§2 A Generalized Proximal Point Algorithm

In this section, we introduce a generalized proximal point algorithm for solving variational inequalities involving general set-valued operators. We first recall the concept of the strongly convex functions introduced by [28] and the some definitions of continuous property as follows.

Definition 2.1. Let C be a closed convex subset of X, f be differentiable on a neighborhood of C, and let ∇f be the gradient of f.

(i) f is *strongly convex* on C, if $\exists\, \alpha > 0$ s.t. $\forall\, x, y \in C$, $f(y) - f(x) \geq \langle y - x, \nabla f(x) \rangle + \alpha \|y - x\|^2$;

(ii) f is *convex* on C, if $\forall\, x, y \in C,\ \ f(y) - f(x) \geq \langle y - x, \nabla f(x) \rangle$.

Also, for a set-valued operator $T : X \longrightarrow X^*$, we shall say that

(iii) T is *upper semicontinuous* (*u.s.c.*) at $x \in X$, if for any open set G containing $T(x)$, there is some neighborhood $V(x)$ of x, such that $T(y) \subset G$ for all $y \in V(x)$;

(iv) T is (*l,w*)-*u.s.c.*, if T is u.s.c. from line segments in X to the weak topology of X^*;

(v) T is (*w,s*)-*u.s.c.*, if T is u.s.c. from the weak topology of X to the norm topology of X^*;

(vi) T is *Lipschitz continuous*, if there exists a constant $m > 0$ such that

$$\|w_1 - w_2\| \leq m\|u - v\|,\ \forall\ w_1 \in T(u), w_2 \in T(v).$$

To establish our proximal point algorithm, we consider a differentiable auxiliary function $M : C \longrightarrow R$, and denoted ∇M by M'. Define $B : C \times C \times C \longrightarrow R$ by

$$B(x, y, z) = \langle M'(x) - M'(y), z - x \rangle.$$

For some $x \in C$, $y \in T(x)$, and a positive number ε, we introduce the auxiliary problem $(P2)$ as follows.

$(P2)$: find $\tilde{x} \in C$ such that

$$\varepsilon\langle y, z - x \rangle + B(\tilde{x}, x, z) \geq 0, \forall z \in C. \tag{2}$$

In fact, if $\tilde{x}$ exists and equal to x, then $B(\tilde{x}, x, z) = 0$, which implies that $\tilde{x}$ is a solution of the original problem $(P1)$ (i.e. $(P2)$ reduces to $(P1)$).

Algorithm 1 to VI(T,C) :

Step 1: Take any $x_0 \in C$ and $y_0 \in T(x_0)$.

Step 2: Knowing (x_k, y_k) and ε_k, compute $x_{k+1} \in C$ such that

$$\varepsilon_k\langle y_k, z - x_{k+1} \rangle + B(x_{k+1}, x_k, z) \geq 0, \forall z \in C. \tag{3}$$

Step 3: Take any $y_{k+1} \in T(x_{k+1})$ and return to Step 2, until $\|x_{k+1} - x_k\|$ is below some threshold.

Remark. In particular, if $M(x) = \frac{1}{2}\|x\|^2$, then $M'(x) = \begin{cases} x, & \text{in a Hilbert space} \\ J(x), & \text{in a Banach space} \end{cases}$. In this case, our algorithm 1 reduces to the classical proximal point algorithm.

We also need the following well-known proposition, due to Karamardian [14, 17], see also Ortoga [20].

Proposition 2.2 [14, Proposition 6.1]. *Let f be a differentiable function on an open convex set C of X, then f is strongly convex on C with constant $\alpha \Longleftrightarrow \nabla f$ is strongly monotone with constant b on C, where $b = 2\alpha$.*

We give hereafter a lemma that will be used to prove the well-definedness of the sequence $\{x_k\}$ generated by our Algorithm 1.

Lemma 2.3. *Let $x \in C$ and T be weakly monotone with constant L. If M' is strongly monotone with constant b on C, and $\varepsilon < \frac{b}{L}$, then the operator $F(y) = \varepsilon T(y) + M'(y) - M'(x)$ is strongly monotone on C with constant $b - \varepsilon L$. Moreover if T is (l,w)-u.s.c. and M' is (l,w)-continuous, then F is also (l,w)-u.s.c..*

Proof : For all $(y_1, w_1),\ (y_2, w_2) \in G(F)$, we have some $z_1 \in T(y_1)$ and $z_2 \in T(y_2)$ such that
$$w_1 = \varepsilon z_1 + M'(y_1) - M'(x),$$
and
$$w_2 = \varepsilon z_2 + M'(y_2) - M'(x).$$
Since T is weakly monotone,
$$\langle z_1 - z_2, y_1 - y_2\rangle \geq -L\|y_1 - y_2\|^2,$$
and M' is strongly monotone,
$$\langle M'(y_1) - M'(y_2), y_1 - y_2\rangle \geq b\|y_1 - y_2\|^2.$$
Thus,
$$\begin{aligned}
\langle w_1 - w_2, y_1 - y_2\rangle &= \varepsilon\langle (z_1 - z_2) + M'(y_1) - M'(y_2), y_1 - y_2\rangle \\
&= \varepsilon\langle z_1 - z_2, y_1 - y_2\rangle + \langle M'(y_1) - M'(y_2), y_1 - y_2\rangle \\
&\geq \varepsilon(-L)\|y_1 - y_2\|^2 + b\|y_1 - y_2\|^2 \\
&= (b - \varepsilon L)\|y_1 - y_2\|^2\ ,
\end{aligned}$$
Hence F is strongly monotone. On the other hand, let G be open in the weak topology of X^* and $F(y) \subset G$, where $y \in L$, and L is a line segment in X. This implies that
$$\varepsilon T(y) \subset G + M'(x) - M'(y).$$
Since T and hence εT is (l, w)-u.s.c. at y, there is an open neighborhood $V_1(x)$ of X such that
$$\varepsilon T(z) \subset G + M'(x) - M'(y), \forall z \in V_1(x) \cap L.$$

Hence, for all $w \in \varepsilon T(z)$,

$$w \in G + M'(x) - M'(y),$$

which implies that

$$M'(y) \in G + M'(x) - w.$$

Since M' is (l, w)-continuous, there is an open neighborhood $V_2(x)$ of X such that

$$M'(u) \in G + M'(x) - w, \ \forall u \in V_2(x) \cap L.$$

Thus above results imply

$$w + M'(u) - M'(x) \in G, \ \forall w \in \varepsilon T(z), \ u \in V_2(x) \cap L, \ z \in V_1(x) \cap L.$$

Let $V(x) = V_1(x) \cap V_2(x)$. Then for all $z \in V(x) \cap L$, we have

$$F(y) = \varepsilon T(z) + M'(z) - M'(x) \subset G.$$

Thus, we complete the proof. □

Remark. Under different assumptions on T, the strongly monotone constants of F are different. For example,

(i) if T is monotone, then F is strongly monotone with constant b;

(ii) if T is strongly monotone with constant a, then F is strongly monotone with constant $\varepsilon a + b$;

(iii) if T has the Dunn property, then F is strongly monotone with constant b.