 2.3.5. By apply the above lemmas and induction on k we prove $D \circ C$ is an identity mapping on RCC_n. The fact that $C \circ D$ is an identity mapping on CC_n is done similarly and both C and D are bijections. $\square$

2.4 The bijection B

The last ingredient is the function B. Let $B : \mathsf{CC}_n \longrightarrow \mathfrak{S}_n$ defined by

$$B(a_1 a_2 \ldots a_n) = B^1 \circ B^2 \circ \cdots \circ B^n(123 \ldots n),$$

where B^i is defined by

$$B^i(\sigma_1 \ldots \sigma_{i-1} \sigma_i \sigma_{i+1} \ldots \sigma_n) = \sigma_1 \ldots \sigma_{i-1} \sigma_{i+1} \ldots \sigma_{i+a_{n-i+1}} \sigma_i \sigma_{i+a_{n-i+1}+1} \ldots \sigma_n.$$

In other words, initially we start from $123 \dots n$. The action B^n moves n to the right with a_1 steps , then B^{n-1} moves $n-1$ to the right with a_2 steps and so on. Generally B^k moves k to the right with a_{n-k+1} steps.

Example 2.4.1. $B : \mathsf{CC}_3 \longrightarrow \mathfrak{S}_3$

σ	$B^3(123) \equiv \tau$	$B^2(\tau) \equiv \tau^{'}$	$B^1(\tau^{'}) = B(\sigma)$
000	123	123	123
010	123	132	132
001	123	123	213
011	123	132	312
012	123	132	321
002	123	123	231

Lemma 2.4.2. *The mapping B is well defined.*

Proof. For any $i \in [n]$ each of $B^{i+1}, B^{i+2}, \dots, B^n$ is to move letters to the right of i further to the right. That is, not until the action of B^i, the letter i will not move. Meanwhile, at the moment i is acted by B^i the letters to the right of i are letters $i+1, i+2, \dots, n$ and there are $n-i$ of them.

We know B^i moves i to the right with a_{n-i+1} step, and a_{n-i+1} is the $(n-i+1)$-th letter of some word in CC_n. Hence from the definition of CC_n we know

$$0 \leq a_{n-i+1} \leq (n-i+1) - 1 = n-i,$$

which means that for the letter i (to its right there are $n-i$ letters) the action of B^i on it will not move it to the right for more than $n-i$ steps. As B is a rearrangment of $[n]$, we have $B(a_1 a_2 \dots a_n) \in \mathfrak{S}_n$ and B is well-defined. ☐

Lemma 2.4.3. *The mapping B is injective.*

Proof. For $i \in [n]$, at the moment that i is acted by B^i, the letters to the right of i will be

$$i+1, i+2, \dots, n.$$

When i moves a_{n-i+1} steps to the right, it crosses letters larger than i. Also the action $B^1 \circ \cdots \circ B^{i-1}$ will not affect letters greater or equal to i. In other words, after the action of B^i, the relative position between i and those greater than i will not change. Hence we know in $B(a_1 \dots a_n)$ to the left of i there will be a_{n-i+1} letters greater than i.

Now suppose $a_1a_2\dots a_n$, $b_1b_2\dots b_n \in \mathsf{CC}_n$, and $a_1\dots a_n \neq b_1\dots b_n$. If $a_{n-i+1} \neq b_{n-i+1}$ for some $i \in [n]$, then in $B(a_1\dots a_n)$ there will be a_{n-i+1} letters larger than i to the left of i. However in $B(b_1\dots b_n)$ there will be b_{n-i+1} letters larger than i to the left of i. As $a_{n-i+1} \neq b_{n-i+1}$ we know

$$B(a_1\dots a_n) \neq B(b_1\dots b_n)$$

and hence B is an injection. □

Lemma 2.4.4. *The mapping B is surjective.*

Proof. For $\sigma = \sigma_1\dots\sigma_n \in \mathfrak{S}_n$, let a_{n-i+1} is the number of letters to the left of i and larger than i in σ. Since there are $n-i$ numbers in $[n]$ greater than i, hence $0 \leq a_{n-i+1} \leq n-i$, that is $0 \leq a_i \leq i-1$ for all i. Hence we have

$$a_1a_2\dots a_n \in \mathsf{CC}_n.$$

Also, from the arguments above we know that in $B(a_1\dots a_n)$ the number of letters to the left of i and greater than i will be a_{n-i+1} for all i. Hence

$$B(a_1\dots a_n) = \sigma$$

and B is a surjection. □

By combining the above two lemmas, we have the following.

Proposition 2.4.5. *The mapping B is a bijection.*

Now we have all the four bijections A, R, C, B at hand. In the next section we will prove that the composition bijection

$$F := B \circ C \circ R \circ A$$

is exactly what we need for proving our equidistribution.

Example 2.4.6. $F : \mathfrak{S}_4 \longrightarrow \mathfrak{S}_4$

σ	$TA(\sigma)$	$A(\sigma)$	$R \circ A(\sigma)$	$C \circ R \circ A(\sigma)$	$F(\sigma)$
1234	1234	0000	0000	0000	1234
1243	1233	0001	1000	0100	1243
1324	1224	0010	0100	0010	1324
1342	1223	0011	1100	0110	1423
1423	1222	0012	2100	0120	1432
1432	1232	0002	2000	0020	1342
2134	1134	0100	0010	0001	2134
2143	1133	0101	1010	0101	2143
2314	1124	0110	0110	0011	3124
2341	1123	0111	1110	0111	4123
2413	1122	0112	2110	0121	4132
2431	1132	0102	2010	0021	3142
3124	1114	0120	0210	0012	3214
3142	1113	0121	1210	0112	4213
3214	1214	0020	0200	0002	2314
3241	1213	0021	1200	0102	2413
3412	1212	0022	2200	0022	3412
3421	1112	0122	2210	0122	4312
4123	1111	0123	3210	0123	4321
4132	1131	0103	3010	0013	3241
4213	1211	0023	3200	0023	3421
4231	1231	0003	3000	0003	2341
4312	1121	0113	3110	0113	4231
4321	1221	0013	3100	0103	2431

Chapter 3

Proof of the Main Result

Recall that our goal is to prove that

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal}) \sim (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})$$

over the symmetric group $\mathfrak{S}_n$. In the previous chapter we introduce four bijections A, R, C, B. In this section we will consider the bijection $F : \mathfrak{S}_n \to \mathfrak{S}_n$ defined by

$$F := B \circ C \circ R \circ A$$

and show that F has the desired property. Namely, we will prove that for $\sigma \in \mathfrak{S}_n$ the following properties hold.

