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Young functions on varifolds

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Abstract

In the thesis, we intend to study the convergence of pairs of surfaces and smooth functions thereon. To capture their limit, we study the convergence of pairs of integral varifolds and Young functions (a measure-theoretic model of surfaces with multiplicity and multiple-valued functions) via their associated graph measures on the product space. To take differentiability into account, we develop the notions of weak differentiability and bounded variation of Young functions; moreover, the compactness properties of pairs of integral varifolds and weakly differentiable or BV Young functions are established.

To this end, we study the topological vector structures of several test function spaces and introduce the concept of integral indecomposability—a notion of indecomposability tailored to our setting. Moreover, an existence theorem for integral decompositions of integral varifolds is established. The analysis of integral decompositions is carried out for a larger class of rectifiable varifolds, for which a compactness theorem analogous to the one for integral varifolds is obtained.

Keywords: varifolds, Young measures, multiple-valued functions, Young functions, graph measures, bounded variation, weak differentiability, compactness, indecomposability, decompositions

Contents

Chapter 1

Introduction

1.1 Motivation

Hypotheses. In this chapter, suppose m and n are positive integers, Y is a finite-dimensional Banach space, U is an open subset of $\mathbf{R}^n$, and V is an m-dimensional varifold in U such that $\|\delta V\|$ is a Radon measure. For terminology and notation, see 1.3.

Our study is motivated by the following question: In a Euclidean space, considering a sequence of smooth surfaces and smooth functions thereon such that over compact subsets, the area of the surfaces and the integrals of mean curvatures, the functions themselves and their derivatives are uniformly bounded, what is the limit of these pairs, and what can we say about the smoothness of the limit? Let us first focus on the first question. Note that varifolds (see [All72]) are a natural candidate to model the convergence of the surfaces to retain information on their mean curvature in the limit process. With this model, we allow surfaces to have multiplicity, and the following example shows that we should also allow functions to have multiple values. Consider two parallel lines approaching each other, and on each of them, the functions have constant values 1 and -1, respectively; in this case, the expected limit should consist of a line of density 2 and a constant two-valued function thereon. Therefore, we model such pairs with varifolds and Young functions as defined below.

Definition (see 3.2.4)**.** Suppose X and Z are locally compact Hausdorff spaces and μ is a Radon measure over X. By a μ *Young function f of type Z*, we mean a μ measurable $\mathbf{P}(Z)$-valued function f, where $\mathbf{P}(Z)$ denotes the space of probability Radon measures over Z endowed with the initial topology induced from the maps $\nu \mapsto \int f \,\mathrm{d}\nu$ for $\nu \in \mathbf{P}(Z)$ corresponding to continuous functions $f : Z \to \mathbf{R}$ with compact support. Moreover, if

$V \in \mathbf{IV}_m(U)$, we term a $\|V\|$ Young function f of type Z to be *integral* if and only if for $\|V\|$ almost all x, letting $k = \boldsymbol{\Theta}^m(\|V\|, x)$, there exist $z_1, z_2, \ldots, z_k \in Z$ such that $f(x) = k^{-1} \sum_{i=1}^{k} \boldsymbol{\delta}_{z_i}$.

Compared with Almgren's Q-valued functions (see [Alm00] or [DLS11]), the values of Young functions can be diffuse; also, the number of values can vary from one point to another as prescribed by the density. Note that $\mathscr{L}^n$ measurable Q-valued functions can be canonically identified as integral $\|V\|$ Young functions, where $V \in \mathbf{IV}_n(\mathbf{R}^n)$ satisfies $\|V\| = Q\mathscr{L}^n$. For pairs of measures and Young functions, we define their associated graph measures.

Definition (see 3.2.8)**.** Suppose X and Z are locally compact Hausdorff spaces, μ is a Radon measure over X, and f is a μ Young function of type Z. We define the graph measure $\mathbf{Y}(\mu, f)$ associated with μ and f to be the Radon measure over $X \times Z$ such that

$$\textstyle\int \phi \, \mathrm{d}\mathbf{Y}(\mu, f) = \int \phi(x, y) \, \mathrm{d}f(x)\, y \, \mathrm{d}\mu\, x$$

whenever $\phi : X \times Z \to \mathbf{R}$ is a continuous function with compact support.

If X and Z have countable bases and g is a μ measurable Z-valued function, then the formula $f(x) = \boldsymbol{\delta}_{g(x)}$ defines a μ Young function of type Z and $\mathbf{Y}(\mu, f)$ is the pushforward of μ by the map $x \mapsto (x, g(x))$. The convergence of pairs of measures and Young functions is then defined to be the weak convergence of their associated graph measures.

To answer the second question in the first paragraph, we should develop notions of differentiability for Young functions on varifolds. For $0 \le s < \infty$, denote by E_s the space of all continuously differentiable functions $\gamma : Y \to \mathbf{R}$ such that $\gamma(0) = 0$ and $\operatorname{spt} \mathrm{D}\, \gamma \subset \mathbf{B}(0, s)$, and endow $E = \bigcup \{E_s : 0 \le s < \infty\}$ with the locally convex final topology induced from the inclusions $E_s \to E$ corresponding to $0 \le s < \infty$, see 3.3.1. A notion of differentiability of Y-valued functions on varifolds was introduced by Menne.

Definition (see [Men16, 8.3] and [MS18, 4.2])**.** A $\|V\| + \|\delta V\|$ measurable Y-valued function g is termed to be *generalized* V *weakly differentiable* if and only if there exists a $\operatorname{Hom}(\mathbf{R}^n, Y)$-valued function $V \, \mathbf{D}\, g$ such that

$$\int_{K \cap \{x \,:\, |g(x)| \le s\}} \|V \, \mathbf{D}\, g\| \, \mathrm{d}\|V\| < \infty$$

whenever K is a compact subset of U and $0 \le s < \infty$, and such that

$$\begin{aligned} \textstyle\int \langle \theta(x), \mathrm{D}\, \gamma(g(x)) \circ V \, \mathbf{D}\, g(x) \rangle \, \mathrm{d}\|V\|\, x &= \textstyle\int \gamma(g(x)) \theta(x) \bullet \boldsymbol{\eta}(V, x) \, \mathrm{d}\|\delta V\|\, x \\ &\quad - \textstyle\int \gamma(g(x)) S \bullet \mathrm{D}\, \theta(x) \, \mathrm{d}V\,(x, S) \end{aligned}$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in E$. We denote by $\mathbf{T}(V, Y)$ the class of such functions g.

In the definition, we require the post-composition of g with functions $\gamma \in E$, instead of merely the function itself, to satisfy the integration-by-parts identity because otherwise, the resulting class of functions is not stable under post-composition (e.g., truncation), see [Men16, 8.27]. Let $W(V, Y)$ denote the space consisting of $g \in \mathbf{T}(V, Y)$ such that $g \in \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$ and $V\,\mathbf{D}\,g \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$; then, we will refer to members of $W(V, Y)$ as Y-valued V weakly differentiable functions. If $V \in \mathbf{IV}_n(\mathbf{R}^n)$ with $\|V\| = \mathscr{L}^n$, then $\mathbf{T}(V, \mathbf{R})$ agrees with the space $\mathscr{T}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ as in [BBG$^+$95, p. 244], where the letter "T" in the name of these spaces stands for truncation; moreover, the space $W(V, Y)$ agrees with the local Sobolev space $\mathbf{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n, Y)$. In stark contrast with $\mathbf{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n, Y)$, it may happen for general V that $f, g \in W(V, \mathbf{R})$ but $f + g \notin W(V, \mathbf{R})$ and $h = (f, g) \notin W(V, \mathbf{R}^2)$, see [Men16, 8.25]. In other words, weak differentiability of components does not imply that of the function itself. Such difficulties are also expected for our general setting; therefore, our definition of that concept for Young functions genuinely includes the case $\dim Y \geq 2$.

Our goal is to obtain a compactness theorem for pairs of integral varifolds and [generalized weakly differentiable] integral Young functions. Following Allard's approach (see [All72]) to the compactness theorem for integral varifolds, we will start with the theory of general Young functions which comprises definitions of different notions of differentiability for general Young functions.

1.2 Results and strategies

We will outline the results by chapter.

Chapter 2. In the theory of weakly differentiable functions, the behavior of functions is closely related to the connectedness properties of the domain; for instance, to establish a Poincaré inequality for weakly differentiable (single-valued or multiple-valued) functions on varifolds, we shall first determine the class of functions with zero derivatives; in fact, we expect that if $V \in \mathbf{IV}_m(U)$ and an integral $\|V\|$ Young function f which is also a $\|\delta V\|$ Young function has zero derivative, then there exists an integral decomposition of V into countably many integrally indecomposable varifolds W such that f is single-valued on each W. Therefore, we start with the integral indecomposability of integral varifolds before diving into the differentiability theory of Young functions. In [Men16, 6.2], the notion of indecomposability of general varifolds V was introduced by means of the distributional V

boundary of sets; more precisely, V is termed *indecomposable* if and only if there exists no $\|V\| + \|\delta V\|$ measurable set B such that $\|V\|(B) > 0$, $\|V\|(U \sim B) > 0$, and the *distributional V boundary $V\partial B$ of B* is identically zero. Accordingly, a plane of multiplicity 2 is indecomposable. For our purpose, we instead make the following definition.

Definition (see 2.2.1)**.** Suppose $V \in \mathbf{IV}_m(U)$. Then V is called *integrally indecomposable* if and only if there exists no $W \in \mathbf{IV}_m(U)$ such that $W \leq V$, $W \neq 0$, $V - W \neq 0$, $\|V\| = \|W\| + \|V - W\|$, and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$.

Roughly speaking, we allow an integral varifold V to be decomposed not only by restriction to subsets of zero distributional V boundary but also by peeling off sheets without producing extra boundary. For instance, a plane of multiplicity 2 can be integrally decomposed into two identical planes of multiplicity 1. It turns out that indecomposability and integral indecomposability are equivalent if $V \in \mathbf{IV}_m(U)$ has unit density $\|V\|$ almost everywhere and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, see 2.2.2 and 2.2.6. Therefore, just as decompositions, integral decompositions are not unique, see 2.3.9. The main goal of this chapter is to show the following existence theorem for integral decompositions:

Theorem A (see 2.3.12)**.** *Suppose $V \in \mathbf{IV}_m(U)$. Then, there exist a countable subset H of $\mathbf{IV}_m(U)$ and a function $\xi : H \to \{1, 2, \dots\}$ such that W is nonzero and integrally indecomposable whenever $W \in H$, and such that*

$$V(k) = \sum_{W \in H} \xi(W) W(k), \quad \|\delta V\|(\ell) = \sum_{W \in H} \xi(W) \|\delta W\|(\ell)$$

whenever $k : U \times \mathbf{G}(n, m) \to \mathbf{R}$ and $\ell : U \to \mathbf{R}$ are continuous functions with compact support.

Moreover, our result can be generalized to larger classes $\mathbf{RV}_m(U, C)$ of rectifiable varifolds V such that $\boldsymbol{\Theta}^m(\|V\|, x) \in C$ for $\|V\|$ almost all x, whenever C is a closed subset of $\mathbf{R}$ such that $\inf C \geq 1$ and $c + d \in C$ for $c, d \in C$; for these classes, we also prove a compactness theorem in an approach analogous to Allard's compactness theorem for integral varifolds, see 2.3.1, 2.3.3(1), and 2.3.4.

Chapter 3. Suppose X and Z are locally compact Hausdorff spaces that possess countable bases. Consider the class of Radon measures Γ over $X \times Z$ such that $\Gamma(K \times Z) < \infty$ whenever K is a compact subset of X, for which we study the compactness property and present a disintegration theorem, see 3.2.17 and 3.2.21. Consequently, we have the following compactness theorem for pairs of varifolds and Young functions.

Theorem B (see 3.2.24)**.** *Suppose $V \in \mathbf{IV}_m(U)$, V_i is a sequence in $\mathbf{IV}_m(U)$ such that $V = \lim_{i\to\infty} V_i$, and f_i is a sequence of $\|V_i\|$ Young function of type Y such that whenever K is a compact subset of U, there holds*

$$\lim_{s\to\infty} \sup\{\mathbf{Y}(\|V_i\|, f_i)(K \times (Y \sim \mathbf{B}(0,s))) : i \in \{1,2,\dots\}\} = 0.$$

Then, there exists a $\|V\|$ Young function f of type Y such that, possibly passing to a subsequence, we have

$$\lim_{i\to\infty} \mathbf{Y}(\|V_i\|, f_i) = \mathbf{Y}(\|V\|, f); \quad \lim_{i\to\infty} \mathbf{Y}(V_i, f_i \circ p) = \mathbf{Y}(V, f \circ p),$$

where $p : U \times \mathbf{G}(n,m) \to U$ is the projection map.

The boundedness condition assures that the mass of graph measures cannot escape to infinity; for instance, we exclude the case that $Y = \mathbf{R}$, $V_i = V \neq 0$, and f_i are constant functions with value $\boldsymbol{\delta}_i$ for each positive integer i.

Next, we define typical operations on the space of probability measures, such as pushforward, Cartesian product, and convolution, see 3.3.19, 3.3.20, and 3.3.25, respectively. Since Dirac-measure-valued μ Young functions of type Y must be of the form $\boldsymbol{\delta}_{g(\cdot)}$ for some μ measurable Y-valued function g, see 3.2.5, these operations acting on the class of Young functions then correspond to the post-composition, join[1], and addition of Y-valued functions. Note that the notion of addition of multiple-valued functions is also new to the theory of Q-valued functions, but the addition of two Q-valued functions will be a Q^2-valued function, see 3.3.22(3).

Finally, the definition of generalized V weakly differentiable Y-valued functions suggests that it is expedient to view Young functions of type Y as E^*-valued functions; indeed, we use E as the space of test functions in the target space of Young functions to define the notions of differentiability of Young functions in Chapter 4. With this in mind, we study the topological vector space structure of E and give a homeomorphic embedding of E into a space of continuous functions with compact support, see 3.4.19. Moreover, we use E to define a pseudo-metric on $\mathbf{P}(Y)$ and the notion of Lipschitz continuity for Young functions, see 3.3.6 and 3.3.9. To study the differentiability of Young functions via their associated graph measures, we define the space H of certain test functions $\eta : U \times Y \to \mathbf{R}^n$; again, a homeomorphic embedding of H into a space of continuous functions with compact support is established, see 3.4.25.

Chapter 4. To present the results in this chapter, we shall define the distributional derivatives of Young functions on varifolds.

[1]Suppose A, B, and C are sets and $f : A \to B$ and $g : A \to C$ are functions. By the join of f and g, we mean the function $A \to B \times C$ given by $a \mapsto (f(a), g(a))$.

Definition (see 4.1.7)**.** Whenever $0 \le s < \infty$ and f is a $\|V\| + \|\delta V\|$ Young function of type Y, there exists a distribution $T^s \in \mathscr{D}'(U, \mathbf{R}^n \otimes E_s)$ uniquely characterized by the requirement that

$$\begin{aligned} T^s_{(x)}(\theta(x) \otimes \gamma) = \textstyle\int \big(\int \gamma \,\mathrm{d} f(x)\big)\, \theta(x) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\|\, x \\ - \textstyle\int \big(\int \gamma \,\mathrm{d} f(x)\big)\, S \bullet \mathrm{D}\,\theta(x) \,\mathrm{d} V\,(x, S) \end{aligned}$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in E_s$. In this case, we say f possesses *generalized V bounded variation (GBV)* if and only if $\|T^s\|$ is a Radon measure whenever $0 \le s < \infty$, see 4.1.9.

If $Y = \mathbf{R}$ and g is a $\|V\| + \|\delta V\|$ measurable $\mathbf{R}$-valued function, such a notion is characterized in [MS25b, 4.2] via the distributional boundary of the subgraph of g; in this case, if f satisfies $f(x) = \boldsymbol{\delta}_{g(x)}$ for $x \in \operatorname{dmn} g$, then the two notions of GBV agree, see 4.1.11. Note that our definition allows for the case $\dim Y \ge 2$ which is new even for the single-valued case. For Young functions of type $\mathbf{R}$, a characterization analogous to the aforementioned one in the single-valued case is established, see 4.1.14. Next, we present the compactness theorem of GBV Young functions.

Theorem C (see 4.1.16)**.** *Additionally to the hypotheses of the compactness theorem for pairs of varifolds and Young functions, Theorem B, suppose that each f_i possesses generalized V_i bounded variation, and that whenever $0 \le s < \infty$, their associated distributions T_i^s satisfy*

$$\sup\{\|T_i^s\|(K) : i \in \{1, 2, \dots\}\} < \infty \quad \textit{whenever } K \textit{ is a compact subset of } U.$$

Then, we can require the limit Young function f in the conclusion to possess generalized V bounded variation.

From the compactness theorem of pairs of varifolds and Young functions, we only obtain a $\|V\|$ Young function f as the limit, and to assure the $\|\delta V\|$ measurability of f, we shall specify the value of f on the singular part of $\|\delta V\|$; we readily check that the choice of values does not affect whether f is GBV. However, in general we do not have weak convergence of derivatives; in fact, the limit function f may have non-zero derivative even if the derivatives of the functions f_i vanish. For instance, consider the example that V_i is the union of two disjoint open rays in $\mathbf{R}$ pointing in opposite directions with endpoints approaching each other; on each ray, f_i has constant values 1 and -1 respectively. In this example, each f_i has zero derivative and V_i has nontrivial first variation, but the first variation of the varifolds transfers through the limit to the variation of the limit function, see 4.1.17.

Next, we define generalized V weak differentiability of Young functions.

Definition (see 4.3.1)**.** We say a $\|V\| + \|\delta V\|$ Young function of type Y is *generalized* V *weakly differentiable* if and only if it possesses generalized V bounded variation and, whenever $0 \leq s < \infty$, the variation measure $\|T^s\|$ of its associated distributions T^s is absolutely continuous with respect to $\|V\|$.

In this case, there exists a $\|V\|$ almost unique $\|V\|$ measurable $(\mathbf{R}^n \otimes E)^*$-valued function F such that

$$T^s_{(x)}(\theta(x) \otimes \gamma) = \textstyle\int \langle \theta(x) \otimes \gamma, F(x) \rangle \, \mathrm{d}\|V\| \, x$$

whenever $0 \leq s < \infty$, $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, and $\gamma \in E_s$, see 4.3.2. Next, we present three examples. The first two thereof demonstrate the connection of our general theory to the main model cases. The third one illustrates a difficulty peculiar to the general setting.

Examples. Consider the isomorphism $(\mathbf{R}^n \otimes E)^* \simeq \operatorname{Hom}(\mathbf{R}^n, E^*)$ and let $v \in \mathbf{R}^n$.

(1) Suppose g is a generalized V weakly differentiable Y-valued function. Then the formula $f(x) = \boldsymbol{\delta}_{g(x)}$ for $x \in \operatorname{dmn} g$ defines a generalized V weakly differentiable Young function of type Y, and their derivatives $V \, \mathbf{D} \, g$ and F are related by the equation

$$\langle v, \textstyle\int \mathrm{D} \, \gamma(y) \circ V \, \mathbf{D} \, g(x) \, \mathrm{d} f(x) \, y \rangle = \langle \gamma, \langle v, F(x) \rangle \rangle \quad \text{whenever } \gamma \in E$$

for $\|V\|$ almost all x, see 4.3.5; for such x, we have

$$\operatorname{spt} \langle v, F(x) \rangle \subset \{g(x)\} = \operatorname{spt} f(x).$$

(2) Suppose Q is a positive integer, g is a Lipschitzian $\mathbf{Q}_Q(Y)$-valued function on $\mathbf{R}^n$, and $V \in \mathbf{IV}_n(\mathbf{R}^n)$ satisfies $\|V\| = Q \mathscr{L}^n$. Then the formula $f(x) = Q^{-1} \|g(x)\|$ for $x \in \mathbf{R}^n$ defines a generalized V weakly differentiable integral Young function of type Y, see 3.2.7 and 4.3.11. Based on [Men10, 2.5], it can be shown that there exists $G \in \mathbf{L}_\infty(\mathbf{Y}(\|V\|, f), \operatorname{Hom}(\mathbf{R}^n, Y))$ such that

$$\langle v, \textstyle\int \mathrm{D} \, \gamma(y) \circ G(x, y) \, \mathrm{d} f(x) \, y \rangle = \langle \gamma, \langle v, F(x) \rangle \rangle \quad \text{whenever } \gamma \in E$$

for $\|V\|$ almost all x; for such x, we have

$$\operatorname{spt} \langle v, F(x) \rangle \subset \operatorname{spt} g(x) = \operatorname{spt} f(x).$$

(3) Suppose $Y = \mathbf{R}^2$, $V \in \mathbf{IV}_n(\mathbf{R}^n)$ with $\|V\| = \mathscr{L}^n$, and $f : \mathbf{R}^n \to \mathbf{P}(\mathbf{R}^2)$ is defined by $f(x) = \mathscr{L}^2 \llcorner C$, where C denotes the unit square in $\mathbf{R}^2$. Then, f is generalized V weakly differentiable with $F = 0$. However, whenever $0 \neq X \in \mathscr{D}_C(\mathbf{R}^2, \mathbf{R}^2)$ with $\operatorname{div} X = 0$, its associated function $G : \mathbf{R}^n \times \mathbf{R}^2 \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^2)$ defined by $\langle v, G(x, y)\rangle = X(y)$ for $(x, y) \in \mathbf{R}^n \times \mathbf{R}^2$ satisfies

$$\langle v, \textstyle\int \mathrm{D}\,\gamma(y) \circ G(x, y) \,\mathrm{d} f(x)\, y\rangle = 0 = \langle \gamma, \langle v, F(x)\rangle\rangle \quad \text{whenever } \gamma \in E$$

for $\|V\|$ almost all x; for such x, we have

$$\operatorname{spt} G(x, \cdot) = \operatorname{spt} X \subset C = \operatorname{spt} f(x).$$

Such vector fields X can be constructed as in 3.4.21.

In the first two examples, for $\|V\|$ almost all x, there exists an $f(x)$ almost unique map $G(x, \cdot) : Y \to \operatorname{Hom}(\mathbf{R}^n, Y)$ representing $F(x) \in \operatorname{Hom}(\mathbf{R}^n, E^*)$ via the equation

$$\langle v, \textstyle\int \mathrm{D}\,\gamma(y) \circ G(x, y) \,\mathrm{d} f(x)\, y\rangle = \langle \gamma, \langle v, F(x)\rangle\rangle \quad \text{whenever } v \in \mathbf{R}^n \text{ and } \gamma \in E.$$

The third example shows that such representations are not unique for general Young functions. Initially, we were targeting only the concept of and compactness theorem for generalized V weak differentiability of integral Young functions; following the two model cases (i.e., the first two examples), we intended to work with the following property which, in retrospect, could be called strong generalized V weakly differentiability: it is satisfied by a $\|V\| + \|\delta V\|$ Young function f of type Y if and only if there exists $G \in \mathbf{L}_1^{\mathrm{loc}}(\mathbf{Y}(\|V\|, f), \operatorname{Hom}(\mathbf{R}^n, Y))$ such that

$$\begin{aligned} &\textstyle\int \langle \theta(x), \mathrm{D}\,\gamma(y) \circ G(x, y)\rangle \,\mathrm{d} f(x)\, y \,\mathrm{d}\|V\|\, x \\ &\qquad = \textstyle\int \big(\int \gamma \,\mathrm{d} f(x)\big)\, \theta(x) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\|\, x \\ &\qquad\quad - \textstyle\int \big(\int \gamma \,\mathrm{d} f(x)\big)\, S \bullet \mathrm{D}\,\theta(x) \,\mathrm{d} V\,(x, S) \end{aligned}$$

whenever $\gamma \in E$ and $\theta \in \mathscr{D}(U, \mathbf{R}^n)$. Although, as the third example shows, such functions G may fail to be unique, the main advantage of this definition would be to avoid heavy functional analytic machinery. However, we could not establish any compactness theorem. Thus, we have made a more functional analytic definition (the one employed in this thesis) which allows us to obtain a compactness theorem and which readily generalizes to include GBV Young functions in our treatment. Clearly, strong generalized weak differentiability implies generalized weak differentiability, but the converse is an open problem even if the Young function is associated with a Y-valued function. That is, if g is a $\|V\| + \|\delta V\|$ measurable Y-valued function and the Young function f defined by $f(x) = \boldsymbol{\delta}_{g(x)}$ for $x \in \operatorname{dmn} g$ is generalized V weakly differentiable, then it is unknown whether g is necessarily generalized V weakly differentiable; in fact, it is not even clear whether, for $v \in \mathbf{R}^n$, we are ensured that

$$\operatorname{spt}\langle v, F(x)\rangle \subset \operatorname{spt} f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

Finally, we shall present the compactness theorem for generalized V weakly differentiable Young functions.

Theorem D (see 4.3.9). *Additionally to the hypotheses of the compactness theorem for pairs of varifolds and Young functions, Theorem B, suppose that* $1 < p < \infty$, $\|\delta V_i\|$ *is absolutely continuous with respect to* $\|V_i\|$ *and* f_i *is generalized* V_i *weakly differentiable for positive integers* i, *and, whenever* K *is a compact subset of* U *and* $0 \le s < \infty$, *we have*

$$\sup\left\{ \textstyle\int_K |\mathbf{h}(V,x)|^p \,\mathrm{d}\|V_i\|\, x : i \in \{1,2,\dots\} \right\} < \infty$$
$$\sup\left\{ \textstyle\int_K \left|F_i(x)|(\mathbf{R}^n \otimes E_s)\right|^p \mathrm{d}\|V_i\|\, x : i \in \{1,2,\dots\} \right\} < \infty$$

where F_i *is the weak derivative of* f_i. *Then, we can further conclude that the limit* $\|V\|$ *Young function* f *in the conclusion is generalized* V *weakly differentiable.*

The additional condition on the first variation of varifolds excludes the aforementioned example of two rays, in which the limit Young function is not generalized V weakly differentiable but only GBV.

Future research. With the foundations of the theory of general Young functions, we shall address the subclass of integral Young functions, for instance, the class of generalized weakly differentiable [or GBV] integral Young functions; also, their compactness theorems should be established. To study such integral Young functions, we may need the representation theorem of derivatives, the constancy theorem, and Poincaré type inequalities.

A common scenario in our theory is that Young functions carry the geometric information of integral varifolds; for instance, Young functions may be the tangent map, the normal map, or height functions. Our study allows us to investigate how the geometric information is carried over to the limit; such a situation often occurs in the blow-up analysis of integral varifolds. On the other hand, the differentiability of Young functions may provide definitions of geometric quantities in the multiple-valued setting.

1.3 Terminology and notation

Mostly, the terminology and the notation agree with [Fed69] and [All72]. We also introduce additional notation and definitions.

Basic notation and definitions. The set of positive integers is denoted by $\mathscr{P}$, see [Fed69, 2.2.6]. The extended real number system is denoted by $\overline{\mathbf{R}}$, see [Fed69, 2.1.1]. The difference of sets A and B is denoted by $A \sim B$. The domain and image of a function f are denoted by $\operatorname{dmn} f$ and $\operatorname{im} f$. The topological closure and interior of a set A are denoted by $\operatorname{Clos} A$ and $\operatorname{Int} A$, respectively. The open and closed balls with center a and radius r are denoted by $\mathbf{U}(a, r)$ and $\mathbf{B}(a, r)$, respectively, see [Fed69, 2.8.1]. If (X, ρ) is a metric space, $A \subset X$, and $x \in X$, then the distance of x to A is defined by $\operatorname{dist}(x, A) = \inf\{\rho(x, a) : a \in A\}$. For integration, the alternate notations $\int f \,\mathrm{d}\mu$, $\int f(x) \,\mathrm{d}\mu\, x$ and $\mu(f)$ are employed; in this respect, μ integrability of f means that $\int f \,\mathrm{d}\mu$ is defined in $\overline{\mathbf{R}}$ and μ summability of f means that $\int f \,\mathrm{d}\mu \in \mathbf{R}$, see [Fed69, 2.4.2]. Inner products are denoted by $\bullet$, see [Fed69, 1.7.1]. If V is a vector space, $v \in V$, and $f \in \operatorname{Hom}(V, \mathbf{R})$, then we denote $\langle v, f \rangle = f(v)$.

Directed set and limit. We say a set A is directed by the order $\preceq$ if and only if $\preceq$ is a reflexive and transitive relation on A such that every subset of A with at most two members has an upper bound, see [Kel75, Chapter 2, p. 65]. A function f mapping a directed set A into a topological space X converge to $x \in X$ if and only if for each neighborhood U of x there exists $\alpha \in A$ such that $f(\beta) \in U$ whenever $\alpha \preceq \beta$, see [Kel75, Chapter 2, p. 66]; in this case, we denote $\lim_{\alpha \in A} f(\alpha) = x$. Similarly, in case $X = \overline{\mathbf{R}}$, we define $\limsup_{\alpha \in A} f(\alpha) = \inf\{\sup\{f(\beta) : \alpha \preceq \beta \in A\} : \alpha \in A\}$.

Numerical summation. Whenever A is a set and f is an $\overline{\mathbf{R}}$-valued function, $\sum_{x \in A} f(x)$ denotes the numerical sum of f over A; in case $A = \operatorname{dmn} f$, we abbreviate $\sum f = \sum_{x \in A} f(x)$, see [Fed69, 2.1.1].

Pushforward of measures. Suppose ϕ measures a set X, Z is a set, and f is a Z-valued function with $\operatorname{dmn} f \subset X$. The *pushforward of ϕ by f* is defined to be the measure $f_{\#}\phi$ over Z such that

$$(f_{\#}\phi)(B) = \phi(f^{-1}[B]) \quad \text{whenever } B \subset Z,$$

see [MS25a, 2.9, 2.10].

The space of locally summable functions. Suppose μ is a measure over an open subset U of $\mathbf{R}^n$ such that every open subset of U is μ measurable. We denote by $\mathbf{L}_1^{\mathrm{loc}}(\mu)$ the space of all real-valued functions f such that $\int_K |f| \,\mathrm{d}\mu < \infty$ whenever K is a compact subset of U. Taking a sequence of compact subsets K_i of U such that $U = \bigcup_{i=1}^{\infty} K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ for all

$i \in \mathscr{P}$, the space $\mathbf{L}_1^{\mathrm{loc}}(\mu)$ endowed with the pseudo-metric

$$\rho(f,g) = \sum_{i=1}^{\infty} 2^{-i} \inf \left\{ \textstyle\int_{K_i} |f-g| \,\mathrm{d}\mu, 1 \right\} \quad \text{for } f,g \in \mathbf{L}_1^{\mathrm{loc}}(\mu)$$

becomes a complete pseudo-metric space.

Locally convex final topology and inductive limits. We employ the terms symmetric, absorbent, locally convex space and locally convex topology, inductive limits of locally convex spaces, and strict inductive limit of locally convex spaces in accordance with [Bou89a, I, §4, No. 1, p. 31], [Bou87, I, §1, No. 5, Definition 4], [Bou87, II, §4, No. 1, Definition 1], [Bou87, II, §4, No. 4, Example II], and [Bou87, II, §4, No. 6], respectively. If F is the inductive limit of locally convex spaces F_α, then we denote $F = \varinjlim F_\alpha$.

Suppose A is a set, F is a vector space, F_α is a locally convex space for $\alpha \in A$, and $f_\alpha : F_\alpha \to F$ is a linear map for $\alpha \in A$. Then, the *locally convex final topology on F induced from the maps f_α* is the finest locally convex topology on F such that f_α is continuous whenever $\alpha \in A$; moreover, a fundamental system of neighborhoods of 0 of F is given by the family of all absorbent, convex, symmetric subsets V of F such that $f_\alpha^{-1}[V]$ is a neighborhood of 0 of F_α whenever $\alpha \in A$, see [Bou87, II, §4, No. 4, Proposition 5]. If $F = \varinjlim F_\alpha$, then F carries the locally convex final topology induced from the maps $f_\alpha : F_\alpha \to F$, see [Bou87, II, §4, No. 4, Example II]. The locally convex final topology satisfies the following universal property: if G is a locally convex space and $L : F \to G$ is a linear map, then L is continuous if and only if $L \circ f_\alpha$ is continuous whenever $\alpha \in A$, see [Bou87, II, §4, No. 4, Proposition 5]

The space of continuous functions with compact support, its dual space, and Daniell integrals. Suppose X is a locally compact Hausdorff space and Z is a locally convex space. The space $\mathscr{K}_K(X,Z)$ of continuous functions from X into Z with compact support in K is endowed with the topology of uniform convergence and the space

$$\mathscr{K}(X,Z) = \bigcup \{ \mathscr{K}_K(X,Z) : K \text{ is a compact subset of } X \}$$

is endowed with the locally convex final topology with respect to the inclusion maps $\mathscr{K}_K(X,Z) \to \mathscr{K}(X,Z)$ for compact subsets K of X, see [Bou04a, III, §1, No. 1]. In case $Z = \mathbf{R}$, we will simply write $\mathscr{K}_K(X)$ and $\mathscr{K}(X)$. Denote $\mathscr{K}(X)^+ = \mathscr{K}(X) \cap \{f : f \geq 0\}$. The topological dual space $\mathscr{K}(X)^*$ equals the space of all Daniell integrals on $\mathscr{K}(X)$, see [Fed69, 2.5.13], and it is endowed with the weak topology; that is, the initial topology induced from the maps $\mu \mapsto \mu(f)$ corresponding to $f \in \mathscr{K}(X)$, see [Fed69, 2.5.19]. For

$\mu \in \mathscr{K}(X)^*$, we define $\mu^+, \mu^-, |\mu| \in \mathscr{K}(X)^*$ such that

$$\mu^+(f) = \sup\{\mu(g) : 0 \le g \le f, g \in \mathscr{K}(X)\} \quad \text{whenever } f \in \mathscr{K}(X)^+,$$
$$\mu^- = (-\mu)^+, \quad |\mu| = \mu^+ + \mu^-,$$

see [Fed69, 2.5.5, 2.5.6]. For $\mu \in \mathscr{K}(X)^*$, we have $\mu^- = 0$ if and only if $\mu(f) \ge 0$ whenever $f \in \mathscr{K}(X)^+$; in this case, there exists a unique Radon measure over X representing μ, see [Fed69, 2.5.13, 2.5.14].

Functions and submanifolds of class k. Functions that are k times continuously differentiable and submanifolds defined by such functions are termed of class k, see [Fed69, 3.1.11, 3.1.19]. A function is termed to be of class ∞ if it is of class k for all $k \in \mathscr{P}$. If U is an open subset of a finite-dimensional Banach space and Z is a Banach space, then $\mathscr{E}(U, Z)$ denotes the space of all functions of class ∞ from U into Z, see [Fed69, 4.1.1].

Absolute continuity. Suppose X is a metric space, ϕ and ψ are Borel regular measures on X such that every bounded subset of X has finite measure, and we define a Borel regular measure by

$$\psi_\phi(A) = \inf\{\psi(B) : B \text{ is a Borel set and } \phi(A \sim B) = 0\}$$

whenever $A \subset X$. In case $\psi_\phi = \psi$, we say ψ is absolutely continuous with respect to ϕ, see [Fed69, 2.9.1, 2.9.2]. In general, following [Men16, Section 1, pp. 991–992], the definition of ψ_ϕ is also employed when X is merely a countable union of open sets on which ψ and ϕ have finite measure.

Approximate tangent cones. Whenever $m \in \mathscr{P}$, μ measures an open subset U of a normed space X, $a \in U$, and $\iota : U \to X$ is the inclusion map, we abbreviate $\operatorname{Tan}^m(\iota_\# \mu, a)$ as $\operatorname{Tan}^m(\mu, x)$, see [Fed69, 3.2.16] and [Men16, Section 1, p. 992].

Distributions. Whenever $n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, and Z is a separable Banach space, $\mathscr{D}(U, Z)$ denotes the vector space of Z-valued functions of class ∞ with compact support, of which the topology is defined as in [Men16, 2.13] and $\mathscr{D}'(U, Z)$ denotes the dual topological vector space to $\mathscr{D}(U, Z)$. For $T \in \mathscr{D}'(U, Z)$, $\|T\|$ is defined to be the largest Borel regular measure over U such that

$$\|T\|(G) = \sup\{T(g) : g \in \mathscr{D}(U, Z), \operatorname{spt} g \subset G, |g| \le 1\}$$

whenever G is an open subset of U, see [Men16, 2.18]. If $\|T\|$ is a Radon measure (equivalently, T is *representable by integration*), this concept agrees with [Fed69, 4.1.5]. In this case, $T(g)$ continues to denote the value of the unique $\|T\|_{(1)}$ continuous extension of T to $\mathbf{L}_1(\|T\|, Z)$ at $g \in \mathbf{L}_1(\|T\|, Z)$, see [Men16, 2.19]; for every $\|T\|$ measurable set A, we define $T \llcorner A \in \mathscr{D}'(U, Z)$

by $(T \llcorner A)(g) = T(g_A)$, where $g_A(x) = g(x)$ for $x \in A$ and $g_A(x) = 0$ for $x \in U \sim A$, see [Men16, 2.20].

Varifolds and distributional boundary of sets. Whenever $n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, and m is a non-negative integer, the space of varifolds, rectifiable varifolds, and integral varifolds of dimension m are denoted by $\mathbf{V}_m(U)$, $\mathbf{RV}_m(U)$, and $\mathbf{IV}_m(U)$, respectively; whenever B is a $\mathscr{H}^m$ measurable subset of $\mathbf{R}^n$ which meets every compact subset of $\mathbf{R}^n$ in an $(\mathscr{H}^m, m)$ rectifiable subset of $\mathbf{R}^n$, we define $\mathbf{v}_m(B) \in \mathbf{V}_m(\mathbf{R}^n)$ by

$$\mathbf{v}_m(A) = \mathscr{H}^m\{x : (x, \operatorname{Tan}^k(\mathscr{H}^m \llcorner B, x)) \in A\}$$

whenever $A \subset \mathbf{R}^n \times \mathbf{G}(n, m)$, see [All72, 3.1, 3.5]. Whenever $V \in \mathbf{V}_m(U)$ such that $\|\delta V\|$ is a Radon measure, there exists an $\mathbf{R}^n$-valued function $\boldsymbol{\eta}(V, \cdot)$ defined by the requirement that, for $x \in U$,

$$\boldsymbol{\eta}(V, x) \bullet u = \lim_{r \to 0+} \frac{\delta V(b_{x,r} \cdot u)}{\|\delta V\| \mathbf{B}(x, r)} \quad \text{whenever } u \in \mathbf{R}^n,$$

where $b_{x,r}$ is the characteristic function of $\mathbf{B}(x, r)$ on U; hence $x \in \operatorname{dmn} \boldsymbol{\eta}(V, \cdot)$ if and only if the above limit exists. Following [MS25b, 3.20, 3.21], this definition adapts [All72, 4.3] in the spirit of [Fed69, 4.1.5]; in particular, $\boldsymbol{\eta}(V, \cdot)$ is $\|\delta V\|$ measurable, $|\boldsymbol{\eta}(V, \cdot)| = 1$ for $\|\delta V\|$ almost all x, and

$$(\delta V)(g) = \textstyle\int \boldsymbol{\eta}(V, x) \bullet g(x) \, \mathrm{d}\|\delta V\| \, x \quad \text{for } g \in \mathbf{L}_1(\|\delta V\|, \mathbf{R}^n).$$

Similarly, following [Men16, p. 992] and [MS25b, 3.21], we also define a $\|V\|$ measurable $\mathbf{R}^n$-valued function $\mathbf{h}(V, \cdot)$ by the requirement that, for $x \in U$,

$$\mathbf{h}(V, x) \bullet u = -\lim_{r \to 0+} \frac{\delta V(b_{x,r} \cdot u)}{\|V\| \mathbf{B}(x, r)} \quad \text{whenever } u \in \mathbf{R}^n$$

which satisfies

$$\delta V(g) = -\textstyle\int \mathbf{h}(V, x) \bullet g(x) \, \mathrm{d}\|V\| \, x + \int \boldsymbol{\eta}(V, x) \bullet g(x) \, \mathrm{d}\left(\|\delta V\| - \|\delta V\|_{\|V\|}\right) x$$

whenever $g \in \mathbf{L}_1(\|\delta V\|, \mathbf{R}^n)$. If B is $\|V\| + \|\delta V\|$ measurable, then the distributional V boundary of B is given by

$$V \partial B = (\delta V) \llcorner B - \delta(V \llcorner B \times \mathbf{G}(n, m)) \in \mathscr{D}'(U, \mathbf{R}^n),$$

see [Men16, 5.1].

Definitions in the thesis. The *strong topology on* $\mathscr{K}(X)$ is defined in 2.1.1. The *integral indecomposability of integral varifolds* is introduced in 2.2.1. The notion of *appropriate classes of rectifiable varifolds* is introduced in 2.3.1. For an appropriate class P, the notions of *indecomposability*, *components of varifolds*, and *decompositions of varifolds* with respect to P are introduced in 2.3.6, 2.3.7, and 2.3.8 respectively. The notion of *projective tensor product* is defined in 3.1.9. The space $\mathbf{P}(X)$ of probability Radon measures and its topology are defined in 3.2.1. The notions of *Young functions* and their associated *graph measures* are defined in 3.2.4 and 3.2.8. The test function space E is introduced in 3.3.1. The pseudo-metric d on $\mathbf{P}(X)$ is defined in 3.3.6. The notion of *Lipschitzian Young functions* is defined in 3.3.9. The *pushforward* and *product* of Young functions are defined in 3.3.19 and 3.3.20. The *convolutions of measures* and the *convolution of Young functions* are defined in 3.3.21 and 3.3.25, respectively. The test function spaces $\widetilde{E}$ and H are introduced in 3.4.8 and 3.4.22, respectively. The *distributions (distributional derivative) associated with Young functions* are defined in 4.1.7. The notions of *Young functions of generalized V bounded variation* and *functions of (V, Z) bounded variation* are introduced in 4.1.9 and 4.1.12, respectively. The appropriate class $\mathbf{RV}_m(U, C)$ of varifolds is introduced in 3.2.22. The notion of functions of class k with values in a locally convex space is defined in 4.2.5. The notions of *generalized V weakly differentiable Young functions* and *(V, Z) weakly differentiable functions* are introduced in 4.3.1 and 4.3.7, respectively.

Chapter 2

Integral decompositions of varifolds

2.1 Topology

In this section, we present the necessary results about the strong topology on the space $\mathscr{K}(X)$.

Definition 2.1.1 (see [Bou87, III.14, Example 4])**.** Suppose X is a locally compact Hausdorff space. There exists a unique locally convex topology on $\mathscr{K}(X)^*$ termed *strong topology* such that the sets

$$\mathscr{K}(X)^* \cap \{\mu : |\mu(f)| < r \text{ for all } f \in B\}$$

corresponding to $r \in \mathbf{R}$, $r > 0$ and bounded subsets B of $\mathscr{K}(X)$ give a local base at 0.

Remark 2.1.2 (see [Bou87, III.12, Examples 1, 3] and [Bou87, III.23, Corollary 1]). The space $\mathscr{K}(X)^*$ equipped with the strong topology is complete.

Remark 2.1.3 (see [Men16, 2.11, 2.12] and [Bou87, III.5, Proposition 6]). Suppose X has a sequence of compact subsets K_i such that $K_i \subset \operatorname{Int} K_{i+1}$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^{\infty} K_i$. Then, the strong topology on $\mathscr{K}(X)^*$ is metrizable; in fact, it is generated by the family of semi-norms, with value

$$\sup\{\mu(f) : f \in \mathscr{K}(X), \operatorname{spt} f \subset K_i, |f| \leq 1\}$$

at $\mu \in \mathscr{K}(X)^*$, corresponding to $i \in \mathscr{P}$.

2.2 Integral indecomposability

Definition 2.2.1. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$ and $\|\delta V\|$ is a Radon measure. Then V is called *integrally indecomposable* if and only if there exists no $W \in \mathbf{IV}_m(U)$ such that $W \le V$, $W \ne 0$, $V - W \ne 0$, $\|V\| = \|W\| + \|V - W\|$, and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$.

The rest of this section contributes to the relation between the integral indecomposability and the distributional boundary of sets.

Lemma 2.2.2. *Suppose that $m, n \in \mathscr{P}$ with $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, E is $\|V\| + \|\delta V\|$ measurable with $V \partial E = 0$, and $W = V \llcorner E \times \mathbf{G}(n, m)$. Then, we have*

$$\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|.$$

Proof. It follows from [MS25b, 5.1]. □

Remark 2.2.3. In contrast to 2.2.2, it may happen that V is an integral varifold, $\|V\|$ has density 1 everywhere, and there exists an integral varifold W such that W and $V - W$ are integrally indecomposable and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$, but there exists no Borel set E such that $W = V \llcorner E \times \mathbf{G}(n, m)$ and $V \partial E \ne 0$. Let

$$R_1 = \mathbf{R}^2 \cap \{(x, y) : x \ge 0, y = 0\},$$
$$R_2 = \mathbf{R}^2 \cap \{(x, y) : |y| = \sqrt{3}x\}.$$

Define $V, W \in \mathbf{IV}_1(\mathbf{R}^2)$ to satisfy $\|V\| = \mathscr{H}^1 \llcorner (R_1 \cup R_2)$ and $\|W\| = \mathscr{H}^1 \llcorner R_1$. Therefore, W and $V - W$ are integrally indecomposable, $\|\delta V\| = 2\boldsymbol{\delta}_{(0,0)}$, $\|\delta W\| = \boldsymbol{\delta}_{(0,0)} = \|\delta(V - W)\|$, and if $W = V \llcorner E \times \mathbf{G}(2, 1)$ for some Borel set E, then

$$\|V \, \partial E\| = \boldsymbol{\delta}_{(0,0)}.$$

Next, we will show that if $V \in \mathbf{IV}_m(U)$ and $\|\delta V\| = \|\delta V\|_{\|V\|}$, then the converse of 2.2.2 holds.

Theorem 2.2.4. *Suppose that $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V, W \in \mathbf{IV}_m(U)$, $\|\delta V\| + \|\delta W\|$ is a Radon measure and*

$$A = \{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0 \text{ and } \boldsymbol{\Theta}^m(\|W\|, x) > 0\}.$$

Then,

$$\mathbf{h}(V, x) = \mathbf{h}(W, x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \text{ in } A.$$

Proof. In view of [All72, 3.5(1b)], the theorem [Men13, 4.8] reduces the problem to the case of submanifolds of class 2. □

Remark 2.2.5. In case $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, we have V is integrally indecomposable if and only if there exists no $W \in \mathbf{IV}_m(U)$ such that $W \le V$, $W \ne 0$, $V - W \ne 0$, and

$$\|\delta W\| \text{ is absolutely continuous with respect to } \|W\|,$$
$$\|\delta(V-W)\| \text{ is absolutely continuous with respect to } \|V-W\|.$$

In fact, from 2.2.4, we have

$$\|\delta V\|_{\|V\|} = \|\delta W\|_{\|W\|} + \|\delta(V-W)\|_{\|V-W\|},$$

and the assertion follows. Therefore, the present definition of indecomposability extends [Mon14, 2.15] when spt $\|V\|$ is compact and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$.

Corollary 2.2.6. *Suppose that $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, $\|\delta V\|$ is a Radon measure, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, E is $\|V\| + \|\delta V\|$ measurable, and $W = V \llcorner E \times \mathbf{G}(n,m)$ satisfies $\|\delta V\| = \|\delta W\| + \|\delta(V-W)\|$. Then, $V \partial E = 0$.*

Proof. In view of 2.2.4, we have

$$\|\delta V\|_{\|V\|} = \|\delta W\|_{\|W\|} + \|\delta(V-W)\|_{\|V-W\|},$$

and hence

$$\|\delta W\| - \|\delta W\|_{\|W\|} = 0 = \|\delta(V-W)\| - \|\delta(V-W)\|_{\|V-W\|}.$$

Therefore, whenever $g \in \mathscr{D}(U, \mathbf{R}^n)$, we have

$$\begin{aligned} ((\delta V) \llcorner E)(g) &= -\textstyle\int_U \mathbf{h}(V,x) \bullet g(x) \,\mathrm{d}\|W\|\, x \\ &= -\textstyle\int_U \mathbf{h}(W,x) \bullet g(x) \,\mathrm{d}\|W\|\, x \\ &= (\delta W)(g) \end{aligned}$$

This shows $V \partial E = 0$. □

2.3 Integral decomposition

Definition 2.3.1. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$ and $P \subset \mathbf{RV}_m(U)$. Then P is called *appropriate* if and only if

(1) If $V, W \in P$, then $V + W \in P$.

(2) If $V \in P$, then $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x.

(3) P is closed with respect to the strong topology.

Now, we aim to provide examples of appropriate classes; for this purpose, the following lemma is a powerful tool to verify the closedness of a class with respect to the strong topology.

Lemma 2.3.2. *Suppose V_i forms a sequence in $\mathbf{RV}_m(U)$. Then the following statements are equivalent.*

(1) V_i is Cauchy with respect to the strong topology.

(2) $\|V_i\|$ is Cauchy with respect to the strong topology.

(3) $\boldsymbol{\Theta}^m(\|V_i\|, \cdot)$ is Cauchy in $\mathbf{L}_1^{\mathrm{loc}}(\mathscr{H}^m)$.

In this case, the limit V of V_i satisfies $V \in \mathbf{RV}_m(U)$ and $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ is the limit of $\boldsymbol{\Theta}^m(\|V_i\|, \cdot)$ in $\mathbf{L}_1^{\mathrm{loc}}(\mathscr{H}^m)$. In particular, $\mathbf{RV}_m(U)$ and $\mathbf{IV}_m(U)$ are closed subsets of $\mathbf{V}_m(U)$ with respect to the strong topology.

Proof. Note that if $W_1, W_2 \in \mathbf{RV}_m(U)$ and G is an open set with compact closure in U, then applying [MS25b, 3.3] with

$$T = \|W_1\| - \|W_2\|, \quad \phi = \mathscr{H}^m \llcorner |\boldsymbol{\Theta}^m(\|W_1\|, \cdot) - \boldsymbol{\Theta}^m(\|W_2\|, \cdot)|,$$
$$k = \operatorname{sign}(\boldsymbol{\Theta}^m(\|W_1\|, \cdot) - \boldsymbol{\Theta}^m(\|W_2\|, \cdot)),$$

we have $\|T\| = \phi$ and

$$\begin{aligned}&\textstyle\int_G |\boldsymbol{\Theta}^m(\|W_1\|, x) - \boldsymbol{\Theta}^m(\|W_2\|, x)| \,\mathrm{d}\mathscr{H}^m\, x \\ &\qquad = \sup\{\|W_1\|(g) - \|W_2\|(g) : g \in \mathscr{K}(U), \operatorname{spt} g \subset G, |g| \leq 1\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}W_1(f) - W_2(f) &= \textstyle\int f(x, \operatorname{Tan}^m(\|W_1 + W_2\|, x)) \,\mathrm{d}\|W_1\|\, x \\ &\quad - \textstyle\int f(x, \operatorname{Tan}^m(\|W_1 + W_2\|, x)) \,\mathrm{d}\|W_2\|\, x \\ &\leq \sup \operatorname{im} |f| \textstyle\int_K |\boldsymbol{\Theta}^m(\|W_1\|, x) - \boldsymbol{\Theta}^m(\|W_2\|, x)| \,\mathrm{d}\mathscr{H}^m\, x\end{aligned}$$

whenever $f \in \mathscr{K}(U \times \mathbf{G}(n, m))$ with $\operatorname{spt} f \subset K \times \mathbf{G}(n, m)$ and K is a compact subset of U, and note that $|p_\#(W_1 - W_2)| \leq p_\#|W_1 - W_2|$, where $p : U \times \mathbf{G}(n, m) \to U$ is the projection map. Now, the main assertion is obvious.

Note that the subset D of $\mathbf{L}_1^{\mathrm{loc}}(\mathscr{H}^m)$, which consists of all non-negative functions θ such that $\{x : \theta(x) > 0\}$ is countably $(\mathscr{H}^m, m)$ rectifiable, is closed. Denote by θ the limit of $\boldsymbol{\Theta}^m(\|V_i\|, \cdot)$ in $\mathbf{L}_1^{\mathrm{loc}}(\mathscr{H}^m)$. Since $\theta \in D$, it follows from [Fed69, 2.10.19(3)] and [MS25a, 2.25] that $\mathscr{H}^m \llcorner \theta$ is the weight of some member V of $\mathbf{RV}_m(U)$. Finally, applying the results in the previous paragraph with $W_1 = V_i$ and $W_2 = V$, we deduce that V is the limit of V_i with respect to the strong topology, and that if V_i is integral whenever $i \in \mathscr{P}$, then so is V. □

Lemma 2.3.3. *Suppose $m, n \in \mathscr{P}$ and U is an open subset of $\mathbf{R}^n$. Then the following statements hold.*

(1) If C is a closed subset of $\mathbf{R}$ satisfying $\inf C \geq 1$ and $c+d \in C$ whenever $c, d \in C$, then the class

$$P = \mathbf{RV}_m(U) \cap \{V : \boldsymbol{\Theta}^m(\|V\|, x) \in C \text{ for } \|V\| \text{ almost all } x\}$$

is appropriate.

(2) If $n' \in \mathscr{P}$, U' is an open subset of $\mathbf{R}^{n'}$, $f : U \to U'$ is of class ∞, and $f|\operatorname{spt}\|V\|$ is proper, then we have

$$\boldsymbol{\Theta}^m(\|f_\# V\|, y) \in C \quad \textit{for } \|f_\# V\| \textit{ almost all } y$$

whenever $V \in P$.

(3) If $m = n$, $U = \mathbf{R}^m$, $0 < r < \infty$, $0 < d < \infty$, and the sequence $W_i \in P$ satisfies that

$$W_i \llcorner \mathbf{U}(0, r) \times \mathbf{G}(m, m) \to d\mathbf{v}_m(\mathbf{U}(0, r)) \quad \textit{as } i \to \infty$$

with respect to the weak topology, and that

$$\lim_{i\to\infty} \sup\{(\delta W_i)(\psi) : \psi \in \mathscr{D}_{\mathbf{B}(0,r)}(\mathbf{R}^m, \mathbf{R}^m), \sup \operatorname{im} \|\operatorname{D}\psi\| \leq 1\} = 0, \quad (\dagger)$$

then $d \in C$.

Proof. Clearly, (1) follows from 2.3.2, and (2) follows from [All72, 3.5(1b), 3.5(3)]. To prove (3), we denote by $e_1, e_2, \ldots, e_m$ the standard base of $\mathbf{R}^m$ and let $T_i \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R})$ be such that

$$T_i(\phi) = (\|W_i\| \llcorner \mathbf{U}(0, r))(\phi) \quad \text{whenever } \phi \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}).$$

Then, we have $(\mathrm{D}_j T_i)(\phi) = (\delta W_i)(\phi e_j)$ whenever $\phi \in \mathscr{D}_{\mathbf{B}(0,r)}(\mathbf{R}^m, \mathbf{R})$ and $j = 1, 2, \ldots, m$. It follows from (†) that there exist numbers $0 < \kappa_i < \infty$ such that $\lim_{i\to\infty} \kappa_i = 0$ and such that

$$|(\mathrm{D}_j T_i)(\phi)| \leq \kappa_i \sup \operatorname{im} \|\mathrm{D}\, \phi\|$$

whenever $\phi \in \mathscr{D}_{\mathbf{B}(0,r)}(\mathbf{R}^m, \mathbf{R})$ and $j = 1, 2, \ldots, m$. Letting $0 < \lambda < 1$ and applying Allard's strong constancy lemma [Men21, 3.7] (for the original formulation, see [All86, Section 1]), there exists $0 \leq c_i < \infty$ such that

$$\lim_{i\to\infty} \sup\{|T_i(\phi) - c_i \textstyle\int \phi \,\mathrm{d}\mathscr{L}^m| : \phi \in \mathscr{D}_{\mathbf{B}(0,\lambda r)}(\mathbf{R}^m, \mathbf{R}), \sup \operatorname{im} |\phi| \leq 1\} = 0.$$

It follows from [Men21, 2.34(3)] and 2.3.2 that $W_i \llcorner \mathbf{U}(0, \lambda r) \times \mathbf{G}(m, m) - c_i \mathbf{v}_m(\mathbf{U}(0, \lambda r)) \to 0$ as $i \to \infty$ with respect to the strong topology. Therefore, we conclude that $\lim_{i\to\infty} c_i = d$ and that $W_i \llcorner \mathbf{U}(0, \lambda r) \times \mathbf{G}(m, m) \to d\mathbf{v}_m(\mathbf{U}(0, \lambda r))$ as $i \to \infty$ with respect to the strong topology. Finally, it follows from (1) that $d\mathbf{v}_m(\mathbf{U}(0, \lambda r)) \in P$; that is, $d \in C$. □

Theorem 2.3.4 (General compactness theorem)**.** *Suppose m, n, U, C, and P are as in 2.3.3, G_i is a sequence of open subsets of U such that $U = \bigcup_{i=1}^{\infty} G_i$, and M_i is a sequence of non-negative real numbers. Then, the class*

$$P \cap \{V : (\|V\| + \|\delta V\|)(G_i) \leq M_i \text{ whenever } i \in \mathscr{P}\}$$

is compact with respect to the weak topology.

Proof. From the proof of [All72, 6.4], it is enough to show the following statement: if $U = \mathbf{R}^n$, $T \in \mathbf{G}(n, m)$, V_i is a sequence in P such that $V_i \to d\mathbf{v}_m(T)$ as $i \to \infty$ with respect to the weak topology and such that

$$\lim_{i\to\infty} \|\delta V_i\| \mathbf{U}(0, 4r) = 0 \quad \text{for some } 0 < r < \infty,$$

then $d \in C$. Denote by $T^\perp \in \mathbf{G}(n, n-m)$ the orthogonal complement of T in $\mathbf{R}^n$. Let $X = \mathbf{R}^n \cap \{x : \sup\{|Tx|, |T^\perp x|\} < 2r\}$. From 2.3.3(2), we have

$$W_i = T_\#(V_i \llcorner X \times \mathbf{G}(n, m)) \in \mathbf{RV}_m(T),$$
$$\boldsymbol{\Theta}^m(\|W_i\|, z) \in C \quad \text{for } \|W_i\| \text{ almost all } z,$$

whenever $i \in \mathscr{P}$. Choose $\gamma \in \mathscr{D}_{\mathbf{B}(0,2r)}(T^\perp, \mathbf{R})$ such that $\gamma|\mathbf{U}(0, r) = 1$. Because $V_i \llcorner X \times \mathbf{G}(n, m) \to \mathbf{v}_m(X \cap T)$ as $i \to \infty$ with respect to the weak topology by [All72, 2.6(2d)], we conclude that

$$W_i \to d\mathbf{v}_m(T \cap \mathbf{U}(0, r)) \quad \text{as } i \to \infty$$

with respect to the weak topology, and observe that

$$\sup\{|(\delta V_i)((\gamma \circ T^\perp)(\psi \circ T)) - (\delta W_i)(\psi)| : \psi \in \mathscr{D}_{\mathbf{B}(0,r)}(T, T), \sup \operatorname{im} \|\mathrm{D}\, \psi\| \leq 1\}$$

tends to 0 as $i \to \infty$. By 2.3.3(3), the assertion follows. □

Remark 2.3.5. The set C mentioned in 2.3.3(1) could be $\mathscr{P}$, $\mathbf{R} \cap \{t : 1 \le t\}$, or

$$\{1\} \cup (\mathbf{R} \cap \{t : 2 \le t\});$$

the last class also occurs in [PS23].

Definition 2.3.6. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, P is an appropriate subset of $\mathbf{RV}_m(U)$, $V \in P$ and $\|\delta V\|$ is a Radon measure. Then V is called *indecomposable with respect to* P if and only if there exists no $W \in P \sim \{0\}$ such that $V - W \in P \sim \{0\}$ and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$.

Definition 2.3.7. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, P is an appropriate subset of $\mathbf{RV}_m(U)$, $W, V \in P$ and $\|\delta V\|$ is a Radon measure. Then W is called a *component of* V *with respect to* P if and only if $W \ne 0$, $W \le V$, $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$ and W is indecomposable with respect to P.

Definition 2.3.8. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, P is an appropriate subset of $\mathbf{RV}_m(U)$, $V \in P$ and $\|\delta V\|$ is a Radon measure. Then a countable subfamily H of P together with a map $\xi : H \to \mathscr{P}$ is called a *decomposition of* V *with respect to* P if and only if

(1) Each member of H is a component of V with respect to P.

(2) $\sum_{W \in H} \xi(W) W(k) = V(k)$ whenever $k \in \mathscr{K}(U \times \mathbf{G}(n, m))$.

(3) $\sum_{W \in H} \xi(W) \|\delta W\|(f) = \|\delta V\|(f)$ whenever $f \in \mathscr{K}(U)$.

Example 2.3.9. Let $0 < \theta < \pi$ be such that $\cos\theta = 1/4$. Consider the six rays

$$\begin{aligned}
R_1 &= \{t(1, 0) : 0 < t < \infty\}, \\
R_2 &= \{t(\cos\theta, \sin\theta) : 0 < t < \infty\}, \\
R_3 &= \{t(\cos(\pi - \theta), \sin\theta) : 0 < t < \infty\}, \\
R_4 &= \{t(-1, 0) : 0 < t < \infty\}, \\
R_5 &= \{t(\cos(\pi - \theta), -\sin\theta) : 0 < t < \infty\}, \\
R_6 &= \{t(\cos\theta, -\sin\theta) : 0 < t < \infty\},
\end{aligned}$$

in $\mathbf{R}^2$ and the associated varifolds $V_i \in \mathbf{IV}_1(\mathbf{R}^n)$ with $\|V_i\| = \mathscr{H}^1 \llcorner R_i$. Note that the integral varifold defined by

$$V = 2(V_1 + V_2 + V_3 + V_4 + V_5 + V_6)$$

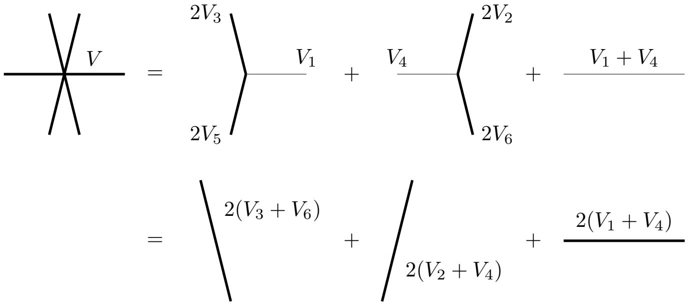

Figure 2.1: Two decompositions of V

is stationary. Let

$$H_1 = \{V_1 + V_4, V_2 + V_5, V_3 + V_6\},$$
$$H_2 = \{V_1 + 2(V_3 + V_5), V_4 + 2(V_2 + V_6), V_1 + V_4\}$$

and define $\xi_i : H_i \to \mathscr{P}$ for $i = 1, 2$ by

$$\operatorname{im} \xi_1 = \{2\} \quad \text{and} \quad \operatorname{im} \xi_2 = \{1\}.$$

Then, (H_i, ξ_i) for $i = 1, 2$ are distinct decompositions of V with respect to $\mathbf{IV}_1(\mathbf{R}^2)$, see figure 2.1. It shows that there may exist different types of decompositions for a varifold, and the components need not have constant density. Furthermore, the decompositions may fail to be unique even if $\|V\|$ has density 1, see also [Men16, 6.13].

To prove the main theorem, the following a priori estimate is a key observation: under smallness conditions on the first variation, the weight measure on a ball has a positive lower bound. This will provide, locally, an upper bound of the number of varifolds in a decomposition; moreover, it also suggests a way to construct a decomposition.