(i) $\mathsf{sort}(\sigma) = \mathsf{inv}(F(\sigma))$,

(ii) $\mathsf{Cyc}(\sigma) = \mathsf{Lmap}(F(\sigma))$,

(iii) $\mathsf{Lmap}(\sigma) = \mathsf{Rmil}(F(\sigma))$,

(iv) $\mathsf{Lmal}(\sigma) = \mathsf{Rmip}(F(\sigma))$.

The first item will be in Section 3.1. The second item will be treated in Section 3.2, and the final two are in Section 3.3. After these our main result is proved.

3.1 $\mathsf{sort}(\sigma) = \mathsf{inv}(F(\sigma))$

First we prove is the following.

Theorem 3.1.1. *We have*

$$\mathsf{sort}(\sigma) = \mathsf{inv}(F(\sigma)).$$

Proof. Given $\sigma \in \mathfrak{S}_n$, set $TA(\sigma) = p_1p_2\dots p_n$. From the definition of A we have

$$A(\sigma) = (1-p_1)(2-p_2)\dots(n-p_n) \equiv a_1a_2\dots a_n.$$

And here $\mathsf{sort}(\sigma) = \sum_{i=1}^{n}(i-p_i) = \sum_{i=1}^{n} a_i$.

We look at the action of $C \circ R$ on $A(\sigma)$. Let $C \circ R \circ A(\sigma) = b_1b_2\dots b_n$, since the effects of $R : \mathsf{CC}_n \longrightarrow \mathsf{RCC}_n$ and $C : \mathsf{RCC}_n \longrightarrow \mathsf{CC}_n$ are to change the positions of letters rather than their magnitude, hence $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.

Now we let

$$F(\sigma) = B(C \circ R \circ A(\sigma)) = B(b_1b_2\dots b_n) = \tau = \tau_1\tau_2\dots\tau_n$$

and consider $\mathsf{inv}(\tau) := \mathsf{inv}(F(\sigma))$. First we can write

$$\mathsf{inv}(\tau) = |\{(i,j) : i < j, \tau_i > \tau_j\}| = \sum_{j=1}^{n} |\{i : i < j, \tau_i > \tau_j\}|.$$

From the property of the mapping B we know that for any $j \in [n]$, there are b_{n-j+1} letters to the left of j and greater than j. Hence

$$\sum_{j=1}^{n} |\{i : i < j, \tau_i > \tau_j\}| = \sum_{j=1}^{n} b_{n-j+1} = \sum_{j=1}^{n} b_j.$$

Again since R and C are just rearrangements of letters, hence

$$\sum_{j=1}^{n} b_j = \sum_{j=1}^{n} a_j = \mathsf{sort}(\sigma)$$

and $\mathsf{inv}(F(\sigma)) = \mathsf{sort}(\sigma)$, as desired. □

3.2 $\mathsf{Cyc}(\sigma) = \mathsf{Lmap}(F(\sigma))$

Our second proposition is the following.

Theorem 3.2.1. *We have*

$$\mathsf{Cyc}(\sigma) = \mathsf{Lmap}(F(\sigma)).$$

For $\sigma \in \mathfrak{S}_n$, we let $A(\sigma) = a_1a_2\dots a_n$. We need the following lemmas to prove this theoerm.

Lemma 3.2.2. *We have*

$$\mathsf{Cyc}(\sigma) = \{i \in [n] : a_i = 0\}$$

Proof. First let us see how a transposition (i, j) affects the cycle notation $\sigma = C_1 C_2 \cdots C_k$. What we need here is the case that i, j are not in the same cycle. The following fact is easy and we contain it here for completeness.

Lemma 3.2.3. *Suppose i, j are not in the same cycle of the $\sigma = C_1 C_2 \cdots C_k$. Then the action of (i, j) on σ will combine two cycles which i, j respectively belongs to into one cycle.*

Proof. Since $C_1, \ldots, C_k$ are disjoint cycles, without loss of generality we can suppose i is in the cycle $C_1 = (i, x_1, \ldots, x_s)$ and j is in $C_2 = (j, y_1, \ldots, y_t)$. A simple calculation shows that

$$(i, j) \circ C_1 \cdots C_k = (i, x_1, \ldots, x_s, j, y_1, \ldots, y_t) C_3 C_4 \cdots C_k.$$

and we are done. □

We continue the proof of Lemma. Write $\sigma = (p_1, 1)(p_2, 2) \cdots (p_n, n)$, $0 \leq p_i \leq i$, for all $i \in [n]$ and we regard each (p_i, i) as a transposition and discuss how this chain of composition acting on the identity $(1)(2) \cdots (n)$.

For $(p_n, n) \circ (1) \cdots (n)$ we have two cases.

(i) Case 1, if $p_n = n$ then nothing happens. hence $(p_n, n) \circ (1) \cdots (n) = (1) \cdots (n)$. Since $p_i \leq i < n$, for $i = 1, \ldots, n-1$, hence $(p_1, 1), \ldots, (p_{n-1}, n-1)$ will not merge the cicle (n) with any other circle. This means in the cycle notation of $(p_1, 1) \cdots (p_n, n)$ the smallest number in the cycle containing n is n itself, hence $n \in \mathsf{Cyc}(\sigma)$.

(ii) Case 2, if $p_n < n$. Now (n) will merge with the cycle in which p_n belongs to. We will explain in the following that n will continue to stay in the same cycle with p_n. As $p_n < n$ hence $n \notin \mathsf{Cyc}(\sigma)$.

Now we apply $(p_{n-1}, n-1)$. First we note that no matter (p_n, n) is in which case above, $1, \ldots, n-1$ are the smallest letter in their own cycles. Again we consider two cases: (i) if $p_{n-1} = n-1$ then nothing happens. Since $p_i \leq i < n-1$, for $i = 1, \ldots, n-2$, the transposition $(p_1, 1), \ldots, (p_{n-2}, n-2)$ will not merge the circle containing $n-1$ with any other cycle. Hence $n-1 \in \mathsf{Cyc}(\sigma)$. (ii) $p_{n-1} < n-1$. As $1, \ldots, n-1$ are the smallest number in their own cycles, $n-1$ and p_{n-1} are in different cycles and $(p_{n-1}, n-1)$ will merge these cycles into one cycle in which the smallest element is p_{n-1}. Later we will continue to explain that $n-1$ and p_{n-1} will remain in the same cycle till the end. Hence $n-1 \notin \mathsf{Cyc}(\sigma)$.

We continue the same arguments for $(p_{n-2}, n-2)$, …, $(p_1, 1)$. Generally, when we apply (p_i, i), we first note that no matter how does $(p_{i+1}, i+$

$1) \circ \cdots \circ (p_n, n)$ do, $1, \ldots, i$ are the smallest letter in their own cycles. We look at two cases similarly: (i) if $p_i = i$ then nothing happens and the by the same argument $i \in \mathsf{Cyc}(\sigma)$. (ii) if $p_i < i$, then (p_i, i) merge the cycles containing p_i and i respectively into one, in which the smallest letter is p_i. These two numbers remain in the same cycle till the end and $i \notin \mathsf{Cyc}(\sigma)$.

Hence, for $i \in [n]$, the effect of the action by (p_i, i) is either staying invariant or merging two cycles. Especially it won't separate any two letters already in the same cycle. Therefore, given $i \in [n]$, (i) if $p_i = i$, since $(p_1, 1) \circ \cdots \circ (p_{i-1}, i-1)$ will not affect the cycle containing i, hence $i \in \mathsf{Cyc}(\sigma)$. (ii) if $p_i \neq i$, then i will get into the circle containing p_i and will remain so till the end. As $p_i < i$ we have $i \notin \mathsf{Cyc}(\sigma)$. Hence

$$\mathsf{Cyc}(\sigma) = \{i \in [n] : p_i = i\},$$

where $p_1 \ldots p_n = TA(\sigma)$. From the definition of A we may set $A(\sigma) = a_1 \ldots a_n$, then

$$\begin{aligned} \mathsf{Cyc}(\sigma) &= \{i \in [n] : p_i - i = 0\} \\ &= \{i \in [n] : a_i = 0\} \end{aligned}$$

and the lemma is proved. $\square$

Example 3.2.4. $A(\sigma) = 0101134$, $TA(\sigma) = 1133433$

$$\begin{aligned}
\langle 3,7\rangle &= (1)(2)(37)(4)(5)(6) \\
\langle 3,6\rangle\langle 3,7\rangle &= (1)(2)(376)(4)(5) \\
\langle 4,5\rangle\langle 3,6\rangle\langle 3,7\rangle &= (1)(2)(376)(45) \\
\langle 3,4\rangle\langle 4,5\rangle\langle 3,6\rangle\langle 3,7\rangle &= (1)(2)(37645) \\
\langle 3,3\rangle\langle 3,4\rangle\langle 4,5\rangle\langle 3,6\rangle\langle 3,7\rangle &= (1)(2)(37645) \\
\langle 1,2\rangle\langle 3,3\rangle\langle 3,4\rangle\langle 4,5\rangle\langle 3,6\rangle\langle 3,7\rangle &= (12)(37645) \\
\langle 1,1\rangle\langle 1,2\rangle\langle 3,3\rangle\langle 3,4\rangle\langle 4,5\rangle\langle 3,6\rangle\langle 3,7\rangle &= (12)(37645) \\
&\Rightarrow \mathsf{Cyc}(\sigma) = \{1,3\}
\end{aligned}$$

Now let us look at $\mathsf{Lmap}(F(\sigma))$.

Lemma 3.2.5. *As $F = B \circ C \circ R \circ A$, suppose that $C \circ R \circ A(\sigma) = b_1 b_2 \ldots b_n \in \mathsf{CC}_n$ and let $B(b_1 \ldots b_n) = \tau = \tau_1 \ldots \tau_n$. then*

$$\mathsf{Lmap}(\tau) = \{n - k + 1 \in [n] : \omega_k = 0\},$$

where $\omega = D(b_1 \ldots b_n)$.

Proof. Our strategy is as follows. Firstly, from b_i we will see which letters be in $\mathsf{Lmal}(\tau)$. Secondly we will trace the position of i via the function D.

From property of B we know after its action there are b_{n-i+1} numbers greater than i and to the left of i. Hence $i \in \mathsf{Lmal}(\tau)$ if and only if $b_{n-i+1} = 0$.

We need to know where does B send these i's. We need to see the action of $B = B^1 \circ \cdots \circ B^n$ on $123 \ldots n$, as well as $D = D^n \circ \cdots \circ D^1$ on $b_1 \ldots b_n$. Note first that if $i \in [n]$ is at the j-th place of $123 \ldots n$ from the left, then b_{n-i+1} is at the j-th place of $b_1 \ldots b_n$ from the right.

Now let us look at the first step. B^n will move n of $123 \ldots n$ to the right by distance b_1, and D^1 will move b_1 to the left by distance b_1. Since the distance moved are of the same, hence the following property hold: For any $i \in [n]$, if i is at the j-th place from the left in $B^n(123 \ldots n)$, Then b_{n-i+1} is at the j-th placce from the right in $D^1(b_1 \ldots b_n)$.

We generalize the above observation. From $k = n-1$ to $k = 1$, the map B^k will move k of $B^{k+1} \circ \cdots \circ B^n(123 \ldots n)$ to the right by distance b_{n-k+1}, and D^{n-k+1} moves b_{n-k+1} to the left by distance b_{n-k+1}. Then, for any $i \in [n]$, if i is at the j-th place from the left in $B^k \circ \cdots \circ B^n(123 \ldots n)$, then b_{n-i+1} is at the j-th place from the right in $D^{n-k+1} \circ \cdots \circ D^1(b_1 \ldots b_n)$.

Therefore after the action of B on $123 \ldots n$ and that of D on $b_1 \ldots b_n$, we know that for any $i \in [n]$, if it is at the j-th place from the left in $B(123 \ldots n)$, then b_{n-i+1} is at the j-th place from the right in $D(b_1 \ldots b_n)$.

Now we can complete our proof. Since $i \in \mathsf{Lmal}(\sigma)$ iff $b_{n-i+1} = 0$, hence

$$\begin{aligned} \mathsf{Lmap}(B(b_1 \ldots b_n)) &= \{k \in [n] : \tau_k \in \mathsf{Lmal}(\tau)\} \\ &= \{k \in [n] : b_{n-\tau_k+1} = 0\}. \end{aligned}$$

We know that τ_k is at the k-th place from the left in $B(b_1 \ldots b_n) = \tau$, hence $b_{n-\tau_k+1}$ is at the k-th place from the right in $D(b_1 \ldots b_n)$, or $(n-k+1)$-th place from the left. Let $D(b_1 \ldots b_n) = \omega = \omega_1 \ldots \omega_n$, we then have

$$b_{n-\tau_k+1} = \omega_{n-k+1},$$

and

$$\begin{aligned} \{k \in [n] : b_{n-\tau_k+1} = 0\} &= \{k \in [n] : \omega_{n-k+1} = 0\} \\ &= \{n-k+1 \in [n] : \omega_k = 0\} \end{aligned}$$

That is,

$$\mathsf{Lmap}(B(b_1 \ldots b_n)) = \{n-k+1 \in [n] : \omega_k = 0\},$$

where $\omega = D(b_1 \ldots b_n)$. This ends the proof. □

Example 3.2.6. $\tau = 00104205 \equiv c_8c_7c_6c_5c_4c_3c_2c_1$

$$\begin{array}{lll||lll}
(\mathbf{0}0104205) & B_8(12345678) = & 1234567\mathbf{8} & D_8 \Rightarrow & \mathbf{0}0104205 & \equiv \mathbf{c_8}c_7c_6c_5c_4c_3c_2c_1 \\
(0\mathbf{0}104205) & B_7(12345678) = & 123456\mathbf{78} & D_7 \Rightarrow & 0\mathbf{0}104205 & \equiv c_8\mathbf{c_7}c_6c_5c_4c_3c_2c_1 \\
(00\mathbf{1}04205) & B_6(12345678) = & 123457\mathbf{68} & D_6 \Rightarrow & 0\mathbf{1}004205 & \equiv c_8\mathbf{c_6}c_7c_5c_4c_3c_2c_1 \\
(001\mathbf{0}4205) & B_5(12345768) = & 1234\mathbf{5}768 & D_5 \Rightarrow & 010\mathbf{0}4205 & \equiv c_8c_6c_7\mathbf{c_5}c_4c_3c_2c_1 \\
(0010\mathbf{4}205) & B_4(12345768) = & 1235768\mathbf{4} & D_4 \Rightarrow & \mathbf{4}0100205 & \equiv \mathbf{c_4}c_8c_6c_7c_5c_3c_2c_1 \\
(00104\mathbf{2}05) & B_3(12357684) = & 1257\mathbf{3}684 & D_3 \Rightarrow & 401\mathbf{2}0005 & \equiv c_4c_8c_6\mathbf{c_3}c_7c_5c_2c_1 \\
(001042\mathbf{0}5) & B_2(12573684) = & 1\mathbf{2}573684 & D_2 \Rightarrow & 401200\mathbf{0}5 & \equiv c_4c_8c_6c_3c_7c_5\mathbf{c_2}c_1 \\
(0010420\mathbf{5}) & B_1(12573684) = & 25736\mathbf{1}84 & D_1 \Rightarrow & 40\mathbf{5}12000 & \equiv c_4c_8\mathbf{c_1}c_6c_3c_7c_5c_2
\end{array}$$

Proof of Theorem 3.2.1. For $\sigma \in \mathfrak{S}_n$, let $A(\sigma) = a_1 \dots a_n$ and from Lemma 3.2.2 we know

$$\mathsf{Cyc}(\sigma) = \{i \in [n] : a_i = 0\}.$$

Since $R(A(\sigma)) = a_n \dots a_1$ and $D \circ C = id_{\mathsf{RCC}_n}$, we have $D \circ C \circ R \circ A(\sigma) = a_n \dots a_1$. Now let $D(C \circ R \circ A(\sigma)) = \omega = \omega_1 \dots \omega_n$ and we will have

$$\begin{aligned}
\{i \in [n] : a_i = 0\} &= \{n-i+1 \in [n] : a_{n-i+1} = 0\} \\
&= \{n-i+1 \in [n] : \omega_i = 0\}
\end{aligned}$$

Plug $C \circ R \circ A(\sigma)$ into $b_1 \dots b_n$ into

$$\mathsf{Lmap}(B(b_1 \dots b_n)) = \{n-k+1 \in [n] : \omega_k = 0\}$$

from Lemma 3.2.5, we have

$$\begin{aligned}
\{n-i+1 \in [n] : \omega_i = 0\} &= \mathsf{Lmap}(B(b_1 \dots b_n)) \\
&= \mathsf{Lmap}(B(C \circ R \circ A(\sigma))) \\
&= \mathsf{Lmap}(F(\sigma)).
\end{aligned}$$

Hence

$$\mathsf{Cyc}(\sigma) = \mathsf{Lmap}(F(\sigma))$$

and the proof is done. $\square$

3.3 $\mathsf{Lmap}(\sigma) = \mathsf{Rmil}(F(\sigma))$ and $\mathsf{Lmal}(\sigma) = \mathsf{Rmip}(F(\sigma))$

In this section we will prove the following.

Theorem 3.3.1. *We have* $\mathsf{Lmap}(\sigma) = \mathsf{Rmil}(F(\sigma))$ *and* $\mathsf{Lmal}(\sigma) = \mathsf{Rmip}(F(\sigma))$.

The proof is quite complicated and is feasible by way of $TA(\sigma)$. More precisely, our strategy is the following and each step is a Lemma.

(i) $\mathsf{Lmap}(\sigma) \subseteq \mathsf{Rmil}(TA(\sigma))$ and $\mathsf{Lmal}(\sigma) \subseteq \mathsf{Rmip}(TA(\sigma))$.

(ii) $\mathsf{Rmil}(TA(\sigma)) \subseteq \mathsf{Lmap}(\sigma)$ and $\mathsf{Rmip}(TA(\sigma)) \subseteq \mathsf{Lmal}(\sigma)$.

(iii) $\mathsf{Rmil}(TA(\sigma)) \subseteq \mathsf{Rmil}(F(\sigma))$ and $\mathsf{Rmip}(TA(\sigma)) \subseteq \mathsf{Rmip}(F(\sigma))$.

(iv) $\mathsf{Rmil}(F(\sigma)) \subseteq \mathsf{Rmil}(TA(\sigma))$ and $\mathsf{Rmip}(F(\sigma)) \subseteq \mathsf{Rmip}(TA(\sigma))$.

If so, then from first two items we have

$$\mathsf{Lmap}(\sigma) = \mathsf{Rmil}(TA(\sigma)), \qquad \mathsf{Lmal}(\sigma) = \mathsf{Rmip}(TA(\sigma)),$$

while from the last two items we have

$$\mathsf{Rmil}(TA(\sigma)) = \mathsf{Rmil}(F(\sigma)), \qquad \mathsf{Rmip}(TA(\sigma)) = \mathsf{Rmip}(F(\sigma))$$

And the proof is completed. □

For convienient's sake in the following write $TA(\sigma) = p_1 p_2 \dots p_n$ in 2-line notation

$$TA(\sigma) = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}.$$

Note that the 2-line notation are also applied on words in CC_n and RCC_n. We also define Rmip, Rmil of $TA(\sigma)$ by

$$\mathsf{Rmip}(TA(\sigma)) := \{i \in [n] : p_i < p_j, \text{ for } j = i+1, \dots, n\},$$

$$\mathsf{Rmil}(TA(\sigma)) := \{p_i : p_i < p_j, \text{ for } j = i+1, \dots, n\}.$$

Lemma 3.3.2. *We have*

$$\mathsf{Lmap}(\sigma) \subseteq \mathsf{Rmil}(TA(\sigma)), \qquad \mathsf{Lmal}(\sigma) \subseteq \mathsf{Rmip}(TA(\sigma)).$$

Proof. For $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$ we let $TA(\sigma) = p_1 \ldots p_n$. Suppose $i \in \mathsf{Lmap}(\sigma)$ for some $i \in [n]$ and meanwhile we have $\sigma_i \in \mathsf{Lmal}(\sigma)$, that is

$$\sigma = \begin{pmatrix} 1 \,\ldots\, i-1 & i & i+1 \,\ldots\, n \\ (<\sigma_i) & \sigma_i & \ldots \end{pmatrix}$$

For $\sigma_i \neq n$, the calculation of $TA(\sigma)$ is by way of 2-line notation of σ: starting from n, consider the position of n is p_n and exchange n with the letter in the n-th place. We can successively perform similar operation with respect to $n-1, n-2, \ldots 1$. Recall that when doing the exchange we are to exchange some letter a with some smaller letter to the right, or exchange a with some larger letter to the left. That is, when we compute p_n, since $\sigma_1, \ldots, \sigma_{i-1}$ are smaller than σ_i, the position of n is greater than i and therefore $p_n > i$. Also n will be exchanged with some smaller letter b to its right. The position of b will still be larger than i and the exchange will not change the facts that (i) σ_i is at the i-th position and (ii) letters to the left of $\sigma(i)$ are all smaller than $\sigma(i)$. Recurrently we have that $p_n, p_{n-1}, \ldots, p_{\sigma_i+1}$ are all greater than i; and when computing p_{σ_i} the letter σ_i remains at the i-th position. Therefore $p_{\sigma_i} = i$ and

$$TA(\sigma) = \begin{pmatrix} \cdots & \sigma_i & \sigma_i + 1 \,\ldots\, n \\ \cdots & i & (>i) \end{pmatrix}$$

Therefore $i \in \mathsf{Rmil}(TA(\sigma))$ and $\sigma_i \in \mathsf{Rmip}(TA(\sigma))$, that is

$$\mathsf{Lmap}(\sigma) \subseteq \mathsf{Rmil}(TA(\sigma)), \qquad \mathsf{Lmal}(\sigma) \subseteq \mathsf{Rmip}(TA(\sigma))$$

For $\sigma_i = n$, we have $p_n = i$, that is

$$TA(\sigma) = \begin{pmatrix} \cdots & \sigma_i & \\ \cdots & i & \end{pmatrix}$$

Therefore $i \in \mathsf{Rmil}(TA(\sigma))$ and $\sigma_i \in \mathsf{Rmip}(TA(\sigma))$, that is

$$\mathsf{Lmap}(\sigma) \subseteq \mathsf{Rmil}(TA(\sigma)), \qquad \mathsf{Lmal}(\sigma) \subseteq \mathsf{Rmip}(TA(\sigma))$$

and the proof is completed. $\square$

Now we look at the other direction of inclusions.

Lemma 3.3.3. *We have*

$$\mathsf{Rmil}(TA(\sigma)) \subseteq \mathsf{Lmap}(\sigma), \qquad \mathsf{Rmip}(TA(\sigma)) \subseteq \mathsf{Lmal}(\sigma).$$

Proof. For $TA(\sigma) = p_1 \dots p_n$, suppose $i \in \mathsf{Rmip}(\sigma)$ for some $i \in [n]$ and $p_i \in \mathsf{Rmil}(\sigma)$. That is,

$$TA(\sigma) = \begin{pmatrix} \cdots & i & i+1 \, \dots \, n \\ \cdots & p_i & (> p_i) \end{pmatrix}$$

We reverse the steps of computing $TA(\sigma)$ and compute σ from $TA(\sigma)$. Starting from $123 \dots n$. As 1 is to move to p_1-th position and since $1 \leq p_i \leq i$, hence the letter 1 (which is at the first place) didn't move rightward and the letters to the right of 1 remain in their places.

Similarly, letters $i, \dots, n$ remain intact until move $i-1$ to p_{i-1}-th position. The next step will move i leftword to p_i-th position and here those letters greater than i will lie to the right of i and those to the left of i are smaller than i.

Now we move $i+1$ to the p_{i+1}-th place. Since $p_{i+1} > p_i$, we know that $i+1$ exchanges with some letter to the right of i, hence those to the left of i are still smaller than i. Similarly when we move $i+2, \dots, n$, those to the left of i keep smaller than i. Therefore in the end we have

$$\sigma = \begin{pmatrix} \cdots & p_i & p_i + 1 \dots n \\ (< i) & i & \begin{pmatrix} \cdots \\ i+1 \dots n \end{pmatrix} \end{pmatrix}$$

and $i \in \mathsf{Lmal}(\sigma), p_i \in \mathsf{Lmap}(\sigma)$. $\square$

Now we prove the third lemma.

Lemma 3.3.4. *We have*

$$\mathsf{Rmil}(TA(\sigma)) \subseteq \mathsf{Rmil}(F(\sigma)), \qquad \mathsf{Rmip}(TA(\sigma)) \subseteq \mathsf{Rmip}(F(\sigma))$$

Proof. Since $1 \leq p_j \leq j$, we suppose $i + x \in \mathsf{Rmip}(TA(\sigma))$ and $p_{i+x} = i \in \mathsf{Rmil}(TA(\sigma))$ for some $x \geq 0$. That is

$$TA(\sigma) = \begin{pmatrix} \cdots & i+x & i+x+1 \, \dots \, n \\ \cdots & i & (\quad > i \quad) \end{pmatrix}.$$

We let $A(\sigma) = a_1 \dots a_n$. Since $a_i = i - p_i$ for $i \in [n]$ and $p_j > i \Rightarrow a_j = j - p_j < j - i$ for $j = i+x+1, \dots, n$, we have

$$A(\sigma) = \begin{pmatrix} \cdots & i+x & i+x+1 & i+x+2 & \cdots & n \\ \cdots & x & < x+1 & < x+2 & \cdots & < n-i \end{pmatrix}.$$

As $R \circ A(\sigma) = a_n \dots a_1$, we have

$$R \circ A(\sigma) = \begin{pmatrix} 1 & \dots & n-(i+x+1)+1 & n-(i+x)+1 & \dots \\ < n-i & \dots & < x+1 & x & \dots \end{pmatrix}.$$

Note that $C^{n-(i+x)+2} \circ \dots \circ C^n$ is to move rightward those letters to the right of x (which is at the $(n-(i+x)+1)$-th position) and it will not affect letters at the $1, \dots, n-(i+x)+1$ positions. Hence we have

$$C^{n-(i+x)+2} \circ \dots \circ C^n(R \circ A(\sigma)) =$$
$$\begin{pmatrix} 1 & \dots & n-(i+x+1)+1 & n-(i+x)+1 & \dots \\ < n-i & \dots & < x+1 & x & \dots \end{pmatrix}.$$

Similarly note that $C^{n-(i+x)+1}$ moves x (at the $(n-(i+x)+1)$-th position) to the right by x and leaves it at the $n-(i+x)+1+x = (n-i+1)$-th position. That is

$$C^{n-(i+x)+1} \circ \dots \circ C^n(R \circ A(\sigma)) =$$
$$\begin{pmatrix} 1 & \dots & n-(i+x+1)+1 & \dots & n-i+1 & \dots \\ < n-i & \dots & < x+1 & \dots & x & \dots \end{pmatrix}.$$

Again, $C^{n-(i+x)}$ moves rightward the letter at the $(n-(i+x))$-th position. But as the distance it moves is smaller than $x+1$ it will not affect x and x remains intact. Again $C^{n-(i+x)-1}$ moves rightward the letter at the $(n-(i+x)-1)$-th position by a distance less than $x+2$. The x is still at the $(n-i+1)$-th position and $C^{n-(i+x)-1}$ will not allow any letter to move to the right of x.

Successively by the same argument we obtain that, those letters to the left of x (which is at the $n-(i+x)+1$-th position in $R \circ A(\sigma)$) will not move to the right of x (which is moved by C to the $(n-i+1)$-th position). Hence, in $C \circ R \circ A(\sigma)$ the $(n-i+1)$-th letter is exactly x and we have

$$C(R \circ A(\sigma)) = \begin{pmatrix} \dots & n-i+1 & \dots \\ \dots & x & \dots \end{pmatrix}$$

Moreover, we will see those letters to the right of x (which itself is at the $(n-i+1)$-th position of $C \circ R \circ A(\sigma)$). We use function D.

First, $D^{n-i} \circ \dots \circ D^1$ is to move leftward those letters to the left of x (at the $(n-i+1)$-th position) and it will not affect x and those letters to its right. That is,

$$D^{n-i}\circ\cdots\circ D^{1}(C\circ R\circ A(\sigma)) = \begin{pmatrix} \dots & n-i+1 & \dots \\ \dots & x & \dots \end{pmatrix}.$$

Now, D^{n-i+1} moves the $(n-i+1)$-position letter x to the left with distance x, and x is now at the position $n-(i+x)+1$. This is reasonable since we suppose $i+x \in \mathsf{Rmip}(TA(\sigma))$ and therfore $i+x \in [n]$ and $n-(i+x)+1 > 0$. Currently we have

$$D^{n-i+1}\circ\cdots\circ D^{1}(C\circ R\circ A(\sigma)) = \\ \begin{pmatrix} \dots & n-(i+x)+1 & \dots & n-i+2 & \dots & n \\ \dots & x & \dots & (& ? &) \end{pmatrix}.$$

Hence this x is back to its position in $R\circ A(\sigma)$.

Now we claim that the $(n-i+1+j)$-th position letter in $C\circ R\circ A(\sigma)$ is smaller than $x+j$. Suppose not, then this letter will move from the right of x to the left of x after the action of D on $C\circ R\circ A(\sigma)$. This will make x not be at the $(n-(i+x)+1)$-th position of $D(C\circ R\circ A(\sigma))$. But by the previous lemma we have $D\circ C = id_{\mathsf{RCC}_n}$ and $D(C\circ R\circ A(\sigma)) = R\circ A(\sigma)$, which means x is at the $(n-(i+x)+1)$-th position, a contradiction.

Hence we know that in $C\circ R\circ A(\sigma)$, the letter at the $(n-i+1+j)$-th position will be less than $x+j$. That is,

$$C(R\circ A(\sigma)) = \begin{pmatrix} \dots & n-i+1 & n-i+2 & n-i+3 & \dots & n \\ \dots & x & <x+1 & <x+2 & \dots & <x+i-1 \end{pmatrix}.$$

Now we investigate the effect of B on $123\dots n$. $B^{n-(n-i)+1}\circ\cdots\circ B^{n}$ moves $i+1,\dots,n$ to the right and does not affect the positions of i and those letters to the left of i, namely

$$B^{i+1}\circ\cdots\circ B^{n}(123\dots n) = \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ 1 & \dots & i & & \dots & \end{pmatrix}.$$

Note that B^{i} is to move i to the right with distance x to the $(i+x)$-th position, that is

$$B^{i}\circ\cdots\circ B^{n}(123\dots n) = \begin{pmatrix} 1 & \dots & i-1 & \dots & i+x & \dots \\ 1 & \dots & i-1 & \dots & i & \dots \end{pmatrix}.$$

Similarly B^{i-1} moves $i-1$ to the right by distance $< x+1$, hence $i-1$ will not move to the right of i, which means that i remains at the $(i+x)$-position. As B^{i-2} moves $i-2$ to the right by distance $< x+2$ and i stays at the $(i+x)$-th position, the letter $i-2$ will not move to the right of i and i remains at the $(i+x)$-th position. Recurrently by the same arguments the action of B will leave the letter i at the $(i+x)$-th position and leave those letters smaller than i staying to the left of i. That is

$$F(\sigma) = B(C \circ R \circ A(\sigma)) = \begin{pmatrix} \cdots & i+x & \cdots \\ \begin{pmatrix} \cdots \\ 1 \ldots i-1 \end{pmatrix} & i & \cdots \end{pmatrix},$$

which means $i \in \mathsf{Rmil}(F(\sigma))$ and $i+x \in \mathsf{Rmip}(F(\sigma))$. Hence

$$\mathsf{Rmil}(TA(\sigma)) \subseteq \mathsf{Rmil}(F(\sigma))$$

and

$$\mathsf{Rmip}(TA(\sigma)) \subseteq \mathsf{Rmip}(F(\sigma)).$$

$\square$

The last lemma we need is the following.

Lemma 3.3.5. *We have*

$$\mathsf{Rmil}(F(\sigma)) \subseteq \mathsf{Rmil}(TA(\sigma)), \qquad \mathsf{Rmip}(F(\sigma)) \subseteq \mathsf{Rmip}(TA(\sigma)).$$

Proof. Let $i \in \mathsf{Rmil}(F(\sigma))$. It is easy to see that the position of i is greater or equal to i (If not, then there exists a letter in $1, \ldots, i-1$ which is to the right of i and then we have $i \notin \mathsf{Rmil}(F(\sigma))$, which is absurd.)

We suppose the position of i is $i+x$ for some $x \geq 0$ and $i+x \in \mathsf{Rmip}(F(\sigma))$. That means

$$F(\sigma) = B(C \circ R \circ A(\sigma)) = \begin{pmatrix} \cdots & i+x & \cdots \\ \begin{pmatrix} \cdots \\ 1 \ldots i-1 \end{pmatrix} & i & \cdots \end{pmatrix}.$$

We investigate $C \circ R \circ A(\sigma)$. From above we know that after the action of B, i moves to the $(i+x)$-th position and those letters smaller than i are to the left of i. Hence i does not move after moved B^i. We infer that B^i moves i by a distance x, so the $(n-i+1)$-th position in $C \circ R \circ A(\sigma)$ is x. That means

$$C \circ R \circ A(\sigma) = \begin{pmatrix} \cdots & n-i+1 & \cdots \\ \cdots & x & \cdots \end{pmatrix}.$$

Let us look the letters in $C \circ R \circ A(\sigma)$ and to the right of x (which is now at the $(n-i+1)$-position). If the letter at the $(n-i+1+j)$-th position is greater or equal to $x+j$, then under the action of B, the letter $i-j$ will move to the right of i and then the position of i in $F(\sigma)$ will not be the $(i+x)$-th, a contradiction. Hence the $(n-i+1+j)$-th letter in $C \circ R \circ A(\sigma)$ will be less than $x+j$ and we have

$$C \circ R \circ A(\sigma) = \begin{pmatrix} \dots & n-i+1 & n-i+2 & n-i+3 & \dots & n \\ \dots & x & < x+1 & < x+2 & \dots & < x+i-1 \end{pmatrix}.$$

We use function D again and apply D on $C \circ R \circ A(\sigma)$. $D^{n-i} \circ \cdots \circ D^1$ is to move leftward those letters to the left of x (which is at the $(n-i+1)$-th position) and keeps x and the letters to the right of x intact. Hence

$$\begin{aligned} &D^{n-i} \circ \cdots \circ D^1(C \circ R \circ A(\sigma)) = \\ &\quad \begin{pmatrix} \dots & n-i+1 & n-i+2 & n-i+3 & \dots & n \\ \dots & x & < x+1 & < x+2 & \dots & < x+i-1 \end{pmatrix}. \end{aligned}$$

Again D^{n-i+1} moves x (at the $(n-i+1)$-th position) to the left by x and results in

$$\begin{aligned} &D^{n-i+1} \circ \cdots \circ D^1(C \circ R \circ A(\sigma)) = \\ &\begin{pmatrix} \dots & n-(i+x)+1 & \dots & n-i+2 & n-i+3 & \dots & n \\ \dots & x & \dots & < x+1 & < x+2 & \dots & < x+i-1 \end{pmatrix}. \end{aligned}$$

As D^{n-i+2} moves the letter at the $(n-i+2)$-th position to the left by a distance less than $x+1$ and so x (at the $(n-(i+x)+1)$-th position) will not change. Again D^{n-i+3} moves the letter at the $(n-i+3)$-th position to the left by a distance less than $x+2$ and will keep x intact. By the same arguments we know that the letters to the right of x (which is originally at the $(n-i+1)$-th position of $C \circ R \circ A(\sigma)$) will stay to the right of x (which is now at the $(n-(i+x)+1)$-th position after the action of D), as

$$D(C \circ R \circ A(\sigma)) = \begin{pmatrix} \dots & n-(i+x)+1 & \dots \\ \dots & x & \dots \end{pmatrix}.$$

Now we apply C on $D(C \circ R \circ A(\sigma))$ to observe those letters to the left of x (which is at the $(n-(i+x)+1)$-th position of $D(C \circ R \circ A(\sigma))$). Firstly, the effect of $C^{n-(i+x)+2} \circ \cdots \circ C^n$ is to move rightward those letters to the right of x, hence it does not affect x and the letters to the left of x. We have now

$$C^{n-(i+x)+2} \circ \cdots \circ C^n(D \circ C \circ R \circ A(\sigma)) = \begin{pmatrix} \dots & n-(i+x)+1 & \dots \\ \dots & x & \dots \end{pmatrix}.$$

Next, $C^{n-(i+x)+1}$ move x (at the $(n-(i+x)+1)$-th position) to the right by x and results in the $(n-i+1)$-th position. Now we have

$$C^{n-(i+x)+1} \circ \cdots \circ C^n(D \circ C \circ R \circ A(\sigma)) = \\ \begin{pmatrix} 1 & \ldots & n-(i+x) & \ldots & n-i+1 & \ldots \\ (& ? &) & \ldots & x & \ldots \end{pmatrix}$$

Note that x comes back to the position it was originally in $C \circ R \circ A(\sigma)$.

We claim below that the letter at the $(n-(i+x)+1-j)$-th position of $D \circ C \circ R \circ A(\sigma)$ will be less than $x+j$. Suppose not, then this number will move from the left of x to the right of x after the action of C on $D \circ C \circ R \circ A(\sigma)$. This will make x not be at the $(n-i+1)$-th position of $C(D \circ C \circ R \circ A(\sigma))$. However from the previous lemma we know $C \circ D = id_{\mathsf{CC}_n}$ and $C(D \circ C \circ R \circ A(\sigma)) = C \circ R \circ A(\sigma)$, so x will be at the $(n-i+1)$-th position of $C(D \circ C \circ R \circ A(\sigma))$, a contradiction.

Therefore we obtain that the letter at the $(n-(i+x)+1-j)$-th position of $D \circ C \circ R \circ A(\sigma)$ will be less than $x+j$. That is,

$$D \circ C \circ R \circ A(\sigma) = \\ \begin{pmatrix} 1 & \ldots & n-(i+x+2)+1 & n-(i+x+1)+1 & n-(i+x)+1 & \ldots \\ < n-i & \ldots & < x+2 & < x+1 & x & \ldots \end{pmatrix}.$$

From lemma we know $D \circ C = id_{\mathsf{RCC}_n}$ and hence $D \circ C \circ R \circ A(\sigma) = R \circ A(\sigma)$ and we have

$$R \circ A(\sigma) = \\ \begin{pmatrix} 1 & \ldots & n-(i+x+2)+1 & n-(i+x+1)+1 & n-(i+x)+1 & \ldots \\ < n-i & \ldots & < x+2 & < x+1 & x & \ldots \end{pmatrix}.$$

Since the inverse of R is R itself, we consider $R \circ R \circ A$ and obtain

$$A(\sigma) = \begin{pmatrix} \ldots & i+x & i+x+1 & i+x+2 & \ldots & n \\ \ldots & x & < x+1 & < x+2 & \ldots & < n-i \end{pmatrix}.$$

Let $A(\sigma) = a_1 \ldots a_n$, $TA(\sigma) = p_1 \ldots p_n$. By defitnion $p_{i+x} = (i+x) - a_{i+x} = (i+x) - x = i$ and $a_{i+x+j} < x+j \Rightarrow p_{i+x+j} = (i+x+j) - a_{i+x+j} > (i+x+j) - (x+j) = i$, which means

$$TA(\sigma) = \begin{pmatrix} \ldots & i+x & i+x+1 & \ldots & n \\ \ldots & i & (& > i &) \end{pmatrix}.$$

That is, $i \in \mathsf{Rmil}(TA(\sigma))$, and $i + x \in \mathsf{Rmip}(TA(\sigma))$. Hence

$$\mathsf{Rmil}(TA(\sigma)) \supseteq \mathsf{Rmil}(F(\sigma)),$$

$$\mathsf{Rmip}(TA(\sigma)) \supseteq \mathsf{Rmip}(F(\sigma))$$

and we are done. $\square$

3.4 Proof of the main result

Now all the ingredients for the proof of the main theorem are ready.

Proof of Theorem 1.6.1 We define $F : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$F := B \circ C \circ R \circ A,$$

and for $\sigma \in \mathfrak{S}_n$ we have $\mathsf{sort}(\sigma) = \mathsf{inv}(F(\sigma))$ (Theorem 3.1.1), $\mathsf{Cyc}(\sigma) = \mathsf{Lmap}(F(\sigma))$ (Theorem 3.2.1) $\mathsf{Lmap}(\sigma) = \mathsf{Rmil}(F(\sigma))$ and $\mathsf{Lmal}(\sigma) = \mathsf{Rmip}(F(\sigma))$ (Theorem 3.3.1). Hence

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal}) \sim (\mathsf{inv}, \mathsf{Lmap}, \mathsf{Rmil}, \mathsf{Rmip})$$

and the proof is completed. $\square$

Chapter 4

Discussion and concluding remarks

It is somewhat unexpected and disappointing to us that after the completion of this work we find our result can be recovered also by combining that of Chen, Gong and Guo [4] and Eu, Lo and Wong [8] together with the inverse mapping between $\mathfrak{S}_n$. We explain it here.

First of all, from the work of Eu, Lo and Wong [8] we know

$$(\mathsf{sort}, \mathsf{Cyc}, \mathsf{Lmap}, \mathsf{Lmal})\sigma = (\mathsf{inv}, \mathsf{Rmil}, \mathsf{Lmap}, \mathsf{Lmal})\phi^{-1}\sigma$$

for all $\sigma \in \mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, if $i \in \mathsf{Lmal}(\sigma)$ then $\sigma_j < \sigma_i$ for $j < i$. We look at σ^{-1}. As $\sigma^{-1}(\sigma_i) = i$, which means in σ^{-1} the σ_i-th position is i. Moreover in σ^{-1} the position of any letter j smaller than i in σ^{-1} will be less than σ_i. In other words, in σ^{-1} the letters smaller than i will be to the left of i, hence $i \in \mathsf{Rmip}(\sigma^{-1})$.

On the other hand, suppose $i \in \mathsf{Rmip}(\sigma^{-1})$, means that for any j with ${\sigma^{-1}}_i > {\sigma^{-1}}_j$ we have $i > j$. Look at $(\sigma^{-1})^{-1} = \sigma$ and we know that in σ the position of i is ${\sigma^{-1}}_i$ and for any j to the left of i in σ there is ${\sigma^{-1}}_j < {\sigma^{-1}}_i$ and hence $i > j$. In other words, in σ the letters to the left of i is smaller than i and $i \in \mathsf{Lmal}(\sigma)$.

Combine the above two paragraphs we get

$$\mathsf{Lmal}(\sigma) = \mathsf{Rmip}(\sigma^{-1}).$$

Together with Theorem 1.5.2 we obtain

$$\begin{array}{ccccc} \mathfrak{S}_n & \xrightarrow{\phi^{-1}} & \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n \\ \begin{pmatrix} \mathsf{sort} \\ \mathsf{Cyc} \\ \mathsf{Lmap} \\ \mathsf{Lmal} \end{pmatrix} & & \begin{pmatrix} \mathsf{inv} \\ \mathsf{Rmil} \\ \mathsf{Lmap} \\ \mathsf{Lmal} \end{pmatrix} & & \begin{pmatrix} \mathsf{inv} \\ \mathsf{Lmap} \\ \mathsf{Rmil} \\ \mathsf{Rmip} \end{pmatrix} \end{array},$$

which is exactly what we've done in this paper.

However, both proofs of Chen, Gong and Guo [4] and Eu, Lo and Wong [8] are rather involved and so far it is still not clear to us that if our bijection is equivalent to the composition above. Computer experiment shows until $n = 5$ both bijections are equivalent. We hope to clarify it in the near future.

Problem 1. *Is our bijection F equivalent to $\phi^{-1} \circ i$?*

A natural question arises once we have a bijection on $\mathfrak{S}_n$. If we apply F continuously, it is clear that each σ belongs to an orbit

$$\sigma \to F(\sigma) \to F \circ F(\sigma) \to \cdots \to F \circ F \circ \cdots \circ F(\sigma) \to \sigma.$$

Elements in each orbit are equivalent, and the set $\mathfrak{S}_n$ can be partitioned into equivalent classes (namely, orbits) under this equivalent relation. Hence it is natural to ask what the orbit structure is. A computer program gives the following data.

n	i	numbers of orbits with length i in $\mathfrak{S}_n$
3	1	3
	3	1
4	1	6
	3	3
	9	1
5	1	10
	3	9
	9	2
	12	1
	19	1
	34	1
6	1	20
	3	25
	4	1
	9	6
	12	2
	19	2
	34	2
	136	1
	142	1
	145	1

Problem 2. *What is the orbit structure $\mathfrak{S}_n$ under this equivalent relation?*

n	i	numbers of orbits with length i in $\mathfrak{S}_n$
7	1	35
	3	68
	4	3
	7	1
	9	18
	12	6
	13	1
	14	2
	19	6
	24	1
	33	1
	34	6
	61	1
	125	1
	136	2
	142	2
	145	2
	206	1
	2894	1

Conjecture 1. *There are* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ *fixed points of* F *on* $\mathfrak{S}_n$.

Conjecture 2. *The fixed points of* F *are the permutations which are both involution and become a SYT which has less than two rows by RSK algorithm.*

In enumerative combinatorics an important goal is the generating function. For our case it seems not trivial to write down the generating function, as three of the statistics are set-valued and should be encoded by nomomials.

Problem 3. *What can we say about the generating function*

$$F(x,y,z,w) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathsf{sort}(\sigma)} y^{\mathsf{Cyc}(\sigma)} z^{\mathsf{Lmap}(\sigma)} w^{\mathsf{Lmal}(\sigma)}.$$

The final problem arises from many instances in the study of statistic. In the study of permutation statistics sometimes an equidistribution identity over $\mathfrak{S}_n$ holds also for some proper subset $A \subset \mathfrak{S}_n$.

Problem 4. *Can our equidistribution result also hold for some interesting proper subset* $A \subset \mathfrak{S}_n$*?*

We leave these problems for future study and also to interested readers.

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