Lemma 2.3.10 (a priori estimate)**.** *Suppose* $0 < c < \infty$*,* $0 < d < \infty$*,* $m, n \in \mathscr{P}$*,* $m \le n$*,* U *is an open subset of* $\mathbf{R}^n$*,* $a \in U$*,* $r > 0$*,* $\mathbf{B}(a, r) \subset U$*,* $V \in \mathbf{V}_m(U)$*,* $\|\delta V\|$ *is a Radon measure and*

$$\boldsymbol{\Theta}^{*m}(\|V\|, a) \ge d,$$
$$\|\delta V\| \mathbf{B}(a, t) \le c \boldsymbol{\alpha}(m) t^m \quad \text{for } 0 < t < r.$$

Then, there holds

$$\|V\| \mathbf{B}(a, r) \ge \boldsymbol{\alpha}(m)(d - cr) r^m.$$

Proof. From [Men16, 4.5, 4.6], we have

$$s^{-m}\|V\|\mathbf{B}(a,s) \le r^{-m}\|V\|\mathbf{B}(a,r) \\ + \textstyle\int_s^r t^{-m-1} \int_{\mathbf{B}(a,t)} (x-a) \bullet \boldsymbol{\eta}(V,x)\, \mathrm{d}\|\delta V\|\, x\, \mathrm{d}\mathscr{L}^1\, t$$

whenever $0 < s \le r$; note that the last term is less than

$$\textstyle\int_s^r t^{-m}\|\delta V\|\mathbf{B}(a,t)\, \mathrm{d}\mathscr{L}^1\, t \le c\boldsymbol{\alpha}(m)(r-s)$$

hence, letting $s \to 0+$,

$$r^{-m}\|V\|\mathbf{B}(a,r) \ge \boldsymbol{\alpha}(m)\boldsymbol{\Theta}^{*m}(\|V\|,a) - c\boldsymbol{\alpha}(m)r \ge \boldsymbol{\alpha}(m)(d-cr)$$

which means $\|V\|\mathbf{B}(a,r) \ge \boldsymbol{\alpha}(m)(d-cr)r^m$. □

Definition 2.3.11. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, and P is an appropriate subset of $\mathbf{RV}_m(U)$. Then, Ξ denotes the class of all functions ξ such that

$$\operatorname{dmn}\xi \text{ is a finite subset of } P \sim \{0\}, \quad \operatorname{im}\xi \subset \mathscr{P},$$
$$\sum_{W \in \operatorname{dmn}\xi} \xi(W)\|\delta W\| = \|\delta \mathbf{v}(\xi)\|,$$

where the map $\mathbf{v} : \Xi \to P$ is defined by

$$\mathbf{v}(\xi)(f) = \sum_{W \in \operatorname{dmn}\xi} \xi(W)W(f) \quad \text{whenever } f \in \mathscr{K}(U \times \mathbf{G}(n,m)).$$

Furthermore, $\xi \in \Xi$ is called *maximal with respect to a Borel set B* if and only if $\|W\|(B) > 0$ for all $W \in \operatorname{dmn}\xi$ and

$$\sum \xi \ge \sum \rho$$

whenever $\rho \in \Xi$ satisfies

$$\mathbf{v}(\rho) = \mathbf{v}(\xi)$$

and $\|X\|(B) > 0$ for all $X \in \operatorname{dmn}\rho$. We say W *splits* V *in* P if and only if $W \in P$, $V \in P$, $V - W \in P$, and $\|\delta W\| + \|\delta(V-W)\| = \|\delta V\|$.

Theorem 2.3.12. *Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $P \subset \mathbf{RV}_m(U)$ is appropriate, $V \in P$ and $\|\delta V\|$ is a Radon measure. Then, there exists a decomposition of V with respect to P.*

Proof. Assume $V \neq 0$. Define $\delta_i = \boldsymbol{\alpha}(m)2^{-m-1}i^{-m}$, $\varepsilon_i = 2^{-1}i^{-1}$ for $i \in \mathscr{P}$ and let A_i denote the the Borel set of $a \in \mathbf{R}^n$ satisfying

$$|a| \le i, \quad \mathbf{U}(a, 2\varepsilon_i) \subset U, \quad 1 \le \boldsymbol{\Theta}^m(\|V\|, a) < \infty,$$
$$\|\delta V\|\mathbf{B}(a, r) \le \boldsymbol{\alpha}(m) i r^m \quad \text{for } 0 < r < \varepsilon_i$$

whenever $i \in \mathscr{P}$. Clearly, $A_i \subset A_{i+1}$ for $i \in \mathscr{P}$ and $\|V\|(U \sim \bigcup_{i=1}^{\infty} A_i) = 0$ by [All72, 3.5 (1a)] and [Fed69, 2.8.18, 2.9.5]. For each $i \in \mathscr{P}$, we infer from 2.3.10 that

$$\begin{gathered}\|W\|\mathbf{B}(a, \varepsilon_i) \ge \delta_i \quad \text{whenever } a \in A_i \text{ and } W \in P \text{ satisfy} \\ W \le V,\ \|\delta W\| \le \|\delta V\|, \text{ and } \boldsymbol{\Theta}^{*m}(\|W\|, a) \ge 1\end{gathered} \tag{2.1}$$

and hence

$$\begin{aligned}\delta_i \sum \xi &\le \sum_{W \in \operatorname{dmn} \xi} \xi(W) \|W\| \{x : \operatorname{dist}(x, A_i) \le \varepsilon_i\} \\ &= \|\mathbf{v}(\xi)\| \{x : \operatorname{dist}(x, A_i) \le \varepsilon_i\} < \infty\end{aligned}$$

whenever $\xi \in \Xi$ satisfies $\mathbf{v}(\xi) \le V$, $\|\delta \mathbf{v}(\xi)\| \le \|\delta V\|$, and $\|W\|(A_i) > 0$ for each $W \in \operatorname{dmn} \xi$.

Since $V \neq 0$, there exists $\lambda \in \mathscr{P}$ such that $\|V\|(A_\lambda) > 0$. From now on, we replace A_i by $A_{i+\lambda}$ for $i \in \mathscr{P}$. Let

$$R = P \cap \{W : W \le V, \|\delta W\| \le \|\delta V\|\},$$
$$P_i = R \cap \{W : \|W\|(A_i) > 0\} \quad \text{whenever } i \in \mathscr{P}.$$

Then, we may select functions $c_i : P_i \to \Xi$ such that $\mathbf{v}(c_i(W)) = W$ and $c_i(W)$ is maximal with respect to A_i; in particular, $\operatorname{dmn} c_i(W) \subset P_i$ whenever $W \in P_i$.

From now on, we will use the convention that $\infty \cdot 0 = 0$. Let Σ be the class of all sequences $Z_1, Z_2, Z_3, \ldots$ in P satisfying $Z_1 = V$ and $Z_{i+1} \in \operatorname{dmn} c_i(Z_i)$ and abbreviate $\lim Z = \lim_{i \to \infty} Z_i \in P$ for $Z \in \Sigma$, where the limit is taken with respect to the strong topology. Let $C = \Sigma \cap \{Z : \lim Z \neq 0\}$ and define $\nu : \Sigma \to \mathscr{P} \cup \{\infty\}$ by

$$\nu(Z) = \prod_{i=1}^{\infty} c_i(Z_i)(Z_{i+1}).$$

Note that for $Z \in \Sigma$, and $i, j \in \mathscr{P}$ with $i \le j$, we have

$$\prod_{k=i}^{j} c_k(Z_k)(Z_{k+1}) Z_{j+1} \le Z_i,$$

hence

$$\nu(Z)\lim Z \le \prod_{k=1}^{j} c_k(Z_k)(Z_{k+1})Z_{j+1} \le V;$$

thus, $\operatorname{im}(\nu|C) \subset \mathscr{P}$.

Now, we aim to prove C is countable,

$$\sum_{Z\in C}\nu(Z)\lim Z \le V, \quad \text{and} \quad \sum_{Z\in C}\nu(Z)\|\delta\lim Z\| \le \|\delta V\|.$$

For $i \in \mathscr{P}$, let F_i be the set of all finite sequences $Z_1, Z_2, \ldots, Z_i$ such that $Z_1 = V$ and $Z_{j+1} \in \operatorname{dmn} c_j(Z_j)$ for $1 \le j \le i-1$. We define, for $i \in \mathscr{P}$, the restriction map

$$R_i : \Sigma \cup \bigcup_{i<j\in\mathscr{P}} F_j \to F_i, \quad R_i(Z) = Z|\mathscr{P} \cap \{k : 1 \le k \le i\}.$$

Observe that whenever $i \in \mathscr{P}$, $Y \in F_i$, and F is a finite subset of Σ satisfying $R_i[F] = \{Y\}$, there exists $j \in \mathscr{P}$ such that $j > i$ and $R_j|F$ is injective and that

$$\sum_{X\in F_j, R_i(X)=Y}\prod_{k=1}^{j-1} c_k(X_k)(X_{k+1})X_j = \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i \tag{2.2}$$

whenever $i, j \in \mathscr{P}$ with $i < j$. Therefore, we have

$$\begin{aligned}\sum_{Z\in F}\nu(Z)\lim Z &\le \sum_{Z\in F}\prod_{k=1}^{j-1} c_k(Z_k)(Z_{k+1})Z_j \\ &\le \sum_{X\in F_j, R_i(X)=Y}\prod_{k=1}^{j-1} c_k(X_k)(X_{k+1})X_j \\ &= \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i;\end{aligned} \tag{2.3}$$

similarly, we have

$$\sum_{Z\in F}\nu(Z)\|\delta\lim Z\| \le \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})\|\delta Y_i\|. \tag{2.4}$$

Choosing by [Men16, 2.2, 2.23] countable dense subsets of $\mathscr{K}(U \times \mathbf{G}(n,m))^+$ and $\mathscr{K}(U)^+$, we conclude from (2.3), (2.4) with $i = 1$ and [Fed69, 2.1.1(3)] that

$$\begin{aligned}\sum_{Z\in\Sigma}\nu(Z)\lim Z(f)\le V(f)\quad\text{whenever } f\in\mathscr{K}(U\times\mathbf{G}(n,m))^+,\\ \sum_{Z\in\Sigma}\nu(Z)\|\delta\lim Z\|(g)\le\|\delta V\|(g)\quad\text{whenever } g\in\mathscr{K}(U)^+;\end{aligned}\tag{2.5}$$

in particular, C is countable by [Fed69, 2.1.1(12)].

Next, to show the equalities in (2.5) hold, we shall first prove that, for $\mathscr{H}^m$ almost all x,

$$\sum_{Z\in C}\nu(Z)\boldsymbol{\Theta}^m(\|\lim Z\|,x)=\boldsymbol{\Theta}^m(\|V\|,x).\tag{2.6}$$

If $x\in U$ satisfies $\boldsymbol{\Theta}^m(\|V\|,x)=0$, then (2.6) is trivial. To the other case, let B consist of all $x\in\bigcup_{i=1}^\infty A_i$ such that

$$\boldsymbol{\Theta}^m(\|W\|,x)\in\{0\}\cup(\mathbf{R}\cap\{t:1\le t\}),$$
$$\boldsymbol{\Theta}^m(\|\lim Z\|,x)=\lim_{i\to\infty}\boldsymbol{\Theta}^m(\|Z_i\|,x)$$

whenever $W\in\bigcup_{Z\in\Sigma}\operatorname{im}Z$ and $Z\in C$; in particular,

$$\boldsymbol{\Theta}^m(\|Z_i\|,x)=\sum_{W\in\operatorname{dmn}c_i(Z_i)}c_i(Z_i)(W)\boldsymbol{\Theta}^m(\|W\|,x)\quad\text{for all } x\in B\tag{2.7}$$

whenever $Z\in\Sigma$ and $i\in\mathscr{P}$. For $x\in B$ and $Z\in\Sigma$, we abbreviate

$$\Theta_Z(x)=\lim_{i\to\infty}\boldsymbol{\Theta}^m(\|Z_i\|,x)$$

and the crucial observation is that

$$\Theta_Z(x)>0\quad\text{implies}\quad Z\in C\text{ and }\Theta_Z(x)=\boldsymbol{\Theta}^m(\|\lim Z\|,x);$$

in fact, if $x\in A_i$, then by [All72, 2.6(2c)] and (2.1), we have

$$\|\lim Z\|\mathbf{B}(x,\varepsilon_i)\ge\delta_i$$

and the assertion follows. Therefore, $E_x=\Sigma\cap\{Z:\Theta_Z(x)>0\}\subset C$ whenever $x\in B$. Similarly as (2.5), we derive from (2.7) that

$$\sum_{Z\in\Sigma}\nu(Z)\Theta_Z(x)\le\boldsymbol{\Theta}^m(\|V\|,x)<\infty,$$

and it follows that E_x is finite whenever $x \in B$. Accordingly, letting $x \in B$, there exists $i \in \mathscr{P}$ such that $R_j | E_x$ is injective for $i < j \in \mathscr{P}$; furthermore, we observe that

$$R_j[E_x] = F_j \cap \{Z : \boldsymbol{\Theta}^m(\|Z_j\|, x) > 0\} \quad \text{whenever } i < j \in \mathscr{P}. \tag{2.8}$$

By (2.7) and (2.8), we have

$$\begin{aligned}\boldsymbol{\Theta}^m(\|V\|, x) &= \sum_{Z \in F_j} \prod_{k=1}^{j-1} c_k(Z_k)(Z_{k+1}) \boldsymbol{\Theta}^m(\|Z_j\|, x) \\ &= \sum_{Z \in E_x} \prod_{k=1}^{j-1} c_k(Z_k)(Z_{k+1}) \boldsymbol{\Theta}^m(\|Z_j\|, x)\end{aligned}$$

whenever $i < j \in \mathscr{P}$; letting $j \to \infty$, we conclude

$$\sum_{Z \in E_x} \nu(Z) \boldsymbol{\Theta}^m(\|\lim Z\|, x) = \boldsymbol{\Theta}^m(\|V\|, x).$$

Therefore, (2.6) holds for all $x \in B$, and it remains to prove

$$\mathscr{H}^m(\{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0\} \sim B) = 0.$$

Note that since $\|V\|(U \sim \bigcup_{i=1}^{\infty} A_i) = 0$, we have

$$\mathscr{H}^m(\{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0\} \sim \bigcup_{i=1}^{\infty} A_i) = 0.$$

Thus, it is enough to show $\mathscr{H}^m(\bigcup_{i=1}^{\infty} A_i \sim B) = 0$. From [All72, 3.5(1b)], we write $\|\lim Z\| = \mathscr{H}^m \llcorner \boldsymbol{\Theta}^m(\|\lim Z\|, \cdot)$ and $\|Z_i\| = \mathscr{H}^m \llcorner \boldsymbol{\Theta}^m(\|Z_i\|, \cdot)$ whenever $Z \in C$ and $i \in \mathscr{P}$. Since $\|Z_i\|$ converges to $\|\lim Z\|$ with respect to the strong topology and $\|Z_i\|$ is non-increasing, it follows from 2.3.2 that

$$\boldsymbol{\Theta}^m(\|\lim Z\|, x) = \lim_{i \to \infty} \boldsymbol{\Theta}^m(\|Z_i\|, x) \quad \text{for } \mathscr{H}^m \text{ almost all } x.$$

Then, the assertion follows from that $C \cup \bigcup_{Z \in C} \operatorname{im} Z$ is countable. From (2.6), we deduce

$$\sum_{Z \in C} \nu(Z) \|\lim Z\| = \|V\|;$$

it follows that $\operatorname{Tan}^m(\|\lim Z\|, x) = \operatorname{Tan}^m(\|V\|, x)$ for $\|\lim Z\|$ almost all x whenever $Z \in C$, hence

$$\begin{gathered}\sum_{Z \in C} \nu(Z) \lim Z = V, \\ \sum_{Z \in C} \nu(Z) \|\delta \lim Z\| = \|\delta V\|.\end{gathered} \tag{2.9}$$

Furthermore, it follows from (2.3) that for $i \in \mathscr{P}$ and $Y \in F_i$,

$$\sum_{Z \in C, R_i(Z)=Y} \nu(Z) \lim Z \leq \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i.$$

hence from [Fed69, 2.1.1(9)] that

$$\begin{aligned}\sum_{Z \in C} \nu(Z) \lim Z &= \sum_{Y \in F_i} \sum_{Z \in C, R_i(Z)=Y} \nu(Z) \lim Z \\ &\leq \sum_{Y \in F_i} \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i \\ &= V\end{aligned}$$

which forces

$$\sum_{Z \in C, R_i(Z)=Y} \nu(Z) \lim Z = \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i \quad \text{for } i \in \mathscr{P} \text{ and } Y \in F_i;$$

similarly, we have

$$\sum_{Z \in C, R_i(Z)=Y} \nu(Z) \|\delta \lim Z\| = \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1}) \|\delta Y_i\| \quad \text{for } i \in \mathscr{P} \text{ and } Y \in F_i.$$

In particular, $\lim Z$ splits Z_i in P whenever $Z \in C$ and $i \in \mathscr{P}$.

Finally, we will prove $\lim Z$ is indecomposable with respect to P whenever $Z \in C$. If it were not the case, there would exist $W \in P \sim \{0\}$ such that

$$\begin{gathered} W \leq \lim Z, \quad \lim Z - W \in P \sim \{0\} \\ \text{and} \quad \|\delta \lim Z\| = \|\delta W\| + \|\delta(\lim Z - W)\|. \end{gathered}$$

We could choose i large such that $\|W\|(A_i) > 0$ and $\|\lim Z - W\|(A_i) > 0$, since $\|W\| + \|\lim Z - W\| \leq \|V\|$. Then, W would split Z_{i+1}, $\|W\|(A_i) > 0$ and

$$\|Z_{i+1} - W\|(A_i) \geq \|\lim Z - W\|(A_i) > 0,$$

in contradiction to the maximality of $c_i(Z_i)$ with respect to A_i.

Let $H = \{\lim Z : Z \in C\}$ and define $\xi : H \to \mathscr{P}$ by

$$\xi(W) = \sum_{C \in Z, \lim Z = W} \nu(Z) \quad \text{for } W \in H,$$

then (H, ξ) is a decomposition of V with respect to P. □

Remark 2.3.13. The structure of the proof of 2.3.12 is similar to the one of [Men16, 6.12]. However, since we allow to decompose varifolds not just by restriction to subsets, we should take the multiplicity of varifolds into account, which makes it much more complicated to verify the condition 2.3.8(2).

Chapter 3

Young functions, graph measures, and test function spaces

3.1 Topological tensor product and inductive limit

In this section, we study the interaction between inductive limit and topological tensor product in the category of locally convex spaces.

Lemma 3.1.1. *Suppose F is the strict inductive limit of a sequence of Banach spaces F_i with $F_i \subset F_{i+1}$ for $i \in \mathscr{P}$. Then, F is complete and Hausdorff, F_i is a closed subspace of F for all $i \in \mathscr{P}$, and every bounded subset (in particular, compact subset) of F is contained in some F_i and bounded there.*

Proof. It follows from [Bou87, II, §4, No. 6, Proposition 9] and [Bou87, III, §1, No. 4, Proposition 6]. □

Lemma 3.1.2 (see [Bou87, II, §4, No. 4, Corollary 2])**.** *Suppose A and L are sets, $\{J_\lambda : \lambda \in L\}$ is a partition of A, G_α is a locally convex space for $\alpha \in A$, F_λ is a vector space for $\lambda \in L$, E is a vector space, $g_{\lambda,\alpha} : G_\alpha \to F_\lambda$ is a linear map for $\alpha \in J_\lambda$ and $\lambda \in L$, $h_\lambda : F_\lambda \to E$ is a linear map for $\lambda \in L$, and we endow F_λ with the locally convex final topology with respect to $g_{\lambda,\alpha}$ for $\alpha \in J_\lambda$ and $\lambda \in L$.*

Then, the locally convex final topology on E induced by h_λ for $\lambda \in L$ agrees with the locally convex final topology induced by $h_\lambda \circ g_{\lambda,\alpha}$ for $\alpha \in J_\lambda$ and $\lambda \in L$.

Definition 3.1.3 (see [Bou87, II, §4, No. 5, Definition 2])**.** Suppose A is a set, F_α is a locally convex space for $\alpha \in A$, and F is the direct sum of F_α. Then, the *direct sum topology on F* is the locally convex final topology induced by the canonical injections $F_\alpha \to F$.

Lemma 3.1.4. *Suppose A and B are sets, F_α^β and F_α for $(\alpha, \beta) \in A \times B$ are locally convex spaces, F_α is endowed with the locally convex final topology induced by the linear maps $f_\alpha^\beta : F_\alpha^\beta \to F_\alpha$ for $\beta \in B$ whenever $\alpha \in A$. Then, the locally convex final topology $\mathcal{T}$ on $\bigoplus_{\alpha\in A} F_\alpha$ induced by the linear maps $\bigoplus_{\alpha\in A} f_\alpha^\beta$ for $\beta \in B$ agrees with the direct sum topology $\mathcal{T}'$ of $\bigoplus_{\alpha\in A} F_\alpha$.*

Proof. Denote the canonical injections by

$$i_\alpha : F_\alpha \to \bigoplus_{a\in A} F_a, \quad i_\alpha^\beta : F_\alpha^\beta \to \bigoplus_{a\in A} F_a^\beta.$$

In view of 3.1.2, $\mathcal{T}$ is the locally convex final topology induced by the linear maps $(\bigoplus_{a\in A} f_a^\beta) \circ i_\alpha^\beta$ for $(\alpha, \beta) \in A \times B$, and $\mathcal{T}'$ is the locally convex final topology induced by the linear maps $i_\alpha \circ f_\alpha^\beta$ for $(\alpha, \beta) \in A \times B$. On the other hand, we have

$$\left(\bigoplus_{a\in A} f_a^\beta\right) \circ i_\alpha^\beta = i_\alpha \circ f_\alpha^\beta$$

whenever $(\alpha, \beta) \in A \times B$. It follows that $\mathcal{T} = \mathcal{T}'$. □

Lemma 3.1.5 (see [Bou87, II, §4, No. 3, Proposition 4])**.** *Suppose A is a set, F_α is a locally convex space for $\alpha \in A$, F is a vector space, and $f_\alpha : F \to F_\alpha$ is a linear map for $\alpha \in A$. Then, the initial topology on F with respect to f_α for $\alpha \in A$ is a locally convex topology.*

Remark 3.1.6 (universal property). The following universal property of the Cartesian product of locally convex spaces follows from the corresponding property of topological vector spaces.

Suppose A is a set, G and F_α for $\alpha \in A$ are locally convex spaces, F is the Cartesian product of F_α for $\alpha \in A$, f_α is the projection map for each $\alpha \in A$, and $g_\alpha : G \to F_\alpha$ is a continuous linear map for each $\alpha \in A$. Then, there exists a unique continuous linear map $h : G \to F$ such that $f_\alpha \circ h = g_\alpha$ whenever $\alpha \in A$.

Lemma 3.1.7 (see [Bou87, II, §4, No. 5, Proposition 7])**.** *Suppose A is a set, F_α is a locally convex space for $\alpha \in A$. Then, the canonical injection*

$$\bigoplus_{\alpha\in A} F_\alpha \to \prod_{\alpha\in A} F_\alpha$$

is continuous. If A is finite, then this map is an isomorphism of locally convex spaces.

Remark 3.1.8. Suppose F is the inductive limit of locally convex spaces F_α. In view of 3.1.4 and 3.1.7, we have

$$\varinjlim (F_\alpha)^n \simeq \left(\varinjlim F_\alpha\right)^n .$$

Definition 3.1.9 (see [SW99, III, 6.1 – 6.3])**.** Suppose V and W are locally convex spaces. The *projective tensor product topology* of $V \otimes W$ is the finest locally convex topology on $V \otimes W$ such that the natural bilinear map $\mu : V \times W \to V \otimes W$ is continuous.

Furthermore, if V and W are normed, then $V \otimes W$ is normed by the *projective tensor product norm* with value

$$\inf \left\{ \sum_{i=1}^{n} |v_i||w_i| : \xi = \sum_{i=1}^{n} v_i \otimes w_i, v_i \in V, w_i \in W, i = 1, \dots, n \right\}$$

at $\xi \in V \otimes W$.

Remark 3.1.10. The projective tensor product topology satisfies the following property: whenever Z is a locally convex space and $g : V \times W \to Z$ is a continuous bilinear map, there exists a unique continuous linear map $h : V \otimes W \to Z$ such that $h = g \circ \mu$. In case that V, W and Z are normed, we have $\|g\| = \|h\|$.

Remark 3.1.11. Suppose $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are continuous linear maps between locally convex spaces. Then, their tensor product $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ is continuous with respect to the projective tensor product topologies.

Remark 3.1.12. Suppose V, W, and Z are normed spaces. It is straightforward to verify the following isometries of normed spaces

$$(V \otimes W) \otimes Z \simeq V \otimes (W \otimes Z), \quad V \otimes W \simeq W \otimes V,$$
$$\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}) \otimes V \simeq \mathrm{Hom}(\mathbf{R}^n, V).$$

Remark 3.1.13. Associating with a locally convex space V, the following isomorphism of locally convex spaces

$$\mathbf{R}^n \otimes V \simeq V^n.$$

yields a natural transformation.

Remark 3.1.14. Suppose F is the inductive limit of locally convex spaces F_α. By 3.1.8, 3.1.13 there holds

$$\varinjlim (\mathbf{R}^n \otimes F_\alpha) \simeq \varinjlim (F_\alpha)^n \simeq \left(\varinjlim F_\alpha\right)^n = F^n \simeq \mathbf{R}^n \otimes F = \mathbf{R}^n \otimes \left(\varinjlim F_\alpha\right).$$

Note that these inductive limits are strict if F is the strict inductive limit of F_i for $i \in \mathscr{P}$.

3.2 Young functions and graph measures

In this section, we introduce Young functions and graph measures, and we will focus on the compactness property of graph measures, leaving the discussion about Young functions to the next section.

Definition 3.2.1. Suppose X is a locally compact Hausdorff space. We define $\mathbf{P}(X)$ to be the space of all probability Radon measures over X, and it is endowed with the subspace topology induced from $\mathscr{K}(X)^*$.

Lemma 3.2.2. *Suppose X is a locally compact Hausdorff space. Then, the weak topology on $\mathbf{P}(X)$ agrees with the initial topology induced from the maps $\mu \mapsto \int g \,\mathrm{d}\mu$ corresponding to bounded continuous functions $g : X \to \mathbf{R}$.*

Consequently, a sequence μ_i converges to μ weakly if and only if $\int g \,\mathrm{d}\mu_i \to \int g \,\mathrm{d}\mu$ whenever $g : X \to \mathbf{R}$ is a bounded continuous function.

Proof. Clearly, the weak topology of $\mathbf{P}(X)$ is weaker than the initial topology. Denoting by $e_g(\mu) = \int g \,\mathrm{d}\mu$, then it is enough to show that $e_g : \mathbf{P}(X) \to \mathbf{R}$ is continuous with respect to the weak topology whenever $g : X \to \mathbf{R}$ is a non-negative bounded continuous function. By [Fed69, 2.2.5] and approximation with functions in $\mathscr{K}(X)$, we can show that

$$\mu(g) = \sup\{\mu(f) : 0 \le f \le g, f \in \mathscr{K}(X)\}$$

whenever $\mu \in \mathbf{P}(X)$ and $g : X \to \mathbf{R}$ is a non-negative bounded continuous function; in this case, it follows that e_g equals the supremum of the family of continuous functions e_f corresponding to $f \in \mathscr{K}(X)$ with $0 \le f \le g$, and therefore e_g is lower semi-continuous. Replacing g with $M - g$, where M is a large number such that $M - g \ge 0$, we have $e_g = -(e_{M-g} - M)$ is upper semi-continuous. □

Remark 3.2.3. The initial topology mentioned in the lemma is usually termed weak topology in texts on probability theory (see [Kle20, 13.14(ii)]), and the lemma shows that these two notions of weak topology agree.

Definition 3.2.4. Suppose X and Y are locally compact Hausdorff spaces and μ is a Radon measure over X. By a μ *Young function f of type Y*, we mean a μ measurable function with values in $\mathbf{P}(Y)$.

Remark 3.2.5. Suppose $\boldsymbol{\delta} : Y \to \mathscr{K}(Y)^*$ is given by $\boldsymbol{\delta}_y(\beta) = \beta(y)$ whenever $y \in Y$ and $\beta \in \mathscr{K}(Y)$, g maps a subset of X into Y, $\mu(X \sim \operatorname{dmn} g) = 0$, and $f = \boldsymbol{\delta} \circ g$. Recall from [Fed69, 2.5.19] that $\boldsymbol{\delta}$ is a homeomorphic embedding, it follows that g is μ measurable if and only if f is μ measurable.

Lemma 3.2.6. *Suppose F is the strict inductive limit of a sequence of separable complete locally convex spaces F_i whose topology is metrizable with $F_i \subset F_{i+1}$ for $i \in \mathscr{P}$, D is a dense subset of F, μ measures a set X, and f maps a subset of X into F^* such that $\mu(X \sim \operatorname{dmn} f) = 0$, where F^* denotes the topological dual space of F endowed with the weak topology.*

Then, f is μ measurable if and only if $\langle \beta, f(\cdot) \rangle$ is μ measurable whenever $\beta \in D$.

Proof. It is immediate from [Men16, 2.22]. □

Example 3.2.7. Let $m, Q \in \mathscr{P}$. Suppose f is a $\mathscr{L}^m$ measurable $\mathbf{Q}_Q(\mathbf{R}^n)$-valued function in the sense of [Alm00, 1.1]. By [DLS11, 0.4], there exists $\mathscr{L}^m$ measurable $\mathbf{R}^n$-valued functions $f_1, f_2, \ldots, f_Q$ such that $f = \boldsymbol{\delta} \circ f_1 + \boldsymbol{\delta} \circ f_2 + \cdots + \boldsymbol{\delta} \circ f_Q$; in particular, $Q^{-1} f$ is a $Q\mathscr{L}^m$ Young function of type $\mathbf{R}^n$ by 3.2.6.

Definition 3.2.8. Suppose X and Y are locally compact Hausdorff spaces, μ is a Radon measure over X, and f is μ Young function of type Y. Then the *graph measure* $\mathbf{Y}(\mu, f)$ *associated with* μ *and* f is the Radon measure over $X \times Y$ such that (see 3.2.9)

$$\mathbf{Y}(\mu, f)(\psi) = \iint \psi(x, y) \, \mathrm{d} f(x) \, y \, \mathrm{d}\mu \, x$$

whenever $\psi \in \mathscr{K}(X \times Y)$.

Remark 3.2.9. The notion of graph measure is well-defined. By [Bou04a, III, §4, No. 1, Lemma 1(ii)], for $\psi \in \mathscr{K}(X \times Y)$ and $i \in \mathscr{P}$, there exist $k_i \in \mathscr{P}$, $\alpha_j \in \mathscr{K}(X)$ and $\beta_j \in \mathscr{K}(Y)$ for $j = 1, \ldots, k_i$ such that

$$\left| \psi(x, y) - \sum_{j=1}^{k_i} \alpha_j(x) \beta_j(y) \right| \leq i^{-1} \quad \text{whenever } (x, y) \in X \times Y$$

and hence

$$\lim_{i \to \infty} \left| \int \psi(x, y) \, \mathrm{d} f(x) \, y - \sum_{j=1}^{k_i} \alpha_j(x) \int \beta_j(y) \, \mathrm{d} f(x) \, y \right| = 0 \quad \text{for } \mu \text{ almost all } x.$$

Thus, the map $x \mapsto \int \psi(x, y) \, \mathrm{d} f(x) \, y$ is μ measurable.

Remark 3.2.10. Suppose $m, n \in \mathscr{P}$ and g is a $\mathscr{L}^m$ measurable function with values in $\mathbf{R}^n$. Then, $f = \boldsymbol{\delta} \circ g$ is a $\mathscr{L}^m$ Young function of type $\mathbf{R}^n$; letting $G(x) = (x, g(x))$ for $x \in \operatorname{dmn} g$, we have that $G_\# \mathscr{L}^m$ is a Radon measure by [MS25a, 2.11] and [Fed69, 2.2.2], and that the graph measure $\mathbf{Y}(\mathscr{L}^m, f)$

associated with the pair $(\mathscr{L}^m, f)$ equals $G_\# \mathscr{L}^m$. Compared with the notion of measure-function pairs introduced by Hutchinson (see [Hut86, 4.1.1]), our setting does not require any summability condition on functions g; also, this example shows our notion of graph measure generalizes the one in [Hut86, 4.3.1].

Remark 3.2.11. Young functions are named after Laurence C. Young for their connection with Young measures. Suppose $m, n \in \mathscr{P}$ and U is a bounded open subset of $\mathbf{R}^m$. Using our terminology, the Young measures associated with a bounded sequence of functions g_i in $\mathbf{L}_\infty(\mathscr{L}^m \llcorner U)$ form a $\mathscr{L}^m \llcorner U$ Young function f of type $\mathbf{R}^n$ such that

$$\mathbf{Y}(\mathscr{L}^m \llcorner U, f) = \lim_{i \to \infty} \mathbf{Y}(\mathscr{L}^m \llcorner U, \boldsymbol{\delta} \circ g_i),$$

see [AFP00, 2.30(i)] and 3.2.9.

Example 3.2.12. Suppose $m, n \in \mathscr{P}$ and U is an open subset of $\mathbf{R}^n$. Employing the notation of Allard's varifold disintegration (see [All72, 3.3]), for each varifold $V \in \mathbf{V}_m(U)$, the formula $x \mapsto V^{(x)}$ defines a $\|V\|$ Young function of type $\mathbf{G}(n, m)$ such that $V = \mathbf{Y}(\|V\|, V^{(\cdot)})$. A similar statement also holds for Young's generalized surfaces (which are termed oriented varifolds in [Hut86, Section 3]), see [Fle20, Part I, Section 4, (b), Section 10] and [Fle57, Section 3].

The following lemma shows that the disintegration formula in the definition of graph measures holds for a much larger class of functions.

Lemma 3.2.13. *Suppose X and Y are second-countable locally compact Hausdorff spaces, μ is a Radon measure over X, and f is a μ Young function of type Y. Then, we have*

$$\textstyle\int \psi \, \mathrm{d}\mathbf{Y}(\mu, f) = \iint \psi(x, y) \, \mathrm{d}f(x)\, y \, \mathrm{d}\mu\, x$$

whenever ψ is a $\mathbf{Y}(\mu, f)$ integrable $\overline{\mathbf{R}}$-valued function.

Proof. Let F consist of all $\mathbf{Y}(\mu, f)$ measurable subsets A of $X \times Y$ such that the formula remains true with ψ replaced by the characteristic function χ_A of A.

Step 1. Note that every open subset of $X \times Y$ is a countable union of compact subsets of $X \times Y$, and the disintegration formula holds for $\psi \in \mathscr{K}(X \times Y)$. In view of [Fed69, 2.4.7], F contains all the open subsets of $X \times Y$ and F is stable under countable increasing unions. If $A_i \in F$ for $i \in \mathscr{P}$ satisfies $\mathbf{Y}(\mu, f)(A_1) < \infty$ and $A_{i+1} \subset A_i$ for $i \in \mathscr{P}$, then $\bigcap_{i=1}^{\infty} A_i \in F$ by [Fed69, 2.4.9].

Step 2. If $\mathbf{Y}(\mu, f)(A) = 0$, there exists a countable nonempty subfamily $G \subset F$ of open subsets in $X \times Y$ such that $A \subset \bigcap G$ and $\mathbf{Y}(\mu, f)(\bigcap G) = 0$. Therefore, we have

$$\int^* \chi_A(x, \cdot)\, \mathrm{d}f(x) \leq \int \chi_{\bigcap G}(x, \cdot)\, \mathrm{d}f(x) = 0 \quad \text{for } \mu \text{ almost all } x$$

and this shows F contains all the sets A with $\mathbf{Y}(\mu, f)(A) = 0$.

Step 3. If A is a $\mathbf{Y}(\mu, f)$ measurable set with $\mathbf{Y}(\mu, f)(A) < \infty$, then there exists a countable nonempty subfamily $G \subset F$ of open subsets in $X \times Y$ such that $A \subset \bigcap G$ and $\mathbf{Y}(\mu, f)((\bigcap G) \sim A) = 0$, hence

$$\int \chi_{(\bigcap G) \sim A}(x, \cdot)\, \mathrm{d}f(x) = 0 \quad \text{for } \mu \text{ almost all } x;$$

it follows that

$$\int \chi_A(x, \cdot)\, \mathrm{d}f(x) = \int \chi_{\bigcap G}(x, \cdot)\, \mathrm{d}f(x) \quad \text{for } \mu \text{ almost all } x,$$

hence $A \in F$. Note that $X \times Y$ is countably $\mathbf{Y}(\mu, f)$ measurable. Therefore, F contains all $\mathbf{Y}(\mu, f)$ measurable sets.

Now, the assertion follows from [Fed69, 2.3.3, 2.4.8, 2.4.4(6)]. □

To prove the compactness theorem and disintegration theorem, we want to reduce the problem by replacing Y with its one-point compactification Z; for this purpose, we study the relation between the Radon measures over Y and their image under the inclusion map $\iota : Y \to Z$.

Lemma 3.2.14. *Suppose Z is a compact Hausdorff space, C is a closed subset of Z, and M is a nonempty compact family of Radon measures over Z such that $\phi(C) = 0$ for all $\phi \in M$. Then, there holds*

$$\lim_{L \in P} \sup\{\phi(Z \sim L) : \phi \in M\} = 0,$$

where P is the family of compact subsets of $Z \sim C$ directed by the order $\preceq$ such that $L_1 \preceq L_2$ if and only if $L_1 \subset L_2$ for $L_1, L_2 \in P$.

Proof. Let $\lambda = \limsup_{L \in P} \sup\{\phi(Z \sim L) : \phi \in M\}$ and let F consist of those $f \in \mathscr{K}(Z)$ satisfying $0 \leq f \leq 1$, and $C \subset \operatorname{Int}\{z : f(z) = 1\}$. Since M is compact, whenever $f \in F$, there exists $\psi_f \in M$ such that

$$\psi_f(f) \geq \phi(f) \quad \text{whenever } \phi \in M;$$

furthermore, letting $L = Z \sim \operatorname{Int}\{z : f(z) = 1\}$, we have $L \in P$ and $\psi_f(f) \geq \phi(Z \sim L)$ whenever $\phi \in M$, hence $\psi_f(f) \geq \lambda$. Observe that $\{\operatorname{Clos}\{\psi_g : f \geq$

$g \in F\} : f \in F\}$ forms a family of closed subsets of M that possesses the finite intersection property, then there exists

$$\phi \in \bigcap_{f \in F} \operatorname{Clos}\{\psi_g : f \geq g \in F\}.$$

Since $\psi_g(f) \geq \psi_g(g) \geq \lambda$ and $f \mapsto \psi(f)$ for $\psi \in M$ is continuous whenever $f \geq g \in F$, we conclude $\phi(f) \geq \lambda$ whenever $f \in F$, thus $\lambda \leq \inf\{\phi(f) : f \in F\} = \phi(C) = 0$. $\square$

Lemma 3.2.15. *Suppose X and Y are locally compact Hausdorff spaces, Z is the one-point compactification of Y, and M is a family of Radon measures over $X \times Z$. Then, the following statements are equivalent.*

(1) $\operatorname{Clos} M$ *is compact and* $\phi(K \times (Z \sim Y)) = 0$ *whenever K is a compact subset of X and* $\phi \in \operatorname{Clos} M$.

(2) $\sup\{\phi(K \times Z) : \phi \in M\} < \infty$ *whenever K is a compact subset of X and*

$$\lim_{L \in P} \sup\{\phi(K \times (Z \sim L)) : \phi \in M\} = 0,$$

where P is the family of compact subsets of Y directed by the order $\preceq$ such that $L_1 \preceq L_2$ if and only if $L_1 \subset L_2$ for $L_1, L_2 \in P$.

Proof. To show (1) implies (2), we observe that for a compact subset K of X and a compact neighborhood G of K, there exists $f \in \mathscr{K}_{G \times Z}(X \times Z)^+$ such that $0 \leq f \leq 1$ and $K \times Z \subset \{x : f(x) = 1\}$, then mapping $\phi \in \operatorname{Clos} M$ onto the finite Radon measure $(\phi \llcorner f)|\mathbf{2}^{G \times Z}$ over $G \times Z$ defines a continuous map; from 3.2.14, we infer

$$\lim_{L \in P} \sup\{\phi(K \times (Z \sim L)) : \phi \in M\} = 0.$$

Conversely, for a compact subset K of X, a compact neighborhood G of K, and $L \in P$, there exists $f_L \in \mathscr{K}(X \times Z)$ such that

$$0 \leq f_L \leq 1, \quad K \times (Z \sim Y) \subset \operatorname{Int}\{x : f_L(x) = 1\}, \quad \operatorname{spt} f_L \subset G \times (Z \sim L);$$

it follows that

$$\psi(K \times (Z \sim Y)) \leq \psi(f_L) \leq \sup\{\phi(G \times (Z \sim L)) : \phi \in M\},$$

hence $\psi(K \times (Z \sim Y)) = 0$ whenever $\psi \in \operatorname{Clos} M$. $\square$

Lemma 3.2.16. *Suppose X and Y are locally compact Hausdorff spaces, Z is the one-point compactification of Y, $\iota : X \times Y \to X \times Z$ is the inclusion map, and the continuous monomorphism $j : \mathscr{K}(X \times Y) \to \mathscr{K}(X \times Z)$ is given by extension by zero to $X \times Z$. Then, there hold the following statements.*

(1) If X is a countable union of compact subsets and ψ is a Radon measure over $X \times Y$ such that $\psi(K \times Y) < \infty$ whenever K is a compact subset of X, then $\iota_\# \psi$ is a Radon measure over $X \times Z$.

(2) If ϕ is a Radon measure over $X \times Z$ such that $\phi(K \times (Z \sim Y)) = 0$ whenever K is a compact subset of X and ψ is the Radon measure representing $j^*(\phi)$, then we have $\psi = \phi | \mathbf{2}^{X \times Y}$ and ϕ is the Radon measure representing the member in $\mathscr{K}(X \times Z)^*$ induced by $\iota_\# \psi$.

(3) Suppose M is a compact family of Radon measures ϕ over $X \times Z$ such that $\phi(K \times (Z \sim Y)) = 0$ whenever K is a compact subset of X. The dual map $j^* : \mathscr{K}(X \times Z)^* \to \mathscr{K}(X \times Y)^*$ embeds M homeomorphically into $\mathscr{K}(X \times Y)^*$.

Proof. To prove (1), we first infer from [Fed69, 2.1.2] that every open subset of $X \times Z$ is $\iota_\# \psi$ measurable. Next, noting that $X \times Y$ is an open subset of $X \times Z$, we have

$$\begin{aligned} (\iota_\# \psi)(W) &= \psi(W \cap (X \times Y)) \\ &= \sup\{\psi(C) : C \text{ is compact, } C \subset W \cap (X \times Y)\} \\ &\leq \sup\{(\iota_\# \psi)(C) : C \text{ is compact, } C \subset W\} \end{aligned}$$

whenever W is an open subset of $X \times Z$. Let $\varepsilon > 0$ and let K_i be a sequence of compact subsets of X such that $X = \bigcup_{i=1}^\infty K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ for $i \in \mathscr{P}$. Since ψ is a Radon measure and $\psi(K_i \times Y) < \infty$, there exists a sequence of compact subsets L_i of Y such that

$$\psi(K_i \times (Y \sim L_i)) < 2^{-i} \varepsilon.$$

Then $W = \bigcup_{i=1}^\infty (\operatorname{Int} K_i) \times (Z \sim L_i)$ is an open superset of $X \times (Z \sim Y)$ such that $(\iota_\# \psi)(W) < \varepsilon$. For $A \subset X \times Z$, writing $A = (A \cap (X \times Y)) \cup (A \cap (X \times (Z \sim Y)))$, from the estimate above and the fact that ψ is a Radon measure, we conclude

$$(\iota_\# \psi)(A) = \inf\{(\iota_\# \psi)(W) : W \text{ is open, } A \subset W\}$$

whenever $A \subset X \times Z$.

As j is continuous, so is j^*. To prove (2), we compute by [Fed69, 2.4.18] that for $f \in \mathscr{K}(X \times Z)$ with $\operatorname{spt} f \subset X \times Y$,

$$\textstyle\int f \, \mathrm{d}(\iota_\# \psi) = \int f \circ \iota \, \mathrm{d}\psi = j^*(\phi)(f \circ \iota) = \phi(j(f \circ \iota)) = \int f \, \mathrm{d}\phi.$$

Then, the first conclusion of (2) follows. For arbitrary $f \in \mathscr{K}(X \times Z)$, letting K be the image of projection of $\operatorname{spt} f$ onto X, there exists a compact subset L of Y such that $\phi(K \times (Z \sim L))$ and $(\iota_\# \psi)(K \times (Z \sim L))$ are small; therefore, it allows us to approximate the integrals of f with the integrals of those functions whose supports are contained in $X \times Y$, and the second conclusion of (2) follows. By (2), $j^* | M$ is injective. Recalling that every continuous bijection from a compact space into a Hausdorff space is a homeomorphism, (3) follows. □

Theorem 3.2.17 (Compactness). *Suppose X and Y are locally compact Hausdorff spaces, X is a countable union of compact subsets, and M is a class of Radon measures over $X \times Y$ satisfying*

$$\sup\{\Gamma(K \times Y) : \Gamma \in M\} < \infty,$$
$$\lim_{\Gamma \in P} \sup\{\Gamma(K \times (Y \sim L)) : \Gamma \in M\} = 0,$$

whenever K is a compact subset of X, where P is the family of compact subsets of Y directed by the order $\preceq$ such that $L_1 \preceq L_2$ if and only if $L_1 \subset L_2$ for $L_1, L_2 \in P$.

Then, $\operatorname{Clos} M$ *is compact, the push-forward $p_\#$ maps* $\operatorname{Clos} M$ *continuously into $\mathscr{K}(X)^*$, and $p_\# \Gamma$ is a Radon measure over X whenever $\Gamma \in \operatorname{Clos} M$, where $p : X \times Y \to X$ is the projection map.*

Proof. Let Z be the one-point compactification of Y, let $\iota : X \times Y \to X \times Z$ be the inclusion map, let j be as in 3.2.16, and let $M' = \{\iota_\# \Gamma : \Gamma \in M\}$. From 3.2.16(1), the members of M' are Radon measures over $X \times Z$. Applying 3.2.15 with M replaced by M', we see that $\operatorname{Clos} M'$ is a compact family of Radon measures ϕ over $X \times Z$ satisfying $\phi(K \times (Z \sim Y)) = 0$ whenever K is a compact subset of X; thus, $\operatorname{Clos} M = j^*[\operatorname{Clos} M']$ is compact and $\iota_\# \circ j^* | \operatorname{Clos} M' = \mathbf{1}_{\operatorname{Clos} M'}$ by 3.2.16(2)(3) applied with M replaced by $\operatorname{Clos} M'$. By [Fed69, 2.2.17], $q_\#(\iota_\# \Gamma) = p_\# \Gamma$ is a Radon measure over X whenever $\Gamma \in \operatorname{Clos} M$, where $q : X \times Z \to X$ is the projection map, and the continuity of $p_\#$ follows from the continuity of $(j^* | \operatorname{Clos} M')^{-1}$. □

Remark 3.2.18. The proof also shows that $\iota_\#$ maps $\operatorname{Clos} M$ continuously into $\mathscr{K}(X \times Z)^*$. In particular, if $X \times Y$ is second-countable (and hence metrizable by [Kel75, Chapter 4, Theorem 16]) and M consists of a convergent sequence

Γ_i and $\Gamma = \lim_{i\to\infty} \Gamma_i$, then we have $\iota_\# \Gamma = \lim_{i\to\infty} \iota_\# \Gamma_i$. It follows from [AFP00, 1.62] that

$$\lim_{i\to\infty} \int g \,\mathrm{d}\Gamma_i = \int g \,\mathrm{d}\Gamma$$

for any bounded Borel function g such that $\operatorname{Clos} p[\operatorname{spt} g]$ is compact and the set of discontinuity points of g has Γ measure zero.

Next, we shall present a disintegration theorem that suits our setting, see also [AFP00, 2.28].

Lemma 3.2.19. *Suppose X and W are locally compact Hausdorff spaces, K and L are compact subsets of X and W, respectively, Z is a Banach space. and the map*

$$\iota : Z^{X\times W} \to (Z^W)^X$$

is characterized by

$$\iota(\eta)(x) = \eta(x, \cdot) \quad \textit{whenever } x \in X \textit{ and } \eta \in Z^{X\times W},$$

where B^A is the set of all functions $f : A \to B$. Then, the restriction

$$\iota | \mathscr{K}_{K\times L}(X \times W, Z) : \mathscr{K}_{K\times L}(X \times W, Z) \to \mathscr{K}_K(X, \mathscr{K}_L(W, Z))$$

defines a norm-preserving isomorphism of Banach spaces.

Proof. Clearly, we have $\sup \operatorname{im} |\eta| = \sup \operatorname{im} |\iota(\eta)|$ whenever $\eta \in \mathscr{K}_{K\times L}(X \times W, Z)$. For topological spaces A and B, we denote by $\mathscr{C}(A, B)$ the space of all continuous functions $f : A \to B$ endowed with the compact-open topology. In case that B is a normed space, it is straightforward to verify that this topology agrees with the locally convex topology induced by the semi-norms with value $\sup |f|[K]$ at $f \in \mathscr{C}(A, B)$ corresponding to compact subsets K of A. It follows that the inclusion map $\mathscr{K}_K(X, B) \to \mathscr{C}(X, B)$ is a homeomorphic embedding whenever B is a normed space and K is a compact subset of X. Since the restriction

$$\iota | \mathscr{C}(X \times W, Z) : \mathscr{C}(X \times W, Z) \to \mathscr{C}(X, \mathscr{C}(W, Z))$$

defines a bijection by [tD08, 2.4.7], the assertion is obvious. □

Remark 3.2.20. The above lemma is a generalization of [Bou04a, III, §4, No. 1, Lemma 1 (i)] to vector-valued functions.

Theorem 3.2.21. *Suppose X and Y are second-countable locally compact Hausdorff spaces, $p : X \times Y \to X$ is the projection map, and Γ is a Radon measure on $X \times Y$. If $\Gamma(K \times Y) < \infty$ for every compact subset K of X, then $p_\# \Gamma$ is a Radon measure over X and there exists a $p_\# \Gamma$ Young function f of type Y such that $\Gamma = \mathbf{Y}(p_\# \Gamma, f)$. Moreover, such a function f is $p_\# \Gamma$ almost unique.*

Proof. Suppose Z is the one-point compactification of Y, $\iota : Y \to Z$ is the inclusion map, and $q : X \times Z \to X$ is the projection map. Then Z is a second-countable compact Hausdorff space and by 3.2.16(1), $\mu = (\mathbf{1}_X \times \iota)_\# \Gamma$ is a Radon measure over $X \times Z$; in particular, $p_\# \Gamma = q_\# \mu$ is a Radon measure by [Fed69, 2.2.17]. Clearly, we have $\mu(X \times (Z \sim Y)) = 0$.

By 3.2.19, the function $S : \mathscr{K}(X, \mathscr{K}(Z)) \simeq \mathscr{K}(X \times Z)$ given by

$$S(u)(x, y) = u(x)(y) \quad \text{for } u \in \mathscr{K}(X, \mathscr{K}(Z))$$

is a linear isomorphism and apply [Fed69, 2.5.12] with $E = \mathscr{K}(Z)$ endowed with the sup-norm topology (which is separable by [Men16, 2.2, 2.23]), $L = \mathscr{K}(X)$, $\Omega = \mathscr{K}(X, \mathscr{K}(Z))$, $T = \mu \circ S$, then there exists a Radon measure ϕ over X and a ϕ measurable function k with values in $\mathscr{K}(Z)^*$ with respect to the weak topology such that $|k(x)| = 1$ for ϕ almost all x and such that

$$\textstyle\int S(u)\, \mathrm{d}\mu = T(u) = \int \langle u(x), k(x) \rangle \, \mathrm{d}\phi\, x \quad \text{whenever } u \in \mathscr{K}(X, \mathscr{K}(Z)).$$

For $\beta \in \mathscr{K}(Z)^+$, we have

$$0 \leq \textstyle\int \alpha(x)\beta(z) \, \mathrm{d}\mu\,(x, z) = \int \alpha(x) \langle \beta, k(x) \rangle \, \mathrm{d}\phi\, x \quad \text{whenever } \alpha \in \mathscr{K}(X)^+;$$

thus, $\langle \beta, k(x) \rangle \geq 0$ for ϕ almost all x. As $\mathscr{K}(Z)^+$ is separable, we conclude for ϕ almost all x, $k(x)$ is monotone, hence $1 = |k(x)| = k(x)(Z)$. Therefore, k is a ϕ Young function of type Z and $\mu = \mathbf{Y}(\phi, k)$, hence $p_\# \Gamma = q_\# \mu = \phi$. By 3.2.13, we conclude that $k(x)(Z \sim Y) = 0$ for ϕ almost all x.

Note that extension by zero gives a continuous monomorphism

$$j : \mathscr{K}(Y) \to \mathscr{K}(Z),$$

hence a continuous linear map

$$j^* : \mathscr{K}(Z)^* \to \mathscr{K}(Y)^*,$$

where $\mathscr{K}(Z)^*$ and $\mathscr{K}(Y)^*$ are endowed with the weak topology. Therefore, $f = j^* \circ k$ is a $p_\# \Gamma$ Young function of type Y such that

$$\begin{aligned} \textstyle\int \psi \, \mathrm{d}\Gamma &= \textstyle\int j(\psi(x, \cdot))(y) \, \mathrm{d}\mu\,(x, y) \\ &= \textstyle\int \langle j(\psi(x, \cdot)), k(x) \rangle \, \mathrm{d}\phi\, x \\ &= \textstyle\int \langle \psi(x, \cdot), f(x) \rangle \, \mathrm{d}\phi\, x \\ &= \textstyle\iint \psi(x, y) \, \mathrm{d}f(x)\, y \, \mathrm{d}(p_\# \Gamma)\, x \end{aligned}$$

whenever $\psi \in \mathscr{K}(X \times Y)$.

Finally, suppose g is a $p_\#\Gamma$ Young function of type Y such that

$$\mathbf{Y}(p_\#\Gamma, g) = \mathbf{Y}(p_\#\Gamma, f).$$

By 3.2.2 and 3.2.16(1), we see $x \mapsto \iota_\#(g(x))$ for $x \in \operatorname{dmn} g$ is a $p_\#\Gamma$ Young function of type Z. Note that $k(x)(Z \sim Y) = 0$, hence by 3.2.16(2), $\iota_\#(f(x)) = k(x)$ for $p_\#\Gamma$ almost all x. By 3.2.13 and [Fed69, 2.4.18], we compute for $u \in \mathscr{K}(X, \mathscr{K}(Z))$

$$\begin{aligned} T(u) &= \textstyle\int \langle u(x), k(x)\rangle \, \mathrm{d}(p_\#\Gamma)\, x \\ &= \textstyle\iint S(u)(x, \iota(y)) \, \mathrm{d}f(x)\, y \, \mathrm{d}(p_\#\Gamma)\, x \\ &= \textstyle\int S(u)(x, \iota(y)) \, \mathrm{d}\mathbf{Y}(p_\#\Gamma, f)\, (x, y) \\ &= \textstyle\int S(u)(x, \iota(y)) \, \mathrm{d}\mathbf{Y}(p_\#\Gamma, g)\, (x, y) \\ &= \textstyle\iint S(u)(x, \iota(y)) \, \mathrm{d}g(x)\, y \, \mathrm{d}(p_\#\Gamma)\, x \\ &= \textstyle\iint S(u)(x, z) \, \mathrm{d}\iota_\#(g(x))\, z \, \mathrm{d}(p_\#\Gamma)\, x \\ &= \textstyle\int \langle u(x), \iota_\#(g(x))\rangle \, \mathrm{d}(p_\#\Gamma)\, x. \end{aligned}$$

From the uniqueness of k, we conclude $\iota_\#(g(x)) = k(x)$, hence by 3.2.16(2), $g(x) = j^*(\iota_\#(g(x))) = (j^* \circ k)(x) = f(x)$ for $p_\#\Gamma$ almost all x. □

Finally, we will finish this section with the compactness theorem for pairs of rectifiable varifolds and Young functions; for this purpose, we need two more items.

Definition 3.2.22. Suppose $m, n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, and C is a closed subset of $\mathbf{R}$ satisfying $\inf C \geq 1$ and

$$c + d \in C \quad \text{whenever } c, d \in C.$$

Then, $\mathbf{RV}_m(U, C)$ denotes the class of m-dimensional rectifiable varifolds V such that $\boldsymbol{\Theta}^m(\|V\|, x) \in C$ for $\|V\|$ almost all x.

The following lemma shows that if $V \in \mathbf{RV}_m(U)$, then there is a one-to-one correspondence between $\|V\|$ Young functions and V Young functions obtained from precomposition with the functions τ and p as in the lemma.

Lemma 3.2.23. *Suppose $V \in \mathbf{RV}_m(U)$ and define τ by*

$$\tau(x) = (x, \operatorname{Tan}^m(\|V\|, x)) \quad \textit{whenever } \operatorname{Tan}^m(\|V\|, x) \in \mathbf{G}(n, m).$$

Then, the following statements hold.

(1) $V = \tau_\# \|V\|$.

(2) *For* $A \subset U \times \mathbf{G}(n,m)$*,* A *is* V *measurable if and only if* $\tau^{-1}[A]$ *is* $\|V\|$ *measurable.*

(3) $\tau(p(x,S)) = (x,S)$ *for* V *almost all* (x,S)*, where* $p : U \times \mathbf{G}(n,m) \to U$ *is the projection map.*

Proof. By [MS25a, 2.11] and [All72, 3.5(1b)], $\tau_\# \|V\|$ is a Radon measure and $V = \tau_\# \|V\|$. Let $A \subset U \times \mathbf{G}(n,m)$. Clearly, if $\tau^{-1}[A]$ is $\|V\|$ measurable, then A is $V = \tau_\# \|V\|$ measurable. If A is V measurable, then there exists a Borel set B such that $A \subset B$ and $V(B \sim A) = 0$, hence $\|V\| \tau^{-1}[B \sim A] = 0$. Therefore,

$$\tau^{-1}[A] = \tau^{-1}[B] \sim \tau^{-1}[B \sim A]$$

is $\|V\|$ measurable and (2) follows. To prove (3), we note that

$$\tau^{-1}[\{(x,S) : (x,S) \neq (x, \operatorname{Tan}^m(\|V\|, x))\}] = \varnothing,$$

and the assertion follows from (1). □

Theorem 3.2.24 (Compactness)**.** *Suppose* m*,* n*,* U*, and* C *are as in 3.2.22,* Y *is a second-countable locally compact Hausdorff space,* L_j *is a sequence of compact subsets of* Y *such that* $\bigcup_{j=1}^\infty L_j = Y$ *and* $L_j \subset \operatorname{Int} L_{j+1}$ *whenever* $j \in \mathscr{P}$*, and* (V_i, f_i) *is a sequence of pairs of* V_i *in* $\mathbf{RV}_m(U,C)$ *and* $\|V_i\|$ *Young functions* f_i *of type* Y *such that*

$$\lim_{j\to\infty} \sup\{\mathbf{Y}(\|V_i\|, f_i)(K \times (Y \sim L_j)) : i \in \mathscr{P}\} = 0,$$
$$\sup\{(\|V_i\| + \|\delta V_i\|)(K) : i \in \mathscr{P}\} < \infty$$

whenever K *is a compact subset of* U*.*

Then, there exists $V \in \mathbf{RV}_m(U,C)$*, a* $\|V\|$ *Young function* f *of type* Y*, and a subsequence* (V_{i_k}, f_{i_k}) *of* (V_i, f_i) *such that, as* $k \to \infty$*,*

$$V_{i_k} \to V, \quad \mathbf{Y}(V_{i_k}, f_{i_k} \circ p) \to \mathbf{Y}(V, f \circ p), \quad \mathbf{Y}(\|V_{i_k}\|, f_{i_k}) \to \mathbf{Y}(\|V\|, f),$$

where $p : U \times \mathbf{G}(n,m) \to U$ *is the projection map.*

Proof. By [All72, 2.6(2a)], [Men16, 2.23], 2.3.4, 3.2.17, and 3.2.21, there exist a subsequence (V_{i_k}, f_{i_k}), $V \in \mathbf{RV}_m(U,C)$, and a V Young function g of type Y such that

$$V_i \to V, \quad \mathbf{Y}(V_{i_k}, f_{i_k} \circ p) \to \mathbf{Y}(V, g) \quad \text{as } k \to \infty.$$

From 3.2.23, we infer that $f(x) = (g \circ \tau)(x)$ for $x \in \operatorname{dmn} \tau$ defines a $\|V\|$ Young function of type Y such that

$$\mathbf{Y}(V_{i_k}, f_{i_k} \circ p) \to \mathbf{Y}(V, f \circ p), \quad \mathbf{Y}(\|V_{i_k}\|, f_{i_k}) \to \mathbf{Y}(\|V\|, f) \quad \text{as } k \to \infty,$$

where τ is as in 3.2.23. □

3.3 The space of probability Radon measures and operations on Young functions

In this section, we first define a metric on the space of probability Radon measures and study its topological and metric structures. Then, we show that the class of Young functions is stable under several operations.

Definition 3.3.1. Suppose Y is a finite-dimensional Banach space. For $0 \le s < \infty$, the space E_s consists of functions $\gamma : Y \to \mathbf{R}$ of class 1 such that $\gamma(0) = 0$ and $\operatorname{spt} \mathrm{D}\,\gamma \subset \mathbf{B}(0, s)$. We endow E_s with the norm whose value equals

$$\sup \operatorname{im} \|\mathrm{D}\,\gamma\| \quad \text{at } \gamma \in E_s,$$

and we endow $E = \bigcup\{E_s : 0 \le s < \infty\}$ with the locally convex final topology induced by the inclusion maps $E_s \to E$.

Remark 3.3.2. Suppose Y is a finite-dimensional Banach space. Letting $k = \dim Y$ and employing the linear isomorphism $Y \simeq \mathbf{R}^k$, the standard mollification argument in Euclidean spaces also works for Y.

Remark 3.3.3. By Taylor's theorem, we readily check that

$$|\gamma(y)| \le \inf\{|y|, s\} \sup \operatorname{im} \|\mathrm{D}\,\gamma\| \quad \text{for } y \in Y$$

whenever $\gamma \in E_s$.

Lemma 3.3.4. *Suppose Y is a finite-dimensional Banach space and $0 \le s < \infty$. Then, the canonical map $\mathbf{P}(Y) \to (E_s)^*$ is continuous with bounded image, where $(E_s)^*$ is endowed with the dual norm topology.*

Proof. By [Fed69, 2.10.21], the function space

$$E_s \cap \{\gamma : \sup \operatorname{im} \|\mathrm{D}\,\gamma\| \le 1\}$$

is contained in the compact space F of Lipschitzian functions $f : Y \to \mathbf{R}$ with $f(0) = 0$ and $\operatorname{Lip} f \le 1$, where F is endowed with the topology of uniform convergence on each compact subset of Y, or equivalently, the topology induced by the metric ρ with value

$$\rho(f, g) = \sum_{i=1}^{\infty} 2^{-i} \cdot \frac{\sup \operatorname{im} |(f-g)|\mathbf{B}(0,i)|}{1 + \sup \operatorname{im} |(f-g)|\mathbf{B}(0,i)|} \quad \text{at } (f, g) \in F \times F.$$

Note that

$$\sup \operatorname{im} |\gamma_1 - \gamma_2| = \sup\{|\gamma_1(y) - \gamma_2(y)| : y \in Y \cap \mathbf{B}(0, s)\}$$

whenever $\gamma_1, \gamma_2 \in E_s$. Then, it is straightforward to show that the function space $E_s \cap \{\gamma : \sup \operatorname{im} \| \mathrm{D}\, \gamma\| \leq 1\}$ is totally bounded with respect to the supremum metric. With the help of 3.2.2, it follows that every convergent sequence in $\mathbf{P}(Y)$ also converges in $(E_s)^*$. Since $\mathbf{P}(Y) \subset \mathscr{K}(Y)^*$ is metrizable by [Men16, 2.23], the continuity of the canonical map follows, and the boundedness of its image is immediate by 3.3.3. □

Lemma 3.3.5. *Suppose Y is a finite-dimensional Banach space. There holds*

$$|y_1 - y_2| = \sup\{\gamma(y_1) - \gamma(y_2) : \gamma \in E, \sup \operatorname{im} \| \mathrm{D}\, \gamma\| \leq 1\}$$

whenever $y_1, y_2 \in Y$.

Proof. Since $|\gamma(y_1) - \gamma(y_2)| \leq |y_1 - y_2| \sup \operatorname{im} \| \mathrm{D}\, \gamma\|$ whenever $\gamma \in E$, it suffices to show there exists a sequence $\gamma_i \in E$ such that

$$\lim_{i \to \infty} \gamma_i(y_1) - \gamma_i(y_2) = |y_1 - y_2|.$$

Note that there exists $L \in \operatorname{Hom}(Y, \mathbf{R})$ such that $L(y_1 - y_2) = |y_1 - y_2|$ and $\|L\| \leq 1$. Whenever $i \in \mathscr{P}$, since $L_i = \sup\{\inf\{L, i\}, -i\}$ is uniformly continuous and $L_i(0) = 0$, there exists $\beta_i : Y \to \mathbf{R}$ of class 1 such that $\beta_i(0) = 0$, $\sup \operatorname{im} \| \mathrm{D}\, \beta_i\| \leq 1$, and $\sup \operatorname{im} |L_i - \beta_i| \leq i^{-1}$. Choose a function $\phi : Y \to \mathbf{R}$ of class 1 such that $0 \leq \phi \leq 1$, $\phi|\mathbf{B}(0,1) = 1$, $\sup \operatorname{im} \| \mathrm{D}\, \phi\| \leq 1$, and $\operatorname{spt} \phi$ is compact, and we define $\phi_i(y) = (1 - i^{-1})\phi(i^{-3}y)$ whenever $y \in Y$ and $i \in \mathscr{P}$. Then, we have $\gamma_i = \phi_i \beta_i \in E$, $\sup \operatorname{im} \| \mathrm{D}\, \gamma_i\| \leq 1$ for $i \geq 1$, and $L(y) = \lim_{i \to \infty} \gamma_i(y)$ for all $y \in Y$ and the assertion follows. □

Definition 3.3.6. Suppose Y is a finite-dimensional Banach space, we define the pseudo-metric d on $\mathbf{P}(Y)$ by

$$d(\mu, \nu) = \sup\left\{\textstyle\int \gamma \,\mathrm{d}\mu - \int \gamma \,\mathrm{d}\nu : \gamma \in E, \sup \operatorname{im} \| \mathrm{D}\, \gamma\| \leq 1\right\}$$

whenever $\mu, \nu \in \mathbf{P}(Y)$.

Remark 3.3.7. If $\mu, \nu \in \mathbf{P}(Y)$ satisfy $d(\mu, \nu) = 0$, we have $\int \gamma \,\mathrm{d}\mu = \int \gamma \,\mathrm{d}\nu$ whenever $\gamma \in \mathscr{K}(Y)$ with $\operatorname{Lip} \gamma < \infty$; it follows that $\mu = \nu$.

Remark 3.3.8. By approximation, we readily show that

$$\begin{aligned} d(\mu, \nu) &= \sup\left\{\textstyle\int \gamma \,\mathrm{d}\mu - \int \gamma \,\mathrm{d}\nu : \gamma : Y \to \mathbf{R}, \operatorname{Lip} \gamma \leq 1\right\} \\ &= \sup\left\{\textstyle\int \gamma \,\mathrm{d}\mu - \int \gamma \,\mathrm{d}\nu : \gamma \in \mathscr{K}(Y), \operatorname{Lip} \gamma \leq 1\right\}. \end{aligned}$$

whenever $\mu, \nu \in \mathbf{P}(Y)$.

Definition 3.3.9. Suppose (X, ρ) is a pseudo-metric space and Y is a finite-dimensional Banach space. A function $f : X \to \mathbf{P}(Y)$ is termed to be *Lipschitzian* if and only if there exists $0 < M < \infty$ such that

$$d(f(x), f(y)) \le M\rho(x, y) \quad \text{whenever } x, y \in X,$$

where d as in 3.3.6. The Lipschitz constant of f is defined to be the infimum of all such numbers M. We say f is *locally Lipschitzian* if and only if whenever $x \in X$, there exists a neighborhood U of x such that $f|U$ is Lipschitzian.

Remark 3.3.10. Note that if X is a metric space and f is a Lipschitzian $\mathbf{P}(Y)$-valued function, then we have $d(f(x), f(y)) < \infty$ whenever $x, y \in X$; in particular, $d|(\operatorname{im} f \times \operatorname{im} f)$ is a metric and f is Lipschitzian in the usual sense.

Remark 3.3.11. By 3.3.8, we have

$$d(\boldsymbol{\delta}_0, \mu) = \textstyle\int |y| \, \mathrm{d}\mu \, y \quad \text{whenever } \mu \in \mathbf{P}(Y).$$

Let

$$\mathbf{P}_1(Y) = \mathbf{P}(Y) \cap \left\{ \mu : \textstyle\int |y| \, \mathrm{d}\mu \, y < \infty \right\}.$$

Then, $(\mathbf{P}_1(Y)), d)$ is a metric space.

Next, we shall investigate the relation between the weak topology and the d topology on $\mathbf{P}_1(Y)$ when Y is a finite-dimensional Banach space.

Remark 3.3.12. The notation $\mathbf{P}_1(Y)$ is taken from [Vil09, 6.4], on which the metric d agree with 1-Wasserstein distance, see [Vil09, 6.1, 6.5]. The following characterization of d convergence on $\mathbf{P}_1(Y)$ is a special case of [Vil09, 6.9].

Lemma 3.3.13 (see [Vil09, 6.8, 6.9])**.** *Suppose Y is a finite-dimensional Banach space, μ_i is a sequence in $\mathbf{P}_1(Y)$, and $\mu \in \mathbf{P}_1(Y)$. Then, the following two statements are equivalent.*

(1) $\lim_{i\to\infty} d(\mu_i, \mu) = 0$.

(2) $\mu_i \to \mu$ *weakly as* $i \to \infty$ *and* $\lim_{i\to\infty} \int |y| \, \mathrm{d}\mu_i \, y = \int |y| \, \mathrm{d}\mu \, y$.

Proof. Suppose $\lim_{i\to\infty} d(\mu_i, \mu) = 0$, then

$$\lim_{i\to\infty} \textstyle\int |y| \, \mathrm{d}\mu_i \, y = \lim_{i\to\infty} d(\boldsymbol{\delta}_0, \mu_i) = d(\boldsymbol{\delta}_0, \mu) = \int |y| \, \mathrm{d}\mu \, y.$$

Whenever $\varepsilon > 0$ and $f \in \mathscr{K}(Y)$, since f is uniformly continuous, there exists a function $\gamma : Y \to \mathbf{R}$ of class 1 such that $\operatorname{spt} \gamma$ is compact and $\sup \operatorname{im} |f - \gamma| < \varepsilon$, hence we have

$$|\mu_i(f) - \mu(f)| \le 2\varepsilon + |\mu_i(\gamma) - \mu(\gamma)| \le 2\varepsilon + \sup \operatorname{im} \| \mathrm{D}\, \gamma \| d(\mu_i, \mu).$$

Therefore, we conclude $\mu_i \to \mu$ weakly as $i \to \infty$.

To prove the converse, letting $\varepsilon > 0$, we will first show that there exists $0 \le s < \infty$ such that $\int_{Y \sim \mathbf{B}(0,s)} |y| \, \mathrm{d}\mu \, y \le \varepsilon$ and

$$\int_{Y \sim \mathbf{B}(0,s)} |y| \, \mathrm{d}\mu_i \, y \le \varepsilon \quad \text{whenever } i \in \mathscr{P}.$$

Since μ is a Radon measure, there exists $f \in \mathscr{K}(Y)$ such that $0 \le f \le 1$ and $\int (1 - f(y))|y| \, \mathrm{d}\mu \, y \le \varepsilon/2$. From the hypothesis, there exists $N \in \mathscr{P}$ such that

$$\left| \int f(y)|y| \, \mathrm{d}\mu_i \, y - \int f(y)|y| \, \mathrm{d}\mu \, y \right| + \left| \int |y| \, \mathrm{d}\mu_i \, y - \int |y| \, \mathrm{d}\mu \, y \right| \le \varepsilon/2$$

whenever $i \ge N$. Then, we have

$$\int_{Y \sim \operatorname{spt} f} |y| \, \mathrm{d}\mu_i \, y \le \varepsilon \quad \text{whenever } i \ge N$$

and the assertion follows. On the other hand, by [Fed69, 2.10.21], the set $\mathscr{K}_{\mathbf{B}(0,r)}(Y) \cap \{\gamma : \operatorname{Lip} \gamma \le 1\}$ is compact in the Banach space $\mathscr{K}_{\mathbf{B}(0,r)}(Y)$; since each member of $\mathbf{P}(Y)$ defines a continuous linear functional over $\mathscr{K}_{\mathbf{B}(0,r)}$, we have

$$\lim_{i \to \infty} \sup \{ \int \gamma \, \mathrm{d}\mu_i - \int \gamma \, \mathrm{d}\mu : \gamma \in \mathscr{K}_{\mathbf{B}(0,r)}(Y), \operatorname{Lip} \gamma \le 1 \} = 0$$

whenever $0 < r < \infty$. It follows that $\lim_{i \to \infty} d(\mu_i, \mu) = 0$. □

Remark 3.3.14. Since the weak topology on the space of Radon measures over Y is metrizable by [Men16, 2.23], it follows that the map

$$(\mathbf{P}_1(Y), d) \to \mathbf{P}(Y) \times \mathbf{R}$$

defined by $\mu \mapsto (\mu, \int |y| \, \mathrm{d}\mu \, y)$ is a homeomorphic embedding.

Remark 3.3.15. The d topology is strictly finer than the weak topology on $\mathbf{P}_1(Y)$ provided $\dim Y \ge 1$; for instance, letting $0 \ne v \in Y$ and $\mu \in \mathbf{P}_1(Y)$, the sequence $\mu_i = (1 - i^{-1})\mu + i^{-1} \boldsymbol{\delta}_{iv}$ converges weakly to μ as $i \to \infty$ but

$$\lim_{i \to \infty} \int |y| \, \mathrm{d}\mu_i \, y - \int |y| \, \mathrm{d}\mu \, y = |v| > 0.$$

However, these two topologies induce the same subspace topology on the weakly closed subsets

$$A(\varepsilon) = \mathbf{P}(Y) \cap \{\mu : \mu\{a\} \ge \varepsilon \text{ whenever } a \in \operatorname{spt} \mu\} \subset \mathbf{P}_1(Y)$$

whenever $\varepsilon > 0$. To show this, suppose $\varepsilon > 0$ and μ_i is a sequence in $A(\varepsilon)$ that converges weakly to some $\mu \in \mathbf{P}(Y)$. Whenever $0 < r < \infty$ and

$a \in \operatorname{spt} \mu$, by [All72, 2.6(2c)], there exists $N \in \mathscr{P}$ such that $\mu_i(\mathbf{U}(a,r)) > 0$, hence $\mu_i(\mathbf{U}(a,r)) \geq \varepsilon$, whenever $i \geq N$; it follows that

$$\mu(\mathbf{B}(a,r)) \geq \limsup_{i\to\infty} \mu_i(\mathbf{B}(a,r)) \geq \varepsilon.$$

Thus, we conclude $\mu \in A(\varepsilon)$. Since $\lim_{i\to\infty} \int (1-f)\,\mathrm{d}\mu_i = \int (1-f)\,\mathrm{d}\mu$ whenever $f \in \mathscr{K}(Y)$ and note that $\nu(S) > 0$ implies $\nu(S) \geq \varepsilon$ whenever $S \subset Y$ and $\nu \in A(\varepsilon)$, there exists $0 < s < \infty$ such that

$$\sup\{\mu_i(Y \sim \mathbf{B}(0,s)) : i \in \mathscr{P}\} = 0 \quad \text{for some } 0 < s < \infty;$$

by 3.2.2,

$$\lim_{i\to\infty} \int |y|\,\mathrm{d}\mu_i\, y = \lim_{i\to\infty} \int \inf\{|y|, s\}\,\mathrm{d}\mu_i\, y = \int \inf\{|y|, s\}\,\mathrm{d}\mu\, y = \int |y|\,\mathrm{d}\mu\, y.$$

The following two lemmas show that the class of Young functions is stable under pushforward by Borel functions pointwise and taking the Cartesian product of measures pointwise.

Lemma 3.3.16. *Suppose X and Y are second-countable locally compact Hausdorff spaces and $f : X \to Y$ is a Borel function. Then, the pushforward*

$$(f_\# \mu)(A) = \mu(f^{-1}[A]) \quad \textit{whenever } \mu \in \mathbf{P}(X) \textit{ and } A \subset Y,$$

defines a Borel function $f_\# : \mathbf{P}(X) \to \mathbf{P}(Y)$. Moreover, if f is continuous, then so is $f_\#$.

Proof. By [Kel75, Chapter 4, Theorem 16], [MS25a, 2.11], and [Fed69, 2.2.2], $f_\#\mu$ is a Radon measure over Y. Note that it is enough to show $\mu \mapsto \int g\,\mathrm{d}(f_\#\mu)$ is a Borel function whenever $g \in \mathscr{K}(Y)$ by [Men16, 2.23]. Since $\int g\,\mathrm{d}(f_\#\mu) = \int (g \circ f)\,\mathrm{d}\mu$ whenever $\mu \in \mathbf{P}(X)$ and $g : Y \to \mathbf{R}$ is a bounded Borel function by [Fed69, 2.4.18], it reduces to show $\mu \mapsto \int h\,\mathrm{d}\mu$ is a Borel function whenever $h : X \to \mathbf{R}$ is a bounded Borel function.

For $i \in \mathscr{P}$, let B_i be the family of all Borel functions $h : X \to \mathbf{R}$ with $\sup \operatorname{im} |h| \leq i$ such that $\mu \mapsto \int h\,\mathrm{d}\mu$ is a Borel function on $\mathbf{P}(X)$. By 3.2.2, the set B_i contains all continuous functions from X into $\mathbf{R} \cap \{r : |r| \leq i\}$. By [Fed69, 2.4.9] and employing the terminology of [Fed69, 2.2.15], B_i is a Baire class, hence contains all Baire functions with image contained in $\mathbf{R} \cap \{r : |r| \leq i\}$; applying [Fed69, 2.2.15] with $Y = \mathbf{R} \cap \{r : |r| \leq i\}$, we see B_i contains all Borel functions $h : X \to \mathbf{R}$ with $\sup \operatorname{im} |g| \leq i$. Therefore, the main conclusion follows.

For the postscript, it is enough to show $\mu \mapsto \int g\,\mathrm{d}(f_\#\mu) = \int (g \circ f)\,\mathrm{d}\mu$ is continuous whenever $g \in \mathscr{K}(Y)$, which is immediate from 3.2.2. □

Lemma 3.3.17. *Suppose X and Y are second-countable locally compact Hausdorff spaces. Then, the product of measures*

$$(\mu, \nu) \mapsto \mu \times \nu \quad \textit{whenever } \mu \in \mathbf{P}(X) \textit{ and } \nu \in \mathbf{P}(Y)$$

defines a continuous function $\mathbf{P}(X) \times \mathbf{P}(Y) \to \mathbf{P}(X \times Y)$.

Proof. By Fubini's theorem [Fed69, 2.6.2], $\mu \times \nu$ is a Borel regular measure, hence by [Fed69, 2.2.2], a Radon measure whenever $\mu \in \mathbf{P}(X)$ and $\nu \in \mathbf{P}(Y)$.

It is enough to show that $e_f(\mu, \nu) = \int f \, \mathrm{d}(\mu \times \nu)$ defines a continuous function on $\mathbf{P}(X) \times \mathbf{P}(Y)$ whenever $f \in \mathscr{K}(X \times Y)$. If $f \in \mathscr{K}(X \times Y)$ satisfies $f(x, y) = \alpha(x)\beta(y)$ whenever $(x, y) \in X \times Y$ for some $\alpha \in \mathscr{K}(X)$ and $\beta \in \mathscr{K}(Y)$, then e_f is continuous. Therefore, in view of [Bou04a, III, §4, No. 1, Lemma 1 (ii)], for each $f \in \mathscr{K}(X \times Y)$, e_f is the uniform limit of a sequence of continuous functions, hence itself a continuous function. □

Remark 3.3.18. The two lemmas above are variants of [Kec95, 17.28, 17.40].

Definition 3.3.19. Suppose Y and Z are second-countable locally compact Hausdorff spaces, μ is a Radon measure over a locally compact Hausdorff space X, $f : Y \to Z$ is a Borel function, and g is a μ Young function of type Y. Then, we define the μ Young function $f_\# g$ of type Z by $(f_\# g)(x) = f_\#(g(x))$ whenever $x \in \operatorname{dmn} g$.

Definition 3.3.20. Suppose X and Y are second-countable locally compact Hausdorff spaces, μ is a Radon measure over X, and f and g are μ Young functions of type Y. Then, we define the μ Young function $f \times g$ by $(f \times g)(x) = f(x) \times g(x)$ whenever $x \in \operatorname{dmn} f \cap \operatorname{dmn} g$.

We finish this section by introducing the sum of two vector-valued functions and the product of two real-valued functions in the context of Young functions.

Definition 3.3.21 (see [Kle20, 14.20])**.** Suppose μ and ν are measures over a vector space Y, and $A : Y \times Y \to Y$ is defined by

$$A(x, y) = x + y \quad \text{whenever } x, y \in Y.$$

Then, the measure $\mu * \nu = A_\#(\mu \times \nu)$ is termed the *convolution of* μ *and* ν.

Remark 3.3.22 (see [Kle20, 14.21]). Suppose Y is a finite-dimensional Banach space. We readily verify the following basic properties of convolutions of measures.

(1) Whenever $\mu, \nu \in \mathbf{P}(Y)$, we have $\mu * \nu = \nu * \mu$.

(2) If $\mu, \nu \in \mathbf{P}(Y)$, then $\mu * \nu \in \mathbf{P}(Y)$.

(3) If $\mu, \nu \in \mathbf{P}(Y)$ satisfy $\mu = \sum_{x \in A} f(x) \boldsymbol{\delta}_x$ and $\nu = \sum_{y \in B} g(y) \boldsymbol{\delta}_y$ for some finite subsets A and B of Y and functions $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, then $\mu * \nu = \sum_{(x,y) \in A \times B} f(x) g(y) \boldsymbol{\delta}_{x+y}$.

Lemma 3.3.23. *Suppose Y is a finite-dimensional Banach space. Then, the convolution of measures*

$$(\mu, \nu) \mapsto \mu * \nu \quad \textit{whenever } \mu, \nu \in \mathbf{P}(Y)$$

defines a continuous function $\mathbf{P}(Y) \times \mathbf{P}(Y) \to \mathbf{P}(Y)$.

Proof. Combine 3.3.17 and 3.3.16. □

Remark 3.3.24. Similarly, if $Y = \mathbf{R}$, the multiplication on $\mathbf{R}$ induces a continuous function $\mathbf{P}(\mathbf{R}) \times \mathbf{P}(\mathbf{R}) \to \mathbf{P}(\mathbf{R})$.

Definition 3.3.25. Suppose X is a locally compact Hausdorff space, Y is a finite-dimensional Banach space, μ is a Radon measure over X, and f and g are μ Young functions of type Y. Then, we define the μ Young function $f * g$ by $(f * g)(x) = f(x) * g(x)$ whenever $x \in \operatorname{dmn} f \cap \operatorname{dmn} g$.

Lemma 3.3.26. *Suppose Y is a finite-dimensional Banach space, d is as in 3.3.6. Then, there holds*

$$d(\mu * \nu, \lambda * \eta) \le d(\mu, \lambda) + d(\nu, \eta) \quad \textit{whenever } \mu, \nu, \lambda, \eta \in \mathbf{P}(Y).$$

Consequently, the convolution $f*g$ of two Lipschitzian $\mathbf{P}(Y)$-valued functions f and g is again Lipschitzian and $\operatorname{Lip} f * g \le \operatorname{Lip} f + \operatorname{Lip} g$.

Proof. The assertion follows from 3.3.8, 3.3.22(1), and the estimate

$$\begin{aligned} \textstyle\int \gamma \, \mathrm{d}(\mu * \lambda) - \int \gamma \, \mathrm{d}(\nu * \lambda) &= \textstyle\int \left(\int \gamma(x+y) \, \mathrm{d}\mu \, x - \int \gamma(x+y) \, \mathrm{d}\nu \, x \right) \, \mathrm{d}\lambda \, y \\ &\le d(\mu, \nu) \end{aligned}$$

whenever $\mu, \nu, \lambda \in \mathbf{P}(Y)$ and $\gamma \in \mathscr{K}(Y)$ with $\operatorname{Lip} \gamma \le 1$. □

3.4 The test function spaces

Hypotheses. In this section, we always assume Y is a finite-dimensional Banach space with $\dim Y \ge 1$, E is the function space associated with Y as in 3.3.1, $n \in \mathscr{P}$, and U is an open subset of $\mathbf{R}^n$.

The goal of this section is to prove an embedding theorem of $\mathscr{K}(X, E)$ into $\mathscr{K}(X \times Y, \operatorname{Hom}(Y, \mathbf{R}))$, see 3.4.17, and the embedding theorems for other test function spaces of interest follow immediately. For this purpose, we first present basic results about the space $\mathscr{K}(X, Z)$.

Lemma 3.4.1. *Suppose X is a locally compact Hausdorff space and W and Z are Hausdorff locally convex spaces. Then, the following four statements hold.*

(1) If $f : W \to Z$ is a continuous linear map, then post-composition with f defines a continuous linear map $\mathscr{K}(X, W) \to \mathscr{K}(X, Z)$.

(2) If Z is the product of finitely many locally convex spaces $Z_1, Z_2, \ldots, Z_n$, then we have the isomorphism of locally convex spaces

$$\mathscr{K}(X, Z) \simeq \prod_{i=1}^{n} \mathscr{K}(X, Z_i);$$

in particular, we have $\mathscr{K}(X, Z^n) \simeq \mathscr{K}(X, Z)^n$.

(3) If Z is normed, then the topology of $\mathscr{K}_K(X, Z)$ is given by the norm with value $\sup \operatorname{im} |f|$ *at $f \in \mathscr{K}_K(X, Z)$.*

(4) If K_i is a sequence of compact subsets of X such that $K_i \subset \operatorname{Int} K_{i+1}$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^{\infty} K_i$, then $\mathscr{K}(X, Z)$ is the strict inductive limit of $\mathscr{K}_{K_i}(X, Z)$ for $i \in \mathscr{P}$; in particular, the given topology on $\mathscr{K}_{K_i}(X, Z)$ agrees with the subspace topology induced by $\mathscr{K}(X, Z)$.

Proof. It is straightforward to verify (1) from the definitions. The statements (2) and (3) are proved in [Bou04a, III, §1, No. 1]. To prove (4), it is enough to check the inclusion map $\mathscr{K}_{K_i}(X, Z) \to \mathscr{K}_{K_j}(X, Z)$ is a homeomorphic embedding whenever $i \leq j$, and this is straightforward from the definitions. Finally, the postscript of (4) follows from 3.1.1. □

Lemma 3.4.2. *Suppose X is a second-countable locally compact Hausdorff space, K is a compact subset of X, and Z is a separable Banach space. Then, the function space $\mathscr{K}_K(X, Z)$ is a separable Banach space.*

Proof. It follows from [Bou04a, III. §1, No. 1, page 1] that $\mathscr{K}_K(X, Z)$ is a Banach space. Since $\mathscr{K}_K(X)$ is separable by [Men16, 2.2, 2.23], it follows from [Bou04a, III, §1, No. 2, Proposition 5] that $\mathscr{K}_K(X, Z)$ is separable. □

Lemma 3.4.3. *Suppose Z is a normed space, X is a locally compact Hausdorff space, K_i is a sequence of compact subsets of X such that $K_0 = \varnothing$, $K_{i-1} \subset \operatorname{Int} K_i$ whenever $i \in \mathscr{P}$, and $X = \bigcup_{i=1}^{\infty} K_i$. Then, the subsets*

$$V_\alpha = \mathscr{K}(X, Z) \cap \{f : |f(x)| \leq \alpha(i)^{-1} \text{ for } i \in \mathscr{P} \text{ and } x \in X \sim K_{i-1}\}$$

corresponding to each $\alpha \in \mathscr{P}^{\mathscr{P}}$ form a fundamental system of neighborhoods of 0 of $\mathscr{K}(X, Z)$.

Proof. Clearly, V_α is a neighborhood of 0 in $\mathscr{K}(X, Z)$ whenever $\alpha \in \mathscr{P}^{\mathscr{P}}$. Let V be a convex neighborhood of 0 in $\mathscr{K}(X, Z)$. There exists $\beta \in \mathscr{P}^{\mathscr{P}}$ such that

$$\mathbf{B}(0, \beta(i)^{-1}) \cap \mathscr{K}_{K_i}(X, Z) \subset V \cap \mathscr{K}_{K_i}(X, Z) \quad \text{whenever } i \in \mathscr{P}.$$

Choose non-negative functions $\phi_i \in \mathscr{K}(X)$ for $i \in \mathscr{P}$ such that

$$\operatorname{spt} \phi_i \subset (\operatorname{Int} K_{i+1}) \sim K_{i-1} \quad \text{and} \quad \sum_{j=1}^{\infty} 2^{-j} \phi_j(x) = 1$$

whenever $i \in \mathscr{P}$ and $x \in X$, and choose $\alpha \in \mathscr{P}^{\mathscr{P}}$ such that

$$\alpha(i)^{-1} \sup \operatorname{im} |\phi_i| \leq \beta(i+1)^{-1} \quad \text{whenever } i \in \mathscr{P}.$$

Note that if $i, j \in \mathscr{P}$ and $f \in V_\alpha \cap \mathscr{K}_{K_i}(X, Z)$, then

$$\operatorname{spt} \phi_j f \subset (\operatorname{Int} K_{j+1}) \sim K_{j-1} \quad \text{and} \quad \sup \operatorname{im} |\phi_j f| \leq \beta(j+1)^{-1};$$

therefore

$$\begin{gathered} \phi_j f \in V \cap \mathscr{K}_{K_{j+1}}(X, Z), \quad \text{if } j \leq i, \\ \phi_j f = 0, \quad \text{if } j > i. \end{gathered}$$

Since V is convex, $0 \in V$, and $f = \sum_{j=1}^{i} 2^{-j}(\phi_j f) + 2^{-i} \cdot 0$, we conclude that $f \in V$, hence $V_\alpha \subset V$. □

Remark 3.4.4. The proof of 3.4.3 is adapted from the proof of [Sch66, p. 66, Théorème II].

Next, we will study the topological vector structure of E.

Remark 3.4.5. We shall verify that E_s is a separable Banach space. Suppose f_i is a Cauchy sequence in E_s. By 3.3.3, the sequence f_i is also Cauchy with respect to the supremum metric. From [Bou89b, X, §1, No. 6, Corollary 1 of Theorem 2], there exist continuous functions $f : Y \to \mathbf{R}$ and $F : Y \to \operatorname{Hom}(Y, \mathbf{R})$ such that $f_i \to f$ and $\operatorname{D} f_i \to F$ uniformly. By Taylor's theorem, we can show that $\operatorname{D} f$ exists and equals F. It follows that E_s is complete. Note that the derivative D provides an isometric embedding of E_s into the separable metric space $\mathscr{K}_{\mathbf{B}(0,s)}(Y, \operatorname{Hom}(Y, \mathbf{R}))$, see 3.4.2; hence E_s is separable.

Remark 3.4.6. Note that for $0 \leq r \leq s < \infty$, the inclusion map $E_r \to E_s$ is a norm-preserving embedding. Therefore, E is the strict inductive limit of the sequence E_i for $i \in \mathscr{P}$. From 3.1.1, we have that E is complete and Hausdorff, that E_s is a closed subspace of E whenever $0 \leq s < \infty$, and that every compact subset of E is contained on some E_s and bounded there.

Remark 3.4.7. The derivative defines a norm-preserving embedding

$$E_s \to \mathscr{K}_{\mathbf{B}(0,s)}(Y, \mathrm{Hom}(Y, \mathbf{R})),$$

hence a continuous injective linear map

$$E \to \mathscr{K}(Y, \mathrm{Hom}(Y, \mathbf{R})).$$

If $Y = \mathbf{R}$, then both maps above are isomorphisms of locally convex spaces since the formula

$$\gamma(y) = \textstyle\int_0^1 \langle y, f(ty) \rangle \,\mathrm{d}t \quad \text{whenever } y \in \mathbf{R}$$

defines a member in E_s whenever $f \in \mathscr{K}_{\mathbf{B}(0,s)}(\mathbf{R}, \mathrm{Hom}(\mathbf{R}, \mathbf{R}))$. If $Y = \mathbf{R}^n$ with $n \geq 2$, then both maps are not surjective; in fact, we may choose a non-negative function $\omega \in \mathscr{D}(\mathbf{R}, \mathbf{R})$ such that $\omega(r) = 1$ for $|r| \leq 1$ and define

$$f(x) = \omega(|x)|)(x_2, -x_1, 0, \ldots, 0) \quad \text{whenever } x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n.$$

Then, there exists no $\gamma \in E$ such that $\mathrm{D}\,\gamma = f$; here we identify $\mathbf{R}^n$ with $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$.

To show the map $E \to \mathscr{K}(Y, \mathrm{Hom}(Y, \mathbf{R}))$ as above is still a homeomorphic embedding for $\dim Y \geq 2$, we need to study the fundamental system of neighborhoods of 0 in E; for this purpose, we introduce another function space $\widetilde{E}$ that is isomorphic to E as locally convex spaces when $\dim Y \geq 2$, but its members have compact support.

Definition 3.4.8. We define

$$\widetilde{E}_s = \{\widetilde{\gamma} : \widetilde{\gamma} : Y \to \mathbf{R} \text{ is of class } 1, \operatorname{spt} \widetilde{\gamma} \subset \mathbf{B}(0, s)\} \quad \text{whenever } 0 \leq s < \infty$$

endowed with the norm $\widetilde{\gamma} \mapsto \sup \operatorname{im} \| \mathrm{D}\,\widetilde{\gamma} \|$, and the space

$$\widetilde{E} = \bigcup \{\widetilde{E}_s : 0 \leq s < \infty\}$$

is endowed with the locally convex final topology induced by the inclusion maps $\widetilde{E}_s \to \widetilde{E}$.

Remark 3.4.9. Similar arguments as in 3.4.5 and 3.4.6 show that $\widetilde{E}$ is the strict inductive limit of the separable Banach spaces $\widetilde{E}_i$ as $i \to \infty$; it follows from 3.1.1 that the given topology on $\widetilde{E}_i$ agrees with the subspaces induced from $\widetilde{E}$.

Remark 3.4.10. The map $\widetilde{\gamma} \to \widetilde{\gamma} - \widetilde{\gamma}(0)$ defines an norm-preserving embedding from $\widetilde{E}_s$ into E_s whenever $0 \le s < \infty$, hence a continuous injective linear map from $\widetilde{E}$ into E. For $\dim Y \ge 2$, these maps are isomorphisms of locally convex spaces.

Lemma 3.4.11. *Suppose $L_0 = \varnothing$ and $L_i = \mathbf{B}(0, i) \cap Y$ for $i \in \mathscr{P}$. Then, the subsets $\widetilde{W}_\alpha$ consisting of $\widetilde{\gamma} \in \widetilde{E}$ satisfying*

$$|\widetilde{\gamma}(y)| + \| \mathrm{D}\, \widetilde{\gamma}(y) \| \le \alpha(i)^{-1} \quad \text{whenever } i \in \mathscr{P} \text{ and } y \in Y \sim L_{i-1}$$

corresponding to each $\alpha \in \mathscr{P}^{\mathscr{P}}$ form a fundamental system of neighborhoods of 0 of $\widetilde{E}$.

Proof. Clearly, the set $\widetilde{W}_\alpha$ is convex, symmetric, and absorbent whenever $\alpha \in \mathscr{P}^{\mathscr{P}}$. Note that if

$$\begin{gathered} i \in \mathscr{P}, \quad \varepsilon = (2i)^{-1} \inf\{\alpha(j)^{-1} : j = 1, 2, \ldots, i\} \\ \widetilde{\gamma} \in \mathbf{B}(0, \varepsilon) \cap \widetilde{E}_i, \quad v \in Y, \quad |v| = 1, \quad \text{and} \quad y(t) = (i - t)v, \end{gathered}$$

then we have

$$\begin{aligned} |\widetilde{\gamma}(y(t))| &\le \textstyle\int_0^t |\langle -v, \mathrm{D}\, \widetilde{\gamma}(y(s)) \rangle| \, \mathrm{d}\mathscr{L}^1 s \\ &\le \textstyle\int_0^t \| \mathrm{D}\, \widetilde{\gamma}(y(s)) \| \, \mathrm{d}\mathscr{L}^1 s \\ &\le i\varepsilon \end{aligned}$$

whenever $0 \le t \le i$. Thus, $\mathbf{B}(0, \varepsilon) \cap \widetilde{E}_i \subset \widetilde{W}_\alpha \cap \widetilde{E}_i$, and we conclude $\widetilde{W}_\alpha$ is a neighborhood of 0 in $\widetilde{E}$ whenever $\alpha \in \mathscr{P}^{\mathscr{P}}$.

Let V be a convex neighborhood of 0 in $\widetilde{E}$. Then, there exists $\beta \in \mathscr{P}^{\mathscr{P}}$ such that

$$\mathbf{B}(0, \beta(i)^{-1}) \cap \widetilde{E}_i \subset V \cap \widetilde{E}_i \quad \text{whenever } i \in \mathscr{P}.$$

Choose non-negative functions $\phi_i : Y \to \mathbf{R}$ of class 1 for $i \in \mathscr{P}$ such that

$$\operatorname{spt} \phi_i \subset (\operatorname{Int} L_{i+1}) \sim L_{i-1} \quad \text{and} \quad \sum_{i=1}^{\infty} 2^{-i} \phi_i(y) = 1$$

whenever $i \in \mathscr{P}$ and $y \in Y$, and choose $\alpha \in \mathscr{P}^{\mathscr{P}}$ such that

$$\alpha(i)^{-1} \sup \operatorname{im}(|\phi_i| + \| \mathrm{D}\, \phi_i \|) \le \beta(i+1)^{-1} \quad \text{whenever } i \in \mathscr{P}.$$

Note that if $i, j \in \mathscr{P}$ and $\widetilde{\gamma} \in \widetilde{W}_\alpha \cap \widetilde{E}_i$, then

$$\operatorname{spt} \phi_j \widetilde{\gamma} \subset (\operatorname{Int} L_{j+1}) \sim L_{j-1} \quad \text{and} \quad \sup \operatorname{im} \| \mathrm{D}(\phi_j \widetilde{\gamma}) \| \le \beta(j+1)^{-1};$$

therefore

$$\phi_j \widetilde{\gamma} \in V \cap \widetilde{E}_{j+1}, \quad \text{if } j \le i,$$
$$\phi_j \widetilde{\gamma} = 0, \quad \text{if } j > i.$$

Since V is convex, $0 \in V$, and $\widetilde{\gamma} = \sum_{j=1}^{i} 2^{-j}(\phi_j \widetilde{\gamma}) + 2^{-i} \cdot 0$, we conclude that $\widetilde{\gamma} \in V$, hence that $\widetilde{W}_\alpha \subset V$. □

Remark 3.4.12. Suppose X is a second-countable locally compact Hausdorff space. Whenever $\widetilde{W}_\alpha$ and V_α are as in 3.4.11 and 3.4.3, respectively, the images of $\widetilde{W}_\alpha \times \widetilde{W}_\alpha$ and $V_\alpha \times V_\alpha$ under multiplication are contained in $\widetilde{W}_\alpha$ and V_α, respectively. It follows from [Bou87, I, §1, No. 6, Proposition 5] that maps $\widetilde{E} \times \widetilde{E} \to \widetilde{E}$ and $\mathscr{K}(X) \times \mathscr{K}(X) \to \mathscr{K}(X)$ induced by multiplication are continuous.

To prove the embedding theorem, we still need several lemmas.

Lemma 3.4.13. *Suppose $\alpha \in \mathscr{P}^{\mathscr{P}}$. Then, there exists a positive convex decreasing function $h : \mathbf{R} \to \mathbf{R}$ of class 1 such that*

$$h(x) < \alpha(i)^{-1} \quad \text{whenever } x \ge i - 1 \text{ and } i \in \mathscr{P}.$$

Proof. Define β and δ in $\mathscr{P}^{\mathscr{P}}$ by

$$\beta(i) = \sup\{\alpha(j) : 1 \le j \le i\} + i;$$
$$\delta(i) = 2^{\beta(i)}$$

whenever $i \in \mathscr{P}$. Note that δ satisfies

$$\alpha(i) < \delta(i) \le \delta(i+1)/2 \quad \text{and} \quad \delta(i)^{-1} - \delta(i+1)^{-1} > \delta(i+1)^{-1} - \delta(i+2)^{-1}$$

whenever $i \in \mathscr{P}$. Then, the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{\delta(1)} - \left(\dfrac{1}{\delta(1)} - \dfrac{1}{\delta(2)}\right) x & \text{if } x \le 0 \\ \dfrac{1}{\delta(i)} - \left(\dfrac{1}{\delta(i)} - \dfrac{1}{\delta(i+1)}\right)(x - (i-1)) & \text{if } i \in \mathscr{P},\ i - 1 \le x \le i \end{cases}$$

is a convex decreasing function such that

$$f(x) < \alpha(i)^{-1} \quad \text{whenever } x \ge i - 1 \text{ and } i \in \mathscr{P}.$$

Choose a function $g : \mathbf{R} \to \mathbf{R}$ of class 1 such that $g \ge 0$, $\int g \, \mathrm{d}\mathscr{L}^1 = 1$, and $\operatorname{spt} g \subset \mathbf{R} \cap \{x : -1 \le x \le 0\}$.

Let h be the convolution $f * g$, and we will verify that h has the desired properties. Clearly, h is a positive function of class 1. Since

$$h(x) - f(x) = \int_{\mathbf{R}\cap\{y : -1\le y\le 0\}} (f(x-y) - f(x))g(y)\,\mathrm{d}\mathscr{L}^1\, y < 0,$$

it follows that $h \le f$. To show that h is decreasing, we compute

$$h(x+t) - h(x) = \int (f((x-y)+t) - f(x-y))g(y)\,\mathrm{d}\mathscr{L}^1\, y < 0$$

whenever $x \in \mathbf{R}$ and $t > 0$. For $x, z \in \mathbf{R}$ and $0 \le \lambda \le 1$, we compute

$$\begin{aligned}(f*g)(\lambda x + (1-\lambda)z) &= \textstyle\int f(\lambda x + (1-\lambda)z - y)g(y)\,\mathrm{d}\mathscr{L}^1\, y \\ &= \textstyle\int f(\lambda(x-y) + (1-\lambda)(z-y))g(y)\,\mathrm{d}\mathscr{L}^1\, y \\ &\le \lambda \textstyle\int f(x-y)g(y)\,\mathrm{d}\mathscr{L}^1\, y \\ &\quad + (1-\lambda)\textstyle\int f(z-y)g(y)\,\mathrm{d}\mathscr{L}^1\, y \\ &= \lambda(f*g)(x) + (1-\lambda)(f*g)(z)\end{aligned}$$

and it follows that h is convex. □

Lemma 3.4.14. *Suppose X is a locally compact Hausdorff space, K is a compact subset of X, and either $Y = \mathbf{R}$, $F_i = \mathscr{K}_{\mathbf{B}(0,i)}(\mathbf{R})$, and $F = \mathscr{K}(\mathbf{R})$, or $F_i = \widetilde{E}_i$ and $F = \widetilde{E}$. Then, the locally convex final topology $\mathcal{T}_1$ on $\mathscr{K}_K(X, F)$ induced by the inclusion maps $\mathscr{K}_K(X, F_i) \to \mathscr{K}_K(X, F)$ for $i \in \mathscr{P}$ is identical to the topology $\mathcal{T}_2$ of uniform convergence on $\mathscr{K}_K(X, F)$.*

Proof. Since the inclusion map

$$\mathscr{K}_K(X, F_i) \to (\mathscr{K}_K(X, F), \mathcal{T}_2)$$

is continuous whenever $i \in \mathscr{P}$ by 3.4.1(1), it follows that $\mathcal{T}_2 \subset \mathcal{T}_1$.

Conversely, let V be a convex neighborhood of 0 of $(\mathscr{K}_K(X, F), \mathcal{T}_1)$. Then, there exists $\beta \in \mathscr{P}^{\mathscr{P}}$ such that

$$\mathbf{B}(0, \beta(i)^{-1}) \cap \mathscr{K}_K(X, F_i) \subset V \cap \mathscr{K}_K(X, F_i) \quad \text{whenever } i \in \mathscr{P}.$$

Let $L_0 = \varnothing$ and $L_i = \mathbf{B}(0, i) \cap Y$ for $i \in \mathscr{P}$. Choose non-negative functions $\phi_i : Y \to \mathbf{R}$ of class 1 such that

$$\operatorname{spt}\phi_i \subset (\operatorname{Int} L_{i+1}) \sim L_{i-1} \quad \text{and} \quad \sum_{i=1}^{\infty} 2^{-i}\phi(y) = 1$$

whenever $i \in \mathscr{P}$ and $y \in Y$, and choose $\alpha \in \mathscr{P}^{\mathscr{P}}$ such that

$$\alpha(i)^{-1} \sup \operatorname{im}(|\phi_i| + \|\operatorname{D}\phi_i\|) \le \beta(i+1)^{-1} \quad \text{whenever } i \in \mathscr{P}.$$

Let $\eta \in \mathscr{K}_K(X, F)$ such that $\operatorname{im} \eta \subset U_\alpha$, where U_α equals either V_α or $\widetilde{W}_\alpha$ as in 3.4.3 with X and Z both replaced by $\mathbf{R}$ and 3.4.11, respectively. By 3.4.1(4), 3.4.9, and 3.1.1, there exists $i \in \mathscr{P}$ such that $\operatorname{im} \eta \subset F_i$. Since multiplication by ϕ_j induces a continuous linear map from F into F_{j+1} by 3.4.12, 3.4.1(4), and 3.4.9, the function η_j defined by $\eta_j(x) = \phi_j \eta(x)$ for $x \in X$ belongs to $\mathscr{K}_K(X, F_{j+1})$ by 3.4.1(1). Observe that

$$\sup \operatorname{im} |\eta_j| \le \beta (j+1)^{-1}$$

whenever $j \in \mathscr{P}$. Note that $\eta_j = 0$ for $j > i$. Therefore, we have

$$\eta = \sum_{j=1}^{i} 2^{-j} \eta_j + 2^{-1} \cdot 0 \in V$$

since V is convex and $0 \in V$. It follows that $\mathcal{T}_1 \subset \mathcal{T}_2$. □

Remark 3.4.15. Suppose F_i and F are as in 3.4.14. From 3.4.1(4), 3.4.9, and 3.1.1, we have that

$$\mathscr{K}_K(X, F) = \bigcup_{i=1}^{\infty} \mathscr{K}_K(X, F) \cap \{f : \operatorname{im} f \subset F_i\}$$

and that

$$\mathscr{K}_K(X, F) \cap \{f : \operatorname{im} f \subset F_i\} = \mathscr{K}_K(X, F_i) \quad \text{whenever } i \in \mathscr{P}.$$

Thus, $\mathscr{K}_K(X, F)$ is the strict inductive limit of $\mathscr{K}_K(X, F_i)$ as $i \to \infty$. Furthermore, in view of 3.1.2 and 3.4.14, if K_i is a sequence of compact subsets of X such that $K_i \subset \operatorname{Int} K_{i+1}$ whenever $i \in \mathscr{P}$ and $\bigcup_{i=1}^{\infty} K_i = X$, then the locally convex final topology on $\mathscr{K}(X, F)$ induced by the inclusion maps $\mathscr{K}_{K_i}(X, F) \to \mathscr{K}(X, F)$ for $i \in \mathscr{P}$ is identical to the locally convex topology induced by the inclusion maps $\mathscr{K}_{K_i}(X, F_j) \to \mathscr{K}(X, F)$ for $i, j \in \mathscr{P}$. Note that the inclusion $\mathscr{K}_{K_i}(X, F_j) \to \mathscr{K}_{K_k}(X, F_k)$ is a homeomorphic embedding whenever $i, j \in \mathscr{P}$ and $k = \sup\{i, j\}$; in particular, $\mathscr{K}(X, F)$ is the strict inductive limit of $\mathscr{K}_{K_i}(X, F_i)$ as $i \to \infty$.

The assertions remain true if we replace F_i and F with E_i and E, respectively because for $\dim Y = 1$, we have $\mathscr{K}_{\mathbf{B}(0,i)}(\mathbf{R}) \simeq E_i$ and $\mathscr{K}(\mathbf{R}) \simeq E$ by 3.4.7, and for $\dim Y \ge 2$, we have $\widetilde{E}_s \simeq E_s$ and $\widetilde{E} \simeq E$ by 3.4.10.

Remark 3.4.16. Whenever K is a compact subset of X and $0 \le s < \infty$, recall from 3.4.7 that the derivative gives a norm-preserving embedding $E_s \to \mathscr{K}_{\mathbf{B}(0,s)}(Y, \operatorname{Hom}(Y, \mathbf{R}))$, hence a norm-preserving embedding

$$\mathscr{K}_K(X, E_s) \to \mathscr{K}_K(X, \mathscr{K}_{\mathbf{B}(0,s)}(Y, \operatorname{Hom}(Y, \mathbf{R}))),$$

and it follows from 3.2.19 that so is

$$\mathscr{K}_K(X, E_s) \to \mathscr{K}_{K\times \mathbf{B}(0,s)}(X \times Y, \operatorname{Hom}(Y, \mathbf{R})).$$

By 3.1.2, 3.4.7, 3.4.10, and 3.4.14, we see $\mathscr{K}(X, E)$ carries the locally convex final topology induced by the inclusion maps $\mathscr{K}_K(X, E_s) \to \mathscr{K}(X, E)$ corresponding to compact subsets K of X and $0 \le s < \infty$; therefore, the last embedding induces a continuous injective linear map

$$\iota : \mathscr{K}(X, E) \to \mathscr{K}(X \times Y, \operatorname{Hom}(Y, \mathbf{R})).$$

By 3.4.1(1) and 3.4.10, the map $I : \mathscr{K}(X, \widetilde{E}) \to \mathscr{K}(X, E)$ defined by $I(\eta)(x) = \eta(x) - \eta(x)(0)$ for $\eta \in \mathscr{K}(X, \widetilde{E})$ and $x \in X$ is a continuous injective linear map and so is the composition $\widetilde{\iota} = \iota \circ I$; moreover, $\widetilde{\iota}$ satisfies

$$\widetilde{\iota}(\eta)(x, y) = \mathrm{D}(\eta(x))(y) \quad \text{for } (x, y) \in X \times Y \text{ and } \eta \in \mathscr{K}(X, \widetilde{E}).$$

If $\dim Y = 1$, then the embeddings in the previous paragraph are norm-preserving isomorphisms and hence ι is an isomorphism of locally convex spaces.

If $\dim Y \ge 2$, then I is an isomorphism. The following theorem shows that if X is a countable union of compact sets, then $\widetilde{\iota}$ is a homeomorphic embedding (even for the case $\dim Y = 1$); therefore, so is ι.

Theorem 3.4.17. *Suppose X is a locally compact Hausdorff space, $K_0 = \varnothing$, K_i is a sequence of compact subsets of X such that $K_i \subset \operatorname{Int} K_{i+1}$ whenever $i \in \mathscr{P}$ and $\bigcup_{i=1}^{\infty} K_i = X$, let $L_0 = \varnothing$ and $L_i = \mathbf{B}(0, i) \cap Y$ whenever $i \in \mathscr{P}$, and let $C_i = K_i \times L_i$ whenever $i \in \mathscr{P} \cup \{0\}$.*

Then, the subsets $\widetilde{W}_\alpha$ consisting of $\eta \in \mathscr{K}(X, \widetilde{E})$ satisfying

$$|\eta(x)(y)| + \|\mathrm{D}(\eta(x))(y)\| \le \alpha(i)^{-1}$$

whenever $i \in \mathscr{P}$ and $(x, y) \in (X \times Y) \sim C_{i-1}$, corresponding to each $\alpha \in \mathscr{P}^{\mathscr{P}}$ form a fundamental system of neighborhoods of 0 of $\mathscr{K}(X, \widetilde{E})$. Furthermore, the map $\widetilde{\iota} : \mathscr{K}(X, \widetilde{E}) \to \mathscr{K}(X \times Y, \operatorname{Hom}(Y, \mathbf{R}))$ defined by

$$\widetilde{\iota}(\eta)(x, y) = \mathrm{D}(\eta(x))(y) \quad \text{for } (x, y) \in X \times Y$$

is a homeomorphic embedding.

Proof. Clearly, the set $\widetilde{W}_\alpha$ is convex and symmetric; by the first paragraph of 3.4.15, $\widetilde{W}_\alpha$ is absorbent whenever $\alpha \in \mathscr{P}^{\mathscr{P}}$. Note that if

$$\begin{gathered} i \in \mathscr{P}, \quad \varepsilon = (2i)^{-1} \inf\{\alpha(j)^{-1} : j = 1, 2, \ldots, i\}, \\ x \in X, \quad v \in Y, \quad |v| = 1, \quad \text{and} \quad y(t) = (i - t)v, \\ \eta \in \mathbf{B}(0, \varepsilon) \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i) \end{gathered}$$

then we have

$$\begin{aligned}|\eta(x)(y(t))| &\le \textstyle\int_0^t |\langle -v, \mathrm{D}(\eta(x))(y(s))\rangle| \,\mathrm{d}\mathscr{L}^1\, s\\ &\le \textstyle\int_0^t \|\mathrm{D}(\eta(x))(y(s))\| \,\mathrm{d}\mathscr{L}^1\, s\\ &\le t\varepsilon\end{aligned}$$

whenever $0 \le t \le i$. Thus, $\mathbf{B}(0,\varepsilon) \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i) \subset \widetilde{W}_\alpha \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i)$, and we conclude $\widetilde{W}_\alpha$ is a neighborhood of 0 in $\mathscr{K}(X, \widetilde{E})$ whenever $\alpha \in \mathscr{P}^{\mathscr{P}}$.

Let V be a convex neighborhood of 0 in $\mathscr{K}(X, \widetilde{E})$. Then, by 3.4.15, there exists $\beta \in \mathscr{P}^{\mathscr{P}}$ such that

$$\mathbf{B}(0, \beta(i)^{-1}) \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i) \subset V \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i) \quad \text{whenever } i \in \mathscr{P}.$$

Choose non-negative continuous functions $f_i : X \to \mathbf{R}$ and non-negative functions $g_i : Y \to \mathbf{R}$ of class 1 for $i \in \mathscr{P}$ such that

$$\operatorname{spt} f_i \subset (\operatorname{Int} K_{i+1}) \sim K_{i-1} \quad \text{and} \quad \operatorname{spt} g_i \subset (\operatorname{Int} L_{i+1}) \sim L_{i-1}$$

whenever $i \in \mathscr{P}$ and such that

$$\sum_{i=1}^\infty f_i(x) = 1 \quad \text{and} \quad \sum_{i=1}^\infty g_i(y) = 1$$

whenever $(x, y) \in X \times Y$. For $i \in \mathscr{P}$, we define a continuous function ϕ_i with values in $\widetilde{E}$ by

$$\phi_i(x) = 2^i \left(\left(\sum_{j=1}^i f_j(x) \right) \left(\sum_{j=1}^i g_j \right) - \left(\sum_{j=1}^{i-1} f_j(x) \right) \left(\sum_{j=1}^{i-1} g_j \right) \right) \quad \text{for } x \in X.$$

Note that

$$(X \times Y) \cap \{(x, y) : \phi_i(x)(y) \ne 0\} \subset (\operatorname{Int} C_{i+1}) \sim C_{i-1} \quad \text{for } i \in \mathscr{P}$$

and that

$$\sum_{i=1}^\infty 2^{-i} \phi_i(x)(y) = 1 \quad \text{for } (x, y) \in X \times Y.$$

Choose $\alpha \in \mathscr{P}^{\mathscr{P}}$ such that

$$\alpha(i)^{-1}(|\phi_i(x)(y)| + \|\mathrm{D}(\phi_i(x))(y)\|) \le \beta(i+1)^{-1}$$

whenever $(x, y) \in X \times Y$ and $i \in \mathscr{P}$. If $i, j \in \mathscr{P}$ and $\eta \in \widetilde{W}_\alpha \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i)$, then the function $\phi_j \eta$ defined by

$$(\phi_j \eta)(x) = \phi_j(x)\eta(x) \quad \text{whenever } x \in X$$

satisfies $\phi_j \eta \in \mathscr{K}_{K_{j+1}}(X, \widetilde{E}_{j+1})$ because of 3.4.12 and 3.4.9, and

$$\sup\{\sup \operatorname{im} \| \operatorname{D}((\phi_j \eta)(x))\| : x \in X\} \le \beta(j+1)^{-1};$$

therefore

$$\begin{gathered} \phi_j \eta \in V \cap \mathscr{K}_{K_{j+1}}(X, \widetilde{E}_{j+1}) \quad \text{if } j \le i, \\ \phi_j \eta = 0 \quad \text{if } j > i. \end{gathered}$$

Since V is convex, $0 \in V$, and $\eta = \sum_{j=1}^{i} 2^{-j}(\phi_j \eta) + 2^{-i} \cdot 0$, we conclude that $\eta \in V$, hence $\widetilde{W}_\alpha \subset V$.

Finally, by 3.4.16, we have $\widetilde{\iota}$ is continuous, injective, and linear; hence, it remains to prove that $\widetilde{\iota}^{-1}$ is continuous. Let $\alpha \in \mathscr{P}^{\mathscr{P}}$ and let $\widetilde{W}_\alpha$ be a neighborhood of 0 of $\mathscr{K}(X, \widetilde{E})$ defined as above. Now, we aim to find $\delta \in \mathscr{P}^{\mathscr{P}}$ such that

$$V_\delta \cap \operatorname{im} \widetilde{\iota} \subset \widetilde{\iota}[\widetilde{W}_\alpha]$$

where V_δ is a neighborhood of 0 of $\mathscr{K}(X \times Y, \operatorname{Hom}(Y, \mathbf{R}))$ as in 3.4.3 with X, K_i, and Z replaced by $X \times Y$, C_i, and $\operatorname{Hom}(Y, \mathbf{R})$, respectively. By 3.4.13, there exists a positive convex decreasing function $h : \mathbf{R} \to \mathbf{R}$ of class 1 such that $h(i-1) \le 2^{-1}\alpha(i)^{-1}$ whenever $i \in \mathscr{P}$. Then, we define $\delta \in \mathscr{P}^{\mathscr{P}}$ to satisfy

$$\delta(i)^{-1} \le \inf\{-h'(i), 2^{-1}\alpha(i)^{-1}\} \quad \text{whenever } i \in \mathscr{P}$$

and let V_δ be as in 3.4.3. If

$$\begin{gathered} i \in \mathscr{P}, \quad \eta \in \widetilde{\iota}^{-1}[V_\delta] \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i), \\ v \in Y, \quad |v| = 1, \quad \text{and} \quad y(t) = (i-t)v, \end{gathered}$$

then we have

$$\begin{aligned} |\eta(x)(y(t))| &\le \textstyle\int_0^t |\langle -v, \operatorname{D}(\eta(x))(y(s))| \, \mathrm{d}\mathscr{L}^1 s \\ &\le \textstyle\int_0^t \| \operatorname{D}(\eta(x))(y(s))\| \, \mathrm{d}\mathscr{L}^1 s \\ &\le \textstyle\int_0^t -h'(i-s) \, \mathrm{d}\mathscr{L}^1 s \\ &\le h(|y(t)|) \end{aligned}$$

whenever $x \in X$ and $0 \le t \le i$. It follows that $|\eta(x)(y)| \le h(|y|)$ whenever $(x, y) \in X \times Y$, hence $\widetilde{\iota}^{-1}[V_\delta] \cap \mathscr{K}_{K_i}(X, \widetilde{E}_i) \subset \widetilde{W}_\alpha$. □

Corollary 3.4.18. *Suppose X is a locally compact Hausdorff space, $K_0 = \varnothing$, K_i is a sequence of compact subsets of X such that $K_i \subset \operatorname{Int} K_{i+1}$ whenever $i \in \mathscr{P}$ and $\bigcup_{i=1}^{\infty} K_i = X$, and let $C_i = K_i \times (\mathbf{B}(0, i) \cap Y)$ whenever $i \in \mathscr{P} \cup \{0\}$.*

Then, the map $\iota : \mathscr{K}(X, E) \to \mathscr{K}(X \times Y, \mathrm{Hom}(Y, \mathbf{R}))$ *defined by*

$$\iota(\eta)(x, y) = \mathrm{D}(\eta(x))(y) \quad \text{for } (x, y) \in X \times Y$$

is a homeomorphic embedding. Furthermore, the subsets W_α *consisting of* $\eta \in \mathscr{K}(X, E)$ *satisfying*

$$\| \mathrm{D}(\eta(x))(y) \| \leq \alpha(i)^{-1} \quad \text{whenever } i \in \mathscr{P} \text{ and } (x, y) \in (X \times Y) \sim C_{i-1}$$

corresponding to each $\alpha \in \mathscr{P}^{\mathscr{P}}$ *form a fundamental system of neighborhoods of* 0 *of* $\mathscr{K}(X, E)$.

Proof. It follows from 3.4.16, 3.4.17, and 3.4.3. □

Corollary 3.4.19. *The map* $E \to \mathscr{K}(Y, \mathrm{Hom}(Y, \mathbf{R}))$ *defined as in 3.4.7 is a homeomorphic embedding.*

Furthermore, the subsets W_α *consisting of* $\gamma \in E$ *satisfying*

$$\| \mathrm{D}\, \gamma(y) \| \leq \alpha(i)^{-1} \quad \text{whenever } i \in \mathscr{P} \text{ and } |y| > i - 1$$

corresponding to each $\alpha \in \mathscr{P}^{\mathscr{P}}$ *form a fundamental system of neighborhoods of* 0 *of* E.

Proof. If $\dim Y = 1$, it is proved in 3.4.7, and the postscript follows from 3.4.3. If $\dim Y \geq 2$, it follows from 3.4.18 applied with X and $K_i = X$ for $i \in \mathscr{P}$ being a singleton. □

Corollary 3.4.20. *The dual map* $\mathscr{K}(Y, \mathrm{Hom}(Y, \mathbf{R}))^* \to E^*$ *is an epimorphism.*

Proof. It follows from 3.4.19 and the Hahn-Banach theorem [Bou87, II, §4, No. 1, Proposition 2]. □

Remark 3.4.21. The following example shows that the dual map is not injective for $\dim Y = k > 1$. Assume $Y = \mathbf{R}^k$. Whenever $\omega \in \mathscr{D}(\mathbf{R}, \mathbf{R})$, $0 \notin \operatorname{spt} \omega'$, and L is a non-zero anti-symmetric endomorphism on $\mathbf{R}^k$, the functional $\mu \in \mathscr{K}(\mathbf{R}^k, \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}))^*$ defined by

$$\mu(\delta) = \textstyle\int \langle X(y), \delta(y) \rangle \, \mathrm{d}\mathscr{L}^k\, y \quad \text{for } \delta \in \mathscr{K}(\mathbf{R}^k, \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}))$$

belongs to the kernel of the dual map $\mathscr{K}(\mathbf{R}^k, \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}))^* \to E^*$, where $X \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ is defined by

$$X(y) = \omega(|y|) L(y) \quad \text{for } y \in \mathbf{R}^k,$$

because $\operatorname{div} X = 0$.

To unify the test function spaces from domain and codomain, we also introduce the test function space on the product.

Definition 3.4.22. Whenever C is a compact subset of $U \times Y$, we define H_C to consist of all functions $\eta : U \times Y \to \mathbf{R}^n$ such that

(1) $\eta(x, \cdot) \bullet v \in E$ whenever $x \in U$ and $v \in \mathbf{R}^n$.

(2) The function

$$(x, y) \mapsto \mathrm{D}(\eta(x, \cdot))(y) \in \mathrm{Hom}(Y, \mathbf{R}^n)$$

is continuous with compact support in C.

We endow H_C with the norm (see 3.4.23) whose value at $\eta \in H_C$ is

$$\sup\{\| \mathrm{D}(\eta(x, \cdot))(y)\| : (x, y) \in U \times Y\},$$

and endow $H = \bigcup\{H_C : C \text{ is a compact subset of } U \times Y\}$ with the locally convex final topology induced by the inclusion maps $H_C \to H$.

Remark 3.4.23. We shall verify that H_C is a normed space. Let $\eta \in H_C$. Note that 3.4.22(1) implies that $\eta(x, 0) = 0$ whenever $x \in U$. If $(x, y) \in U \times Y$, then we have

$$\eta(x, y) = \textstyle\int_0^1 \langle y, \mathrm{D}(\eta(x, \cdot))(ty)\rangle \, \mathrm{d}\mathscr{L}^1 \, t;$$

in particular, η is continuous. If η satisfies

$$\sup\{\| \mathrm{D}(\eta(x, \cdot))(y)\| : x \in U \text{ and } y \in Y\} = 0,$$

then $\eta = 0$. Thus, H_C is a normed space.

Remark 3.4.24. Let K be a compact subset of U and $0 \le s < \infty$. From 3.4.22(2) and 3.4.23, the map

$$H_{K \times \mathbf{B}(0,s)} \to \mathscr{K}_{K \times \mathbf{B}(0,s)}(U \times Y, \mathrm{Hom}(Y, \mathbf{R}^n))$$

defined by

$$\eta \mapsto [(x, y) \mapsto \mathrm{D}(\eta(x, \cdot))(y)]$$

is a homeomorphic embedding of Banach spaces; noting from 3.4.1(2) that

$$\mathscr{K}_{K \times \mathbf{B}(0,s)}(U \times Y, \mathrm{Hom}(Y, \mathbf{R}^n)) \simeq \mathscr{K}_{K \times \mathbf{B}(0,s)}(U \times Y, \mathrm{Hom}(Y, \mathbf{R}))^n,$$

so is the map

$$f : H_{K \times \mathbf{B}(0,s)} \to \mathscr{K}_{K \times \mathbf{B}(0,s)}(U \times Y, \mathrm{Hom}(Y, \mathbf{R}))^n.$$

On the other hand, recall from 3.4.16 that the map

$$\iota : \mathscr{K}_K(U, E_s) \to \mathscr{K}_{K\times \mathbf{B}(0,s)}(U \times Y, \operatorname{Hom}(Y, \mathbf{R}))$$

defined by

$$\iota(\eta)(x, y) = \mathrm{D}(\eta(x))(y) \quad \text{whenever } (x, y) \in U \times Y$$

is a norm-preserving isomorphism of Banach spaces, and so is the product map $g = \prod_{j=1}^{n} \iota$. Since $\operatorname{im} f = \operatorname{im} g$, it follows that $g^{-1} \circ f$ defines an isomorphism between the normed spaces $H_{K\times \mathbf{B}(0,s)}$ and $\mathscr{K}_K(U, E_s)^n$; in particular, $H_{K\times \mathbf{B}(0,s)}$ is a separable Banach space.

Consequently, letting K_i be a sequence of compact subsets of U such that $U = \bigcup_{i=1}^{\infty} K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ whenever $i \in \mathscr{P}$, we see H is the strict inductive limit of $H_{K_i\times \mathbf{B}(0,i)}$, hence by 3.1.1, a complete Hausdorff space such that H_C is a closed subspace of H whenever C is a compact subset of $U \times Y$.

Theorem 3.4.25. *The map $H \to \mathscr{K}(U \times Y, \operatorname{Hom}(Y, \mathbf{R}^\mathbf{n}))$ defined by*

$$\eta \mapsto (\mathrm{D}\, \eta(x, \cdot))(y) \quad \textit{whenever } \eta \in H \textit{ and } (x, y) \in U \times Y$$

is a homeomorphic embedding.

Furthermore, if $K_0 = \varnothing$ and K_i is a sequence of compact subsets of U such that $U = \bigcup_{i=1}^{\infty} K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ whenever $i \in \mathscr{P}$, then the subsets W_α consisting of $\eta \in H$ satisfying

$$\sup \operatorname{im} \|(\mathrm{D}\, \eta(x, \cdot))(y)\| \le \alpha(i)^{-1}$$

whenever $i \in \mathscr{P}$ and $(x, y) \in (U \times Y) \sim (K_{i-1} \times \mathbf{B}(0, i-1))$, corresponding to each $\alpha \in \mathscr{P}^{\mathscr{P}}$ form a fundamental system of neighborhoods of 0 of H.

Proof. Let K_i be as in the postscript. With the help of 3.4.24, 3.1.8, 3.4.15, 3.4.1(2), we have the following isomorphisms

$$\begin{aligned} H &= \varinjlim H_{K_i\times \mathbf{B}(0,i)} \\ &\simeq \varinjlim \mathscr{K}_{K_i}(U, E_i)^n \\ &\simeq \big(\varinjlim \mathscr{K}_{K_i}(U, E_i)\big)^n \\ &\simeq \mathscr{K}(U, E)^n. \end{aligned}$$

Thus, by 3.4.18, 3.1.8 and 3.4.1(2), the composition

$$H \simeq \mathscr{K}(U, E)^n \to \mathscr{K}(U \times Y, \operatorname{Hom}(Y, \mathbf{R}))^n \simeq \mathscr{K}(U \times Y, \operatorname{Hom}(Y, \mathbf{R}^n))$$

defines a homeomorphic embedding, and the postscript follows from 3.4.18. □

Remark 3.4.26. In case $Y = \mathbf{R}$, the homeomorphic embedding

$$H \to \mathscr{K}(U \times \mathbf{R}, \operatorname{Hom}(\mathbf{R}, \mathbf{R}^n)) \simeq \mathscr{K}(U \times \mathbf{R}, \mathbf{R}^n)$$

is surjective; in fact, for $\phi \in \mathscr{K}(U \times \mathbf{R}, \mathbf{R}^n)$, the function $\eta : U \times \mathbf{R} \to \mathbf{R}^n$ defined by

$$\eta(x, y) = \textstyle\int_0^y \phi(x, z) \, \mathrm{d}\mathscr{L}^1 z$$

is a member in H whose image under the embedding equals ϕ.

Lemma 3.4.27. *Suppose A is a directed set, $(F_\alpha, f_{\beta\alpha})$ is an inductive system of locally convex spaces relative to A, $(D_\alpha, \delta_{\beta\alpha})$ is an inductive system of vector spaces relative to A, we denote by (F, f_α) and (D, δ_α) the inductive limit of $(F_\alpha, f_{\beta\alpha})$ and $(D_\alpha, \delta_{\beta\alpha})$ respectively, for $\alpha \in A$, D_α is a dense subspace of F_α and the inclusion map $\iota_\alpha : D_\alpha \to F_\alpha$ satisfies*

$$f_{\beta\alpha} \circ \iota_\alpha = \iota_\beta \circ \delta_{\beta\alpha} \quad \textit{whenever } \alpha \le \beta \in A.$$

Then, the vector space $D = \bigcup_{\alpha \in A} \delta_\alpha[D_\alpha]$ can be identified as a dense subspace of F. Moreover, if D_α is endowed with the subspace topology induced by ι_α, G is a complete Hausdorff locally convex space, and the family of continuous linear maps $g_\alpha : D_\alpha \to G$ for $\alpha \in A$ satisfies

$$g_\alpha = g_\beta \circ \delta_{\beta\alpha} \quad \textit{whenever } \alpha \le \beta \in A,$$

then the inductive limit of the unique extensions of g_α equals the unique extension of the inductive limit of g_α; here, by convention, we also view G as the limit of the constant inductive system $(G, \mathbf{1}_G)$ of locally convex spaces relative to A.

Proof. From [Bou04b, III, §7, No. 6, Proposition 7], the inductive limit ι of the inclusion maps $\iota_\alpha : D_\alpha \to F_\alpha$ is a monomorphism, and we identify D as its image under ι in F. Since D_α is dense in F_α, we have

$$\operatorname{im} f_\alpha \subset \operatorname{Clos} \operatorname{im}(f_\alpha \circ \iota_\alpha) = \operatorname{Clos} \operatorname{im}(\iota \circ \delta_\alpha) \quad \text{whenever } \alpha \in A.$$

It follows that

$$\begin{aligned} F = \bigcup \{\operatorname{im} f_\alpha : \alpha \in A\} &\subset \bigcup \{\operatorname{Clos} \operatorname{im}(\iota \circ \delta_\alpha) : \alpha \in A\} \\ &\subset \operatorname{Clos} \bigcup \{\operatorname{im}(\iota \circ \delta_\alpha) : \alpha \in A\} \\ &= \operatorname{Clos} \iota[D]. \end{aligned}$$

To prove the postscript, we denote by h_α the unique continuous extension of g_α for $\alpha \in A$, and these h_α satisfy

$$h_\alpha = h_\beta \circ f_{\beta\alpha} \quad \text{whenever } \alpha \le \beta \in A.$$

Then, the inductive limit h of h_α exists and satisfies

$$h \circ (\iota \circ \delta_\alpha) = h \circ (f_\alpha \circ \iota_\alpha) = g_\alpha \quad \text{whenever } \alpha \in A.$$

It follows that $h \circ \iota$ equals the inductive limit of g_α. Finally, if D is endowed with the subspace topology, then h is the unique continuous extension of the inductive limit of g_α. □

Remark 3.4.28. Let K be a compact subset of U and $0 \le s < \infty$. Then $\mathscr{D}_K(U, \mathbf{R}^n)$ and $\mathscr{E}(Y, \mathbf{R}) \cap E_s$ are dense in $\mathscr{K}_K(U, \mathbf{R}^n)$ and E_s, respectively. By considering the composition of the two monomorphisms

$$\mathscr{D}_K(U, \mathbf{R}^n) \otimes (\mathscr{E}(Y, \mathbf{R}) \cap E_s) \to \mathscr{K}_K(U, \mathbf{R}^n) \otimes E_s \to \mathscr{K}_K(U, \mathbf{R}^n \otimes E_s),$$

where the former one is the tensor product of inclusions, and the latter one ι is characterized by

$$\iota(\theta \otimes \gamma)(x) = \theta(x) \otimes \gamma \quad \text{whenever } \theta \in \mathscr{K}(U, \mathbf{R}^n) \text{ and } \gamma \in E,$$

we identify $\mathscr{D}_K(U, \mathbf{R}^n) \otimes (\mathscr{E}(Y, \mathbf{R}) \cap E_s)$ as a subspace of $\mathscr{K}_K(U, \mathbf{R}^n \otimes E_s) \simeq H_{K \times \mathbf{B}(0,s)}$. From [Bou04a, III, §1, No. 2, Proposition 5], the image of the canonical monomorphism

$$\mathscr{K}_K(U, \mathbf{R}) \otimes E_s \to \mathscr{K}_K(U, E_s)$$

is dense; hence the image of $\mathscr{K}_K(U, \mathbf{R}^n) \otimes E_s \simeq (\mathscr{K}_K(U, \mathbf{R}) \otimes E_s)^n$ under ι is dense in $\mathscr{K}_K(U, \mathbf{R}^n \otimes E_s) \simeq \mathscr{K}_K(U, E_s)^n$. Since $\mathscr{D}_K(U, \mathbf{R}^n) \otimes (\mathscr{E}(Y, \mathbf{R}) \cap E_s)$ is dense (with respect to the subspace topology induced from $\mathscr{K}_K(U, \mathbf{R}^n \otimes E_s)$) in $\mathscr{K}_K(U, \mathbf{R}^n) \otimes E_s$, hence in $\mathscr{K}_K(U, \mathbf{R}^n \otimes E_s)$.

Let K_i be a sequence of compact subset of U such that $U = \bigcup_{i=1}^\infty K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ whenever $i \in \mathscr{P}$. Applying 3.4.27 with

$$A = \mathscr{P}, \quad D_i = \mathscr{D}_{K_i}(U, \mathbf{R}^n) \otimes (\mathscr{E}(Y, \mathbf{R}) \cap E_i),$$
$$F_i = \mathscr{K}_{K_i}(U, \mathbf{R}^n \otimes E_i) \simeq H_{K_i \times \mathbf{B}(0,i)},$$

we conclude that $\mathscr{D}(U, \mathbf{R}^n) \otimes (\mathscr{E}(Y, \mathbf{R}) \cap E)$ is dense in $\mathscr{K}(U, \mathbf{R}^n \otimes E) \simeq H$.

Chapter 4

Differentiability of Young functions

Hypotheses. Throughout this chapter, we suppose $m, n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, and Y is a finite-dimensional Banach space. Whenever Z is a finite-dimensional Banach space, we denote, by $E(Z)$, $E_s(Z)$, $\widetilde{E}(Z)$, $\widetilde{E}_s(Z)$ and $H(Z)$, the test function spaces associated with Z as in Section 3.4; when it is clear from the context, we will omit the variable Z.

4.1 Young functions of generalized bounded variation

In this section, we define the distributional derivatives of Young functions on varifolds and the notion of generalized bounded variation of Young functions on varifolds, see 4.1.7 and 4.1.9. Also, we present the basic properties of Young functions of generalized bounded variation, and the compactness theorem is established.

Definition 4.1.1. Suppose μ is a probability Radon measure over Y and Z is a vector space. Then, we define a linear map $\mathbf{1}_Z \otimes \mu : Z \otimes \mathbf{L}_1(\mu) \to Z$ characterized by

$$(\mathbf{1}_Z \otimes \mu)(v \otimes \gamma) = (\textstyle\int \gamma \, \mathrm{d}\mu) v \quad \text{whenever } v \in Z \text{ and } \gamma \in \mathbf{L}_1(\mu).$$

We will instead write $(\mathbf{1}_Z \otimes \mu)(\xi) = \int \xi \, \mathrm{d}\mu$ whenever $\xi \in Z \otimes \mathbf{L}_1(\mu)$.

Remark 4.1.2. Suppose $0 \le s < \infty$, Z is a normed space, $\xi \in Z \otimes E_s$, and write $\xi = \sum_{i=1}^k v_i \otimes \gamma_i$ for some $k \in \mathscr{P}$, $v_i \in Z$, and $\gamma_i \in E_s$. Then, we have

$$\left| \textstyle\int \xi \, \mathrm{d}\mu \right| \le \sum_{i=1}^k |v_i| \left| \textstyle\int \gamma_i \, \mathrm{d}\mu \right| \le s \sum_{i=1}^k |v_i| \sup \operatorname{im} \| \mathrm{D}\, \gamma_i \|$$

by 3.3.3, and it follows that $|\int \xi \, \mathrm{d}\mu| \le s|\xi|$.

Definition 4.1.3. Whenever X is a normed space and Z is an inner product space, we define the bilinear form

$$\blacksquare : Z \times (Z \otimes X) \to X$$

to satisfy

$$v \blacksquare (w \otimes x) = (v \bullet w)x$$

whenever $v, w \in Z$, and $x \in X$.

Remark 4.1.4. If $v \in Z$ and $\xi \in Z \otimes X$ with $\xi = \sum_{i=1}^{k} w_i \otimes x_i$ for some $k \in \mathscr{P}$, $w_i \in Z$, and $x_i \in X$, then we have

$$|v \blacksquare \xi| \le \sum_{i=1}^{k} |v \bullet w_i||x_i| \le |v| \sum_{i=1}^{k} |w_i||x_i|$$

It follows that $|v \blacksquare \xi| \le |v||\xi|$, or equivalently, $\| \blacksquare \| \le 1$; in particular, $\blacksquare$ is continuous.

Remark 4.1.5. Suppose μ is a probability Radon measure on Y. Then, we have $\int v \blacksquare \xi \, \mathrm{d}\mu = v \bullet \left(\int \xi \, \mathrm{d}\mu\right)$ whenever $\xi \in \mathbf{R}^n \otimes E$ and $v \in \mathbf{R}^n$.

Lemma 4.1.6. *Suppose Z is a normed space. Then, the map $(Z \otimes E_s) \times \mathbf{P}(Y) \to Z$ defined by*

$$(\xi, \mu) \mapsto \int \xi \, \mathrm{d}\mu \quad \text{for } \xi \in Z \otimes E_s \text{ and } \mu \in \mathbf{P}(Y)$$

is continuous.

Proof. We endow $(E_s)^*$ with the dual norm topology. Since the dual pairing defines a continuous bilinear map $E_s \times (E_s)^* \to \mathbf{R}$, by 3.1.10, its associated linear map $E_s \otimes (E_s)^* \to \mathbf{R}$ is continuous. On the other hand, the natural bilinear map $(Z \otimes E_s) \times (E_s)^* \to (Z \otimes E_s) \otimes (E_s)^*$ is continuous by definition. Thus, the map in the conclusion can be expressed as the composition of continuous maps

$$\begin{aligned}(Z \otimes E_s) \times \mathbf{P}(Y) \to (Z \otimes E_s) \times (E_s)^* \to (Z \otimes E_s) \otimes (E_s)^* \\ \simeq Z \otimes (E_s \otimes (E_s)^*) \to Z \otimes \mathbf{R} \simeq Z\end{aligned}$$

with the help of 3.3.4, 3.1.12, and 3.1.11. □

Similar to [Men16, 8.1], for $0 \le s < \infty$ and Young functions f of type Y on varifolds, there exists a unique distribution related to the derivative of f; the notions of differentiability of f will be characterized by the behavior of these distributions corresponding to $0 \le s < \infty$.

Lemma 4.1.7. *Suppose $V \in \mathbf{V}_m(U)$, f is a $\|V\| + \|\delta V\|$ Young function of type Y, and $0 \le s < \infty$. Then, there exists $T^s \in \mathscr{D}'(U, \mathbf{R}^n \otimes E_s)$ uniquely characterized by the requirement that*

$$T^s_{(x)}(\theta(x) \otimes \gamma) = \int \left(\int \gamma \,\mathrm{d} f(x)\right) \theta(x) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\| \, x \\ - \int \left(\int \gamma \,\mathrm{d} f(x)\right) S \bullet \mathrm{D}\,\theta(x) \,\mathrm{d} V\,(x, S)$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in E_s$.

Proof. Define $T^s \in \mathscr{D}'(U, \mathbf{R}^n \otimes E_s)$ by

$$T^s(\psi) = \int \left(\int \psi(x) \,\mathrm{d} f(x)\right) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\| \, x \\ - \int S \bullet \left(\int \mathrm{D}\,\psi(x) \,\mathrm{d} f(x)\right) \,\mathrm{d} V\,(x, S)$$

whenever $\psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_s)$, where we employ from 3.1.12 the isomorphism of normed spaces

$$\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n \otimes E_s) \simeq \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \otimes E_s$$

and the $\|V\| + \|\delta V\|$ measurability of the functions $x \mapsto \int \psi(x) \,\mathrm{d} f(x)$ and $x \mapsto \int \mathrm{D}\,\psi(x) \,\mathrm{d} f(x)$ for $x \in \operatorname{dmn} f$ is assured by 4.1.6. The uniqueness follows from [Men16, 3.1]. □

Lemma 4.1.8. *Suppose $B : \mathscr{D}(U, \mathbf{R}^n) \times (\mathscr{E}(Y, \mathbf{R}) \cap E) \to \mathbf{R}$ is a bilinear map. Then, the following statements are equivalent.*

(1) There exists $T \in H^*$ such that $T_{(x,y)}(\gamma(y)\theta(x)) = B(\theta, \gamma)$ whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(Y, \mathbf{R}) \cap E$.

(2) For $0 \le s < \infty$, there exists $T^s \in \mathscr{D}'(U, \mathbf{R}^n \otimes E_s)$ representable by integration such that $T^s_{(x)}(\theta(x) \otimes \gamma) = B(\theta, \gamma)$ whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(Y, \mathbf{R}) \cap E_s$.

Proof. It follows from 3.4.28. □

Definition 4.1.9. Whenever $V \in \mathbf{V}_m(U)$ and f is a $\|V\| + \|\delta V\|$ Young function of type Y, we define the bilinear map B by

$$B(\theta, \gamma) = \int \left(\int \gamma \,\mathrm{d} f(x)\right) \theta(x) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\| \, x \\ - \int \left(\int \gamma \,\mathrm{d} f(x)\right) S \bullet \mathrm{D}\,\theta(x) \,\mathrm{d} V\,(x, S)$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(Y, \mathbf{R}) \cap E$. Then, f is termed to possess *generalized V bounded variation* if and only if B satisfies one of the conditions in 4.1.8.

Remark 4.1.10. In view of 4.1.7, the Young function f is of generalized V bounded variation if and only if T^s is representable by integration whenever $0 \le s < \infty$, where T^s is the distribution associated with f as in 4.1.7.

Similarly, we may define $T : H \to \mathbf{R}$ by

$$\begin{aligned} T(\phi) = \textstyle\iint \phi(x,y) \bullet \boldsymbol{\eta}(V,x)\, \mathrm{d}f(x)\, y \,\mathrm{d}\|\delta V\|\, x \\ - \textstyle\iint S \bullet \mathrm{D}(\phi(\cdot,y))(x)\, \mathrm{d}f(x)\, y\, \mathrm{d}V\,(x,S) \end{aligned}$$

whenever $\phi \in H$. Then, by 4.1.8, we see f is of generalized V bounded variation if and only if T is continuous.

We shall compare our definition with the one in the single-valued case when $Y = \mathbf{R}$.

Remark 4.1.11. Suppose $Y = \mathbf{R}$, g is a $\|V\| + \|\delta V\|$ measurable real-valued function, $f = \boldsymbol{\delta} \circ g$ is as in 3.2.5, $G = \{(x,y) : g(x) > y\} \subset U \times \mathbf{R}$, and $\widetilde{T} \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n)$ satisfies

$$\widetilde{T}(\phi) = \textstyle\int V\partial\{x : (x,y) \in G)\}(\phi(\cdot,y))\,\mathrm{d}\mathscr{L}^1\, y \quad \text{whenever } \phi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n),$$

where

$$\begin{aligned} V\partial\{x : (x,y) \in G\} = (\delta V) \llcorner \{x : (x,y) \in G\} \\ - \delta(V \llcorner \{x : (x,y) \in G\} \times \mathbf{G}(n,m)). \end{aligned}$$

It follows from [Men16, 8.1, 8.2] that

$$\widetilde{T}_{(x,y)}(\gamma'(y)\theta(x)) = B(\theta,\gamma) \quad \text{whenever } \theta \in \mathscr{D}(U, \mathbf{R}^n) \text{ and } \gamma \in \mathscr{E}(\mathbf{R},\mathbf{R}) \cap E.$$

Let I be the isomorphism $H \simeq \mathscr{K}(U \times \mathbf{R}, \mathbf{R}^n)$ described in 3.4.26. If $\widetilde{T}$ is representable by integration, then $\widetilde{T} \circ I \in H^*$ satisfies

$$(\widetilde{T} \circ I)_{(x,y)}(\gamma(y)\theta(x)) = B(\theta,\gamma)$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(\mathbf{R},\mathbf{R}) \cap E$. Conversely, letting T be as in 4.1.10, then we have

$$\widetilde{T}_{(x,y)}(\gamma'(y)\theta(x)) = B(\theta,\gamma) = T_{(x,y)}(\gamma(y)\theta(x))$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(\mathbf{R},\mathbf{R}) \cap E$. Since T is continuous, we infer from 3.4.28 and [Men16, 3.1] that $\widetilde{T} = (T \circ I^{-1}) | \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n)$, hence $\widetilde{T}$ is representable by integration.

This shows 4.1.9 extends the definition [MS25b, 4.2] of real-valued functions of generalized V bounded variation.

Next, to extend the subgraph characterization of functions of generalized bounded variation in [MS25b, 4.2] to our setting, we first define functions of bounded variation on a varifold whose values are contained in the topological dual of some separable Banach space.

Definition 4.1.12. Suppose $V \in \mathbf{V}_m(U)$, Z is a separable Banach space, and f is a $\|V\| + \|\delta V\|$ measurable Z^*-valued function with respect to the Z topology. We say f is of *locally* (V, Z) *bounded variation* if and only if $\|f\| \in \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|)$ and there exists $T \in \mathscr{D}(U, \mathbf{R}^n \otimes Z)$ representable by integration such that

$$T_x(\theta(x) \otimes z) = \textstyle\int \langle z, f(x)\rangle \theta(x) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\|\, x \\ - \textstyle\int \langle z, f(x)\rangle S \bullet \mathrm{D}\,\theta(x) \,\mathrm{d}V\,(x, S)$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $z \in Z$.

The following lemma shows that every Young function of type Y can be viewed as a measurable function with values in $(E_s)^*$ whenever $0 \le s < \infty$, and provides an equivalent definition of Young functions of generalized bounded variation.

Lemma 4.1.13. *Suppose $V \in \mathbf{V}_m(U)$ and f is a $\|V\| + \|\delta V\|$ Young function of type Y. Then, f can be viewed as a member of $\mathbf{L}_\infty(\|V\| + \|\delta V\|, (E_s)^*)$ whenever $0 \le s < \infty$.*

Moreover, f is of generalized V bounded variation if and only if f is of locally (V, E_s) bounded variation whenever $0 \le s < \infty$.

Proof. The main assertion is a consequence of 3.3.4 and the postscript follows immediately. □

Theorem 4.1.14. *Suppose $V \in \mathbf{V}_m(U)$, f is a $\|V\| + \|\delta V\|$ Young function of type $\mathbf{R}$, $g : (\operatorname{dmn} f) \times \mathbf{R} \to \mathbf{R}$ is defined by $g(x, y) = f(x)\{t : y \le t < \infty\}$ for $x \in \operatorname{dmn} f$ and $y \in \mathbf{R}$, and $W \in \mathbf{V}_{m+1}(U \times \mathbf{R})$ satisfies*

$$W(k) = \textstyle\iint k((x, y), S \times \mathbf{R}) \,\mathrm{d}\mathscr{L}^1\, y \,\mathrm{d}V\,(x, S)$$

whenever $k \in \mathscr{K}((U \times \mathbf{R}) \times \mathbf{G}(n + 1, m + 1))$.

Then, g is a $\|W\| + \|\delta W\|$ measurable $\mathbf{R}$-valued function and we have

$$T_g(\psi) = T_f(I^{-1}(p \circ \psi)) - \textstyle\int (q \circ \psi)(x, y) \,\mathrm{d}f(x)\, x \,\mathrm{d}\|V\|\, x$$

whenever $\psi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n \times \mathbf{R})$, where T_f is the linear map associated with f as in 4.1.10, $T_g \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n \times \mathbf{R})$ is the distribution associated with g as in 4.1.12 with V replaced by W and $Z = \mathbf{R} \simeq \mathbf{R}^*$, $I : H \to \mathscr{K}(U \times \mathbf{R}, \mathbf{R}^n)$

denotes the linear isomorphism described in 3.4.26, and $p : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ *and* $q : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ *are the projection maps.*

In particular, f *is of generalized* V *bounded variation if and only if* g *is of locally* $(W, \mathbf{R})$ *bounded variation.*

Proof. We first prove that g is $\|W\| + \|\delta W\|$ measurable. For $y \in \mathbf{R}$, we denote by $[y]$ the largest integer that does not exceed y and define $g_i : (\operatorname{dmn} f) \times \mathbf{R} \to \mathbf{R}$ by

$$g_i(x, y) = f(x)\{t : [iy]/i \le t < \infty\} \quad \text{for } x \in \operatorname{dmn} f \text{ and } y \in \mathbf{R}.$$

By 3.2.13, we see $g_i(\cdot, y)$ is $\|V\| + \|\delta V\|$ measurable whenever $y \in \mathbf{R}$. On the other hand, $g_i(x, y) = f(x)\{t : k/i \le t < \infty\}$ whenever $x \in \operatorname{dmn} f$, $k \in \mathscr{P}$, and $k/i \le y < (k+1)/i$. We conclude from [Fed69, 2.6.2(2)] that g_i is $(\|V\| + \|\delta V\|) \times \mathscr{L}^1$ measurable. Since $\|W\| + \|\delta W\| = (\|V\| + \|\delta V\|) \times \mathscr{L}^1$ by [KM17, 3.6(1)(5)], and $g(x, y) = \lim_{i \to \infty} g_i(x, y)$ for $x \in \operatorname{dmn} f$ and $y \in \mathbf{R}$, we see g is $\|W\| + \|\delta W\|$ measurable.

If $\gamma \in \mathscr{E}(\mathbf{R}, \mathbf{R})$ satisfies $\operatorname{spt} \mathrm{D}\,\gamma$ is compact and $\inf \operatorname{spt} \gamma > -\infty$, then by Fubini's theorem, we have whenever $x \in \operatorname{dmn} f$,

$$\textstyle\int \gamma \,\mathrm{d} f(x) = \int \int_{\{y : -\infty < y \le t\}} \gamma'(y) \,\mathrm{d}\mathscr{L}^1 y \,\mathrm{d} f(x)\, t = \int \gamma'(y) g(x, y) \,\mathrm{d}\mathscr{L}^1 y. \quad (*)$$

Then, we compute by [KM17, 3.6(4)(5)(6)] and $(*)$ that

$$\begin{aligned} T_g(\psi) &= \textstyle\iint g(x, y)(p \circ \psi)(x, y) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\|\, x \,\mathrm{d}\mathscr{L}^1 y \\ &\quad - \textstyle\iint g(x, y) S \bullet \mathrm{D}(p \circ \psi)(\cdot, y)(x) \,\mathrm{d}V\,(x, S)\, \mathscr{L}^1 y \\ &\quad - \textstyle\iint g(x, y)((q \circ \psi)(x, \cdot))'(y) \,\mathrm{d}V\,(x, S)\, \mathscr{L}^1 y \\ &= T_f(I^{-1}(p \circ \psi)) - \textstyle\iint (q \circ \psi)(x, y) \,\mathrm{d} f(x)\, y \,\mathrm{d}\|V\|\, x. \end{aligned}$$

Since $I[H \cap \mathscr{E}(U \times \mathbf{R}, \mathbf{R}^n)] = \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n)$, we conclude from 3.4.28 that T_g is representable by integration if and only if T_f is continuous. □

Remark 4.1.15. The lemma above is an analogy of [MS25b, 4.2] in the multiple-valued function setting.

Next, we will present the compactness theorem for Young functions of generalized bounded variation.

Theorem 4.1.16 (Compactness)**.** *Suppose* C *is as in 3.2.22,* V_i *form a sequence in* $\mathbf{RV}_m(U, C)$ *such that* $\|\delta V_i\|$ *is a Radon measure,* f_i *is a Young function of type* Y *of generalized* V_i *bounded variation whenever* $i \in \mathscr{P}$*, and*

$$\lim_{t \to \infty} \sup\{\mathbf{Y}(\|V_i\|, f_i)(K \times (Y \sim \mathbf{B}(0, t))) : i \in \mathscr{P}\} = 0, \quad (4.1)$$

$$\sup\{(\|V_i\| + \|\delta V_i\| + \|T_i^s\|)(K) : i \in \mathscr{P}\} < \infty, \quad (4.2)$$

whenever K is a compact subset of U and $0 \le s < \infty$, where T_i^s is the distribution associated with f_i as in 4.1.7.

Then, there exists $V \in \mathbf{RV}_m(U, C)$, a $\|V\| + \|\delta V\|$ Young function f of type Y of generalized V bounded variation, and a subsequence (V_{i_k}, f_{i_k}) of (V_i, f_i) such that, as $k \to \infty$,

$$V_{i_k} \to V, \quad \mathbf{Y}(V_{i_k}, f_{i_k} \circ p) \to \mathbf{Y}(V, f \circ p), \quad \mathbf{Y}(\|V_{i_k}\|, f_{i_k}) \to \mathbf{Y}(\|V\|, f),$$

where $p : U \times \mathbf{G}(n, m) \to U$ is the projection map.

Proof. By 3.2.24, there exist a varifold $V \in \mathbf{RV}_m(U, C)$, a $\|V\|$ Young function g of type Y, and a subsequence (V_{i_k}, f_{i_k}) of (V_i, f_i) such that

$$V_{i_k} \to V, \quad \mathbf{Y}(V_{i_k}, f_{i_k} \circ p) \to \mathbf{Y}(V, g \circ p), \quad \mathbf{Y}(\|V_{i_k}\|, f_{i_k}) \to \mathbf{Y}(\|V\|, g),$$

By [Fed69, 2.9.2], there exists a Borel subset of U such that $\|\delta V\|_{\|V\|} = \|\delta V\| \llcorner B$ and $\|V\|(U \sim B) = 0$. Define for $x \in \operatorname{dmn} g$

$$f(x) = \begin{cases} g(x) & \text{if } x \in B \cap \operatorname{dmn} g, \\ \boldsymbol{\delta}_0 & \text{if } x \in U \sim B. \end{cases}$$

Then, f is a $\|V\| + \|\delta V\|$ Young function of type Y such that $f(x) = g(x)$ for $\|V\|$ almost all x, and

$$\mathbf{Y}(\|V\|, f) = \mathbf{Y}(\|V\|, g) \quad \text{and} \quad \mathbf{Y}(V, f \circ p) = \mathbf{Y}(V, g \circ p).$$

Next, we will show that f has generalized V bounded variation.

Suppose $0 \le s < \infty$, T^s is the distribution associated with f as in 4.1.7, $\psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_s)$, and G is an open set such that $\operatorname{spt} \psi \subset G$ and $\operatorname{Clos} G$ is a compact subset of U. From 4.1.2 and [All72, 4.11], we estimate

$$\begin{aligned} &\textstyle\int \left(\int \psi(x) \, \mathrm{d} f(x) \right) \bullet \boldsymbol{\eta}(V, x) \, \mathrm{d}\|\delta V\| \, x \\ &\qquad \le \textstyle\int s |\psi(x)| \, \mathrm{d}\|\delta V\| \, x \\ &\qquad \le s \sup \operatorname{im} |\psi| \|\delta V\|(G) \\ &\qquad \le s \sup \operatorname{im} |\psi| \sup\{\|\delta V_i\|(\operatorname{Clos} G) : i \in \mathscr{P}\}, \end{aligned}$$

and by 3.2.18

$$\begin{aligned} &\left| \textstyle\iint S \bullet \mathrm{D}\, \psi(x) \, \mathrm{d} f(x) \, \mathrm{d}V\,(x, S) \right| \\ &\qquad = \limsup_{k \to \infty} \left| \textstyle\iint S \bullet \mathrm{D}\, \psi(x) \, \mathrm{d} f_{i_k}(x) \, \mathrm{d}V_{i_k}\,(x, S) \right| \\ &\qquad = \limsup_{k \to \infty} \left| \textstyle\int \left(\int \psi(x) \, \mathrm{d} f_{i_k}(x) \right) \bullet \boldsymbol{\eta}(V_{i_k}, x) \, \mathrm{d}\|\delta V_{i_k}\| \, x - T_{i_k}^s(\psi) \right| \\ &\qquad \le \sup \operatorname{im} |\psi| \sup\{(s\|\delta V_i\| + \|T_i^s\|)(\operatorname{Clos} G) : i \in \mathscr{P}\}. \end{aligned}$$

Therefore, it follows from (4.2) that T^s is representable by integration, hence f possesses generalized V bounded variation. □

Remark 4.1.17. We always have

$$\iint S \bullet \mathrm{D}\,\psi(x)\,\mathrm{d}f(x)\,\mathrm{d}V\,(x,S) = \lim_{k\to\infty} \iint S \bullet \mathrm{D}\,\psi(x)\,\mathrm{d}f_{i_k}(x)\,\mathrm{d}V_{i_k}\,(x,S)$$

for $\psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_s)$ and $0 \le s < \infty$. However, it may happen that $T^s_{i_k}$ fails to converge to T^s; for instance, we may take V_i to be the union of two open rays $\mathbf{R} \cap \{r : i^{-1} \le r < \infty\}$ and $\mathbf{R} \cap \{r : -\infty < r \le -i^{-1}\}$ on which f_i is single-valued and has valued 1 and -1, respectively.

The following two lemmas show the relation between the distributions associated with a Young function f and its pushforward $h_\# f$ by a function h.

Lemma 4.1.18. *Suppose $V \in \mathbf{V}_m(U)$, Z is a finite-dimensional Banach space, f is a $\|V\| + \|\delta V\|$ Young function of type Y, $h : Y \to Z$ is a map of class 1, $0 \le r < \infty$, $0 \le s < \infty$, $h^{-1}[Z \cap \mathbf{B}(0,r)] \subset Y \cap \mathbf{B}(0,s)$, and $\Omega : E_r(Z) \to E_s(Y)$ is defined by*

$$\Omega(\gamma) = \gamma \circ h - (\gamma \circ h)(0) \quad \textit{whenever } \gamma \in E_r(Z).$$

Then, we have

$$T^r_{h_\# f}(\psi) = T^s_f((\mathbf{1}_{\mathbf{R}^n} \otimes \Omega) \circ \psi) \quad \textit{whenever } \psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_r(Z)),$$

where $T^r_{h_\# f}$ and T^s_f are the distributions associated with the $\|V\| + \|\delta V\|$ Young functions $h_\# f$ and f as in 4.1.7.

Proof. Let $M = \sup\{\|\mathrm{D}\,h(y)\| : y \in \mathbf{B}(0,s)\}$. Then, the linear map Ω satisfies $\|\Omega\| \le M$, hence

$$\mathrm{Lip}(\mathbf{1}_{\mathbf{R}^n} \otimes \Omega) = \|\mathbf{1}_{\mathbf{R}^n} \otimes \Omega\| \le M;$$

it follows that Ω and $\mathbf{1}_{\mathbf{R}^n} \otimes \Omega$ are continuous. For $x \in \operatorname{dmn} f$ and $\psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_r(Z))$, writing $\psi(x) = \sum_{i=1}^k v_i \otimes \gamma_i$, we employ from 3.1.12 the isomorphism of Banach spaces

$$\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n \otimes B) \simeq \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \otimes B \quad \text{whenever } B \text{ is a Banach space,}$$

and compute

$$\begin{aligned}
\textstyle\int \psi(x)\,\mathrm{d}(h_\# f)(x) &= \sum_{i=1}^{k} \textstyle\int v_i \otimes \gamma_i \,\mathrm{d}(h_\# f)(x) \\
&= \sum_{i=1}^{k} (\textstyle\int (\gamma_i \circ h)\,\mathrm{d}f(x)) v_i \\
&= \sum_{i=1}^{k} (\textstyle\int \Omega(\gamma_i)\,\mathrm{d}f(x)) v_i + (\gamma_i \circ h)(0) v_i \\
&= \sum_{i=1}^{k} \textstyle\int (\mathbf{1}_{\mathbf{R}^n} \otimes \Omega)(v_i \otimes \gamma_i)\,\mathrm{d}f(x) + \sum_{i=1}^{k} \int v_i \otimes \gamma_i \,\mathrm{d}\boldsymbol{\delta}_{h(0)} \\
&= \textstyle\int (\mathbf{1}_{\mathbf{R}^n} \otimes \Omega)(\psi(x))\,\mathrm{d}f(x) + \int \psi(x)\,\mathrm{d}\boldsymbol{\delta}_{h(0)};
\end{aligned}$$

on the other hand, since $\mathbf{1}_{\mathbf{R}^n} \otimes \Omega$ is linear, we have

$$S \bullet \mathrm{D}((\mathbf{1}_{\mathbf{R}^n} \otimes \Omega) \circ \psi)(x) = \Omega(S \bullet \mathrm{D}\,\psi(x)) \quad \text{whenever } (x, S) \in U \times \mathbf{G}(n, m).$$

Finally, note that the distribution associated with the constant Young function $\boldsymbol{\delta}_{h(0)}$ as in 4.1.7 vanishes, and the assertion follows. □

Theorem 4.1.19. *Suppose $V \in \mathbf{V}_m(U)$, Z is a finite-dimensional Banach space, f is a $\|V\| + \|\delta V\|$ Young function of type Y, and $h : Y \to Z$ is a locally Lipschitzian function, $0 \le r < \infty$, $0 \le s < \infty$, and $h^{-1}[Z \cap \mathbf{B}(0, r)] \subset Y \cap \mathbf{B}(0, s)$. Then, we have*

$$\|T^r_{h_\# f}\| \le \operatorname{Lip}(h|\mathbf{B}(0, s + \varepsilon))\|T^s_f\| \quad \textit{for all } \varepsilon > 0,$$

where $T^r_{h_\# f}$ and T^s_f are the the associated distributions of $h_\# f$ and f as in 4.1.7.

Proof. If $\dim Y = 0$, then $Y = \{0\}$ and both f and $h_\# f$ are constant Young functions. It follows that $T^s_f = 0$ and $T^r_{h_\# f} = 0$.

Let $\varepsilon > 0$, $k = \dim Y > 0$, $I : \mathbf{R}^k \to Y$ be a linear isomorphism, and $\phi \in \mathscr{D}(\mathbf{R}^k)$ be such that $\phi \ge 0$ and $\int \phi \,\mathrm{d}\mathscr{L}^k = 1$. Then, the smooth functions $h_i : Y \to Z$ defined by

$$h_i(y) = \textstyle\int h(y - I(x)) \cdot i^k \phi(ix)\,\mathrm{d}\mathscr{L}^k x \quad \text{for } y \in Y \text{ and } i \in \mathscr{P}$$

satisfy $h(y) = \lim_{i\to\infty} h_i(y)$ for $y \in Y$, and $\operatorname{Lip}(h_i|\mathbf{B}(0, s)) \le \operatorname{Lip}(h|\mathbf{B}(0, s + \varepsilon))$ for i sufficiently large. By 4.1.2 and [Fed69, 2.4.9], we have

$$\textstyle\int \xi \,\mathrm{d}(h_\# f)(x) = \lim_{i\to\infty} \int \xi \,\mathrm{d}(h_{i\#} f)(x)$$

whenever $x \in \operatorname{dmn} f$ and $\xi \in \mathbf{R}^n \otimes E_r(Z)$; employing [Fed69, 2.4.9] again, there holds

$$T^r_{h_\# f}(\psi) = \lim_{i \to \infty} T^r_{h_{i\#} f}(\psi) \quad \text{whenever } \psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_r(Z)).$$

Therefore, the assertion follows from 4.1.18. □

Corollary 4.1.20. *Suppose $V \in \mathbf{V}_m(U)$, Z is a finite-dimensional Banach space, f is a Young function of type Y of generalized V bounded variation, and $h : Y \to Z$ is a proper locally Lipschitzian function. Then, the pushforward Young function $h_\# f$ is of generalized V bounded variation.*

4.2 Characterizations of GBV functions

In this section, we investigate other equivalent definitions of Young functions of generalized bounded variation on varifolds.

Lemma 4.2.1. *Suppose* $\dim Y > 0$. *The locally convex space E does not have a countable base at* 0*; in particular, its topology is not metrizable.*

Proof. Suppose $U_1, U_2, \ldots$ is a sequence of neighborhoods of 0 in E and $r_1, r_2, \ldots$ are positive numbers such that

$$\mathbf{B}(0, r_i) \cap E_i \subset U_i \quad \text{whenever } i \in \mathscr{P}.$$

Let $v \in Y$ with $|v| = 1$, let $y_i = (i - 1/2)v$, and define $\mu_i \in E^*$ by $\mu_i(\gamma) = \langle v, \mathrm{D}\,\gamma(y_i)\rangle$ whenever $i \in \mathscr{P}$. Let $g : Y \to \mathbf{R}$ be a function of class 1 such that $\langle v, \mathrm{D}\, g(0)\rangle \neq 0$, $\operatorname{spt} g \subset \mathbf{B}(0, 1/2)$, and $\sup \operatorname{im} \| \mathrm{D}\, g\| = 1$. Note that the linear functional $\mu : E \to \mathbf{R}$ defined by

$$\mu(\gamma) = \sum_{i=1}^{\infty} (r_i \| \mathrm{D}\, g(0)\|)^{-1} \mu_i(\gamma) \quad \text{for } \gamma \in E$$

satisfies $\mu | E_j = \sum_{i=1}^{j} (r_i \| \mathrm{D}\, g(0)\|)^{-1} \mu_i | E_j \in (E_j)^*$ for $j \in \mathscr{P}$ and hence is a member of E^*. If we define

$$f_i(y) = r_i g(y - y_i) \quad \text{for } y \in Y,$$

then $f_i \in \mathbf{B}(0, r_i) \cap E_i \subset U_i$, $\mathrm{D}\, f_i(y_j) = 0$ for $i \neq j$, and $\mathrm{D}\, f_i(y_i) = r_i \,\mathrm{D}\, g(0)$. It follows that

$$\sup\{|\mu(f)| : f \in U_i\} \geq \frac{|\langle v, \mathrm{D}\, g(0)\rangle|}{\| \mathrm{D}\, g(0)\|} > 0 \quad \text{whenever } i \in \mathscr{P}.$$

Therefore, if $U_1, U_2, \ldots$ formed a local base of 0 in E, then μ would fail to be continuous at 0. □

Remark 4.2.2. To study the differentiability of functions $f : U \to E$, the notion of differentiation in Fréchet spaces (see [Ham82, Part I]) is not available for our setting.

Definition 4.2.3. If F is the strict inductive limit of a sequence of Banach spaces F_i with $F_i \subset F_{i+1}$ for $i \in \mathscr{P}$, then a function $f : U \to F$ is said to be of class k if and only if, for any $x \in U$, there exists $i \in \mathscr{P}$ and an open neighborhood V of x such that $f[V] \subset F_i$ and $f|V : V \to F_i$ is of class k.

Remark 4.2.4. If $i \le j \in \mathscr{P}$, then the inclusion map $F_i \to F_j$ is a homeomorphic embedding and its image is a closed subspace of F_j. It is thus straightforward to verify that if f is of class k and V is an open subset of U such that $f[V] \subset F_i$ for some $i \in \mathscr{P}$, then $f|V : V \to F_i$ is of class k.

Definition 4.2.5. We denote by $\mathscr{C}^k(U, \mathbf{R}^n \otimes E)$ the vector space of functions $f : U \to \mathbf{R}^n \otimes E$ of class k and let

$$\mathscr{C}^k_c(U, \mathbf{R}^n \otimes E) = \mathscr{C}^k(U, \mathbf{R}^n \otimes E) \cap \{f : \operatorname{spt} f \text{ is compact}\}.$$

Remark 4.2.6. Recall from 3.1.13, 3.1.14, 3.4.1(2), and 3.4.15 that if K_i is a sequence of compact subsets of U such that $U = \bigcup_{i=1}^{\infty} K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ for $i \in \mathscr{P}$, then $\mathscr{K}(U, \mathbf{R}^n \otimes E)$ is the strict inductive limit of $\mathscr{K}_{K_i}(U, \mathbf{R}^n \otimes E_i)$. Similarly as in 3.4.28, since $\mathscr{C}^1_c(U, \mathbf{R}) \cap \mathscr{K}_{K_i}(U, \mathbf{R})$ and $\mathscr{K}_{K_i}(U, \mathbf{R}) \otimes (\mathbf{R}^n \otimes E_i)$ are dense in $\mathscr{K}_{K_i}(U, \mathbf{R})$ and $\mathscr{K}_{K_i}(U, \mathbf{R}^n \otimes E_i)$, respectively, we conclude that $\mathscr{C}^1_c(U, \mathbf{R}^n \otimes E) \cap \mathscr{K}_{K_i}(U, \mathbf{R}^n \otimes E_i)$ is dense in $\mathscr{K}_{K_i}(U, \mathbf{R}^n \otimes E_i)$. It follows from 3.4.27 that $\mathscr{C}^1_c(U, \mathbf{R}^n \otimes E)$ is dense in $\mathscr{K}(U, \mathbf{R}^n \otimes E)$.

Lemma 4.2.7. *Suppose* $L : \mathscr{C}^1_c(U, \mathbf{R}^n \otimes E) \to \mathbf{R}$ *is linear. Then, the following statements are equivalent.*

(1) L *possesses a continuous linear extension to* $\mathscr{K}(U, \mathbf{R}^n \otimes E)$*.*

(2) Whenever K *is a compact subset of* U *and* $0 \le s < \infty$*, there exists* $0 \le M < \infty$ *such that*

$$|L(\psi)| \le M \sup \operatorname{im} |\psi|$$

whenever $\psi \in \mathscr{C}^1_c(U, \mathbf{R}^n \otimes E_s)$ *and* $\operatorname{spt} \psi \subset K$*.*

Proof. Combine 4.2.6 and 3.4.27. □

Definition 4.2.8 (Alternative definition 1)**.** Suppose $V \in \mathbf{V}_m(U)$ and f is a $\|V\| + \|\delta V\|$ Young function of type Y. We define the linear map $L : \mathscr{C}^1_c(U, \mathbf{R}^n \otimes E) \to \mathbf{R}$ by

$$\begin{aligned} L(\psi) = \textstyle\iint & \boldsymbol{\eta}(V, x) \bullet \psi(x) \, \mathrm{d}f(x) \, \mathrm{d}\|\delta V\| x \\ & - \textstyle\iint S \bullet \mathrm{D}\,\psi(x) \, \mathrm{d}f(x) \, \mathrm{d}V(x, S) \end{aligned}$$

whenever $\psi \in \mathscr{C}_c^1(U, \mathbf{R}^n \otimes E)$. Then, f is termed to possess *generalized* V *bounded variation* if and only if L satisfies 4.2.7(1) or 4.2.7(2).

Remark 4.2.9. Since $L_x(\theta(x) \otimes \gamma) = B(\theta, \gamma)$ whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(Y, \mathbf{R}) \cap E$, where B is as in 4.1.9, we conclude from 3.4.27 and 3.4.28 that 4.2.8 is equivalent to 4.1.9.

Remark 4.2.10. We define the linear map $\iota : \mathscr{C}_c^1(U, \mathbf{R}^n \otimes E) \to H \cap \mathscr{C}^1(U \times Y, \mathbf{R}^n)$ by

$$u \bullet \iota(\psi)(x, y) = (u \blacksquare \psi(x))(y) \quad \text{whenever } (x, y) \in U \times Y \text{ and } u \in \mathbf{R}^n.$$

Note that the map $E_s \times Y \to \mathbf{R}$ defined by $(\gamma, y) \mapsto \gamma(y)$ is of class 1 because $\gamma \mapsto \gamma(y)$ is linear whenever $y \in Y$, hence ι is well-defined. However, ι is not surjective; in fact, the mixed second order derivatives $\mathrm{D}_w \mathrm{D}_v \iota(\psi)$ exist whenever $\psi \in \mathscr{C}_c^1(U, \mathbf{R}^n \otimes E)$, $v \in \mathbf{R}^n$, and $w \in Y$, where D_v and D_w denote the operators of directional derivatives.

Definition 4.2.11 (Alternative definition 2)**.** Suppose $V \in \mathbf{V}_m(U)$ and f is a $\|V\| + \|\delta V\|$ Young function of type Y. We define the linear map $T : \mathscr{E}(U \times Y, \mathbf{R}^n) \cap H \to \mathbf{R}$ by

$$\begin{aligned} T(\phi) = & \textstyle\iint \boldsymbol{\eta}(V, x) \bullet \phi(x, y) \, \mathrm{d} f(x) \, y \, \mathrm{d} \|\delta V\| \, x \\ & - \textstyle\iint S \bullet (\mathrm{D} \, \phi(x, y) \circ \iota) \, \mathrm{d} f(x) \, y \, \mathrm{d} V \, (x, S) \end{aligned}$$

whenever $\phi \in \mathscr{E}(U \times Y, \mathbf{R}^n) \cap H$, where $\iota : \mathbf{R}^n \to \mathbf{R}^n \times Y$ satisfies $\iota(x) = (x, 0)$ for $x \in \mathbf{R}^n$.

Then, f is termed to possess *generalized* V *bounded variation* if and only if whenever K is a compact subset of U and $0 \leq s < \infty$, there exists $0 \leq M < \infty$ such that

$$|T(\phi)| \leq M \sup\{\| \mathrm{D} \, \phi(x, y) \circ \kappa \| : x \in U, y \in Y\}$$

whenever $\phi \in \mathscr{E}(U \times Y, \mathbf{R}^n) \cap H_{K \times \mathbf{B}(0,s)}$, where $\kappa : Y \to \mathbf{R}^n \times Y$ is given by $\kappa(y) = (0, y)$ for $y \in Y$.

Remark 4.2.12. Note that $T_{(x,y)}(\gamma(y)\theta(x)) = B(\theta, \gamma)$ whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(Y, \mathbf{R}) \cap E$. From 3.4.28, we see $\operatorname{dmn} T \cap H_{K \times \mathbf{B}(0,s)}$ is dense in $H_{K \times \mathbf{B}(0,s)}$ whenever K is a compact subset of U and $0 \leq s < \infty$; it follows from 3.4.27 that 4.2.11 is equivalent to 4.1.9.

4.3 Generalized weakly differentiable Young functions

In this section, we present the definition of generalized weakly differentiable Young functions on varifolds as well as the compactness theorem for such functions.

Definition 4.3.1. Whenever $V \in \mathbf{V}_m(U)$, f is of generalized V bounded variation, and T^s is associated with f as in 4.1.7 for $0 \le s < \infty$, we term f *generalized* V *weakly differentiable* if and only if $\|T^s\|$ is absolutely continuous with respect to $\|V\|$ for $0 \le s < \infty$. In this case, there exists a $\|V\|$ measurable function F^s with values in $(\mathbf{R}^n \otimes E_s)^*$ endowed with the $\mathbf{R}^n \otimes E_s$ topology such that

$$\textstyle\int_K |F^s| \, \mathrm{d}\|V\| < \infty \quad \text{whenever } K \text{ is a compact subset of } U$$

and

$$T^s(\psi) = \textstyle\int \langle \psi(x), F^s(x) \rangle \, \mathrm{d}\|V\| \, x$$

whenever $\psi \in \mathbf{L}_1(\|T^s\|, \mathbf{R}^n \otimes E_s)$ and $0 \le s < \infty$ by [MS25b, 3.2]. Such F^s is $\|V\|$ almost unique by [MS25b, 3.3].

Remark 4.3.2. Note that $T^r | \mathscr{D}(U, \mathbf{R}^n \otimes E_s) = T^s$ whenever $0 \le s \le r < \infty$. From the uniqueness of F^s, we have

$$F^r(x) | \mathbf{R}^n \otimes E_s = F^s(x) \quad \text{for } \|V\| \text{ almost all } x$$

whenever $0 \le s \le r < \infty$. Let D consist of all $x \in U$ such that

$$F^i(x) | \mathbf{R}^n \otimes E_j = F^j(x) \quad \text{whenever } i, j \in \mathscr{P} \text{ with } j \le i,$$

then $\|V\|(U \sim D) = 0$. Therefore, the relation

$$\langle \eta, F(x) \rangle = \lim_{i \to \infty} \langle \eta, F^i(x) \rangle \quad \text{whenever } \eta \in \mathbf{R}^n \otimes E \text{ and } x \in D$$

defines a $\|V\|$ measurable function with values in $(\mathbf{R}^n \otimes E)^*$ with respect to the weak topology by 3.1.14 and 3.2.6. Accordingly, we have

$$T(\psi) = \textstyle\int \langle \psi(x), F(x) \rangle \, \mathrm{d}\|V\| \, x$$

whenever $\psi \in \mathscr{K}(U, \mathbf{R}^n \otimes E)$.

Remark 4.3.3. Consider the isomorphism $(\mathbf{R}^n \otimes E)^* \simeq \operatorname{Hom}(\mathbf{R}^n, E^*)$ and let $v \in \mathbf{R}^n$. By 3.4.20, we may represent $\langle v, F(x)\rangle$ by a member μ in the dual space $\mathscr{K}(Y, \operatorname{Hom}(Y, \mathbf{R}))^*$ such that

$$\langle \gamma, \langle v, F(x)\rangle\rangle = \mu(\mathrm{D}\,\gamma) \quad \text{whenever } \gamma \in E.$$

However, if $\dim Y \geq 2$, such representations are far from unique even when constraint to satisfy $\operatorname{spt}\mu \subset \operatorname{spt} f(x)$; for instance, suppose $V \in \mathbf{IV}_1(\mathbf{R})$ satisfies $\|V\| = \mathscr{L}^1$, $Y = \mathbf{R}^2$, and $f(x) = \mathscr{L}^2 \llcorner C$ for $x \in \mathbf{R}$, where C denotes the unit square in $\mathbf{R}^2$, then f is generalized V weakly differentiable with derivative $F = 0$, but there exists $0 \neq \mu \in \mathscr{K}(Y, \operatorname{Hom}(Y, \mathbf{R}))^*$ representing $\langle 1, F(x)\rangle \in E^*$ such that $\operatorname{spt}\mu \subset \operatorname{spt} f(x)$, see 3.4.21.

Remark 4.3.4. Let $I : \widetilde{E} \to E$ be the linear map defined by $\widetilde{\gamma} \mapsto \widetilde{\gamma} - \widetilde{\gamma}(0)$ for $\widetilde{\gamma} \in \widetilde{E}$. Note that $\sup \operatorname{im} \|\mathrm{D}\, I(\widetilde{\gamma})\| = \sup \operatorname{im} \|\mathrm{D}\,\widetilde{\gamma}\|$ whenever $\widetilde{\gamma} \in \widetilde{E}$. If f is generalized V weakly differentiable, then for $0 \leq s < \infty$, there exists a $\|V\|$ measurable function $\widetilde{F}^s$ with values in $(\mathbf{R}^n \otimes \widetilde{E}_s)^*$ with respect to the $\mathbf{R}^n \otimes \widetilde{E}_s$ topology such that

$$\textstyle\int_K |\widetilde{F}^s(x)| \,\mathrm{d}\|V\|\, x < \infty \quad \text{whenever } K \text{ is a comapct subset of } U$$

and

$$\begin{aligned}\textstyle\int \langle \widetilde{\psi}(x), \widetilde{F}^s(x)\rangle \,\mathrm{d}\|V\|\, x = \int \Big(\int \widetilde{\psi}(x)\,\mathrm{d}f(x)\Big) \bullet \boldsymbol{\eta}(V,x)\,\mathrm{d}\|\delta V\|\, x \\ - \textstyle\int S \bullet \int \mathrm{D}\,\widetilde{\psi}(x)\,\mathrm{d}f(x)\,\mathrm{d}V\,(x,S)\end{aligned}$$

whenever $\widetilde{\psi} \in \mathscr{D}(U, \mathbf{R}^n \otimes \widetilde{E}^s)$ and $0 \leq s < \infty$; in fact, we may take $\widetilde{F}^s(x) = F^s(x) \circ (\mathbf{1}_{\mathbf{R}^n} \otimes I)$ whenever $x \in \operatorname{dmn} F^s$. The converse is true when $\dim Y \geq 2$ because $I|\widetilde{E}_s$ is a norm-preserving isomorphism onto E_s whenever $0 \leq s < \infty$.

Remark 4.3.5. Suppose g is a $\|V\| + \|\delta V\|$ measurable Y-valued function and $f = \boldsymbol{\delta} \circ g$ is as in 3.2.5. If g is a generalized V weakly differentiable function in the sense of [MS18, 4.2], then f is generalized V weakly differentiable in the sense of 4.3.1 and their derivatives $V\,\mathbf{D}\,g$ and F^s are related by the equation

$$\langle v, \mathrm{D}\,\gamma(g(x)) \circ V\,\mathbf{D}\,g(x)\rangle = \langle v \otimes \gamma, F^s(x)\rangle$$

whenever $0 \leq s < \infty$, $\gamma \in E_s$, and $V \in \mathbf{R}^n$.

Remark 4.3.6. By 4.1.19, if Z is a finite-dimensional Banach space, $h : Y \to Z$ is a proper locally Lipschitzian map, and f is generalized V weakly differentiable, then so is $h_\# f$.

Similar to 4.1.12 and 4.1.13, we define the weak differentiability of measurable functions with values contained in the topological dual of some separable Banach space and the relation of such functions to generalized weakly differentiable Young functions is studied.

Definition 4.3.7. Suppose $V \in \mathbf{V}_m(U)$, Z is a separable Banach space, and f is a function of locally (V, Z) bounded variation. We say f is (V, Z) *weakly differentiable* if and only if $\|T\|$ is absolutely continuous with respect to $\|V\|$, where T is the distribution associated with f as in 4.1.12.

In this case, there exists a $\|V\|$ measurable $(\mathbf{R}^n \otimes Z)^*$-valued function F with respect to the $\mathbf{R}^n \otimes Z$ topology such that $|F| \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|)$ and

$$T_x(\theta(x) \otimes z) = \textstyle\int \langle \theta(x) \otimes z, F(x) \rangle \, \mathrm{d}\|V\| \, x$$

whenever $z \in Z$ and $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ by [MS25b, 3.2]. Such F is $\|V\|$ almost unique by [MS25b, 3.3].

Lemma 4.3.8. *Suppose $V \in \mathbf{V}_m(U)$ and f is a $\|V\| + \|\delta V\|$ Young function of type Y. Then, f is generalized V weakly differentiable if and only if f is (V, E_s) weakly differentiable whenever $0 \le s < \infty$.*

Proof. By 4.1.13, the assertion is obvious. □

Theorem 4.3.9 (Compactness)**.** *Suppose C is as in 3.2.22, $V_i \in \mathbf{RV}_m(U, C)$, $\|\delta V_i\|$ is a Radon measure, $\|\delta V_i\|$ is absolutely continuous with respect to $\|V_i\|$, and f_i is a generalized V_i weakly differentiable Young function of type Y whenever $i \in \mathscr{P}$, such that*

$$\lim_{t \to \infty} \sup\{\mathbf{Y}(\|V_i\|, f_i)(K \times (Y \sim \mathbf{B}(0, t))) : i \in \mathscr{P}\} = 0, \tag{4.3}$$

$$\sup\{\|V_i\|(K) : i \in \mathscr{P}\} < \infty, \tag{4.4}$$

$$\lim_{t \to \infty} \sup\left\{\textstyle\int_{K \cap \{x : |\mathbf{h}(V_i, x)| > t\}} |\mathbf{h}(V_i, x)| \, \mathrm{d}\|V_i\| \, x : i \in \mathscr{P}\right\} = 0, \tag{4.5}$$

$$\lim_{t \to \infty} \sup\left\{\textstyle\int_{K \cap \{x : |F_i^s(x)| > t\}} |F_i^s| \, \mathrm{d}\|V_i\| : i \in \mathscr{P}\right\} = 0, \tag{4.6}$$

whenever K is a compact subset of U and $0 \le s < \infty$, where F_i^s is associated with f_i as in 4.3.1.

Then, there exist $V \in \mathbf{RV}_m(U, C)$, a generalized V weakly differentiable Young function f of type Y, and a subsequence (V_{i_k}, f_{i_k}) of (V_i, f_i) such that, as $k \to \infty$,

$$V_{i_k} \to V, \quad \mathbf{Y}(V_{i_k}, f_{i_k} \circ p) \to \mathbf{Y}(V, f \circ p), \quad \mathbf{Y}(\|V_{i_k}\|, f_{i_k}) \to \mathbf{Y}(\|V\|, f),$$

where $p : U \times \mathbf{G}(n, m) \to U$ is the projection map.

Proof. Note that from (4.4), (4.5) and (4.6) we have

$$\sup\{(\|V\| + \|\delta V_i\| + \|T_i^s\|)(K) : i \in \mathscr{P}\} < \infty$$

whenever K is a compact subset of U, where T_i^s is associated with f_i as in 4.1.7. Then, most of the conclusions follow from 4.1.16, and it remains to show that f is generalized V weakly differentiable, or equivalently, that $\|T^s\|$ is absolutely continuous with respect to $\|V\|$ whenever $0 \le s < \infty$, where T^s is the distribution associated with f as in 4.1.7.

Let V_{i_k} and f_{i_k} be the subsequences of V_i and f_i obtained in the last paragraph. Suppose $\varepsilon > 0$, $0 \le s < \infty$, and A is a subset of U such that $\operatorname{Clos} A$ is a compact subset of U and $\|V\|(A) = 0$. From (4.5) and (4.6), there exist $t > 0$ and an open set G such that

$$\begin{gathered}\operatorname{Clos} G \text{ is a compact subset of } U, \quad A \subset G, \quad \|V\|(G) \le t^{-1}\varepsilon,\\ \sup\left\{ \textstyle\int_{(\operatorname{Clos} G)\cap\{x : |\mathbf{h}(V_i,x)|>t\}} |\mathbf{h}(V_i,x)| \,\mathrm{d}\|V_i\|\, x : i \in \mathscr{P} \right\} \le \varepsilon,\\ \sup\left\{ \textstyle\int_{(\operatorname{Clos} G)\cap\{x : |F_i^s(x)|>t\}} |F_i^s| \,\mathrm{d}\|V_i\| : i \in \mathscr{P} \right\} \le \varepsilon.\end{gathered}$$

For $g \in \mathscr{D}(U, \mathbf{R}^n)$ with $\operatorname{spt} g \subset G$ and $\sup \operatorname{im} |g| \le 1$, we estimate by [All72, 2.6(2c)]

$$\begin{aligned}|\delta V(g)| &= \lim_{i\to\infty} |\textstyle\int g(x) \bullet \mathbf{h}(V_i,x) \,\mathrm{d}\|V_i\|\, x|\\ &\le t \limsup_{i\to\infty} \|V_i\|(\operatorname{spt} g) + \varepsilon\\ &\le t\|V\|(\operatorname{spt} g) + \varepsilon\\ &\le 2\varepsilon,\end{aligned}$$

hence $\|\delta V\|(A) \le \|\delta V\|(G) \le 2\varepsilon$, and we conclude $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Let $\psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_s)$ with $\operatorname{spt}\psi \subset G$ and $\sup \operatorname{im} |\psi| \le 1$. By 3.2.18, [MS25b, 3.21], 4.1.2, and [All72, 2.6(2c)], we have

$$\begin{aligned}&|\textstyle\int S \bullet \int \mathrm{D}\,\psi(x) \,\mathrm{d}f(x) \,\mathrm{d}V\,(x,S)|\\ &\qquad = \limsup_{k\to\infty} |\textstyle\int S \bullet \int \mathrm{D}\,\psi(x) \,\mathrm{d}f_{i_k}(x) \,\mathrm{d}V_{i_k}\,(x,S)|\\ &\qquad \le \limsup_{k\to\infty} |\textstyle\int \big(\int \psi(x) \,\mathrm{d}f_{i_k}(x)\big) \bullet \mathbf{h}(V_{i_k},x) \,\mathrm{d}\|V_{i_k}\|\, x|\\ &\qquad\quad + \limsup_{k\to\infty} |T_{i_k}^s(\psi)|\\ &\qquad \le s(t \limsup_{k\to\infty} \|V_{i_k}\|(\operatorname{spt}\psi) + \varepsilon)\\ &\qquad\quad + t \limsup_{k\to\infty} \|V_{i_k}\|(\operatorname{spt}\psi) + \varepsilon\\ &\qquad \le 2\varepsilon(s+1).\end{aligned}$$

On the other hand, by [MS25b, 3.21] and 4.1.2, we have

$$\textstyle\int \big(\int \psi(x)\,\mathrm{d}f(x)\big) \bullet \boldsymbol{\eta}(V,x)\,\mathrm{d}\|\delta V\|\,x \le s\int_G |\mathbf{h}(V,x)|\,\mathrm{d}\|V\|\,x.$$

Therefore, we conclude $\|T^s\|$ is absolutely continuous with respect to $\|V\|$. □

Remark 4.3.10. The example in 4.1.17 shows that the hypothesis on absolute continuity of the first variation with respect to the weight measure of varifolds cannot be omitted; in fact, the limit function in this example has a jump discontinuity at $0 \in \mathbf{R}$.

Lemma 4.3.11. *Suppose $V \in \mathbf{RV}_m(U)$. Then, every locally Lipschitzian $\|V\|+\|\delta V\|$ Young function f of type Y is generalized V weakly differentiable. Moreover, for $0 \le s < \infty$ and $\gamma \in E_s$, the function F^s associated with f as in 4.3.1 satisfies, for $\|V\|$ almost every x,*

$$\langle v \otimes \gamma, F^s(x)\rangle = \langle v, V\,\mathbf{D}\,f_\gamma(x)\rangle \quad \textit{whenever } v \in \mathbf{R}^n,$$

where $f_\gamma(x) = \int \gamma\,\mathrm{d}f(x)$.

Proof. Let $0 \le s < \infty$. For $\gamma \in E_s$, and $\theta \in \mathscr{D}(U,\mathbf{R}^n)$, letting $\tau(x) = \operatorname{Tan}^m(\|V\|,x)$ whenever $\operatorname{Tan}^m(\|V\|,x) \in \mathbf{G}(n,m)$, we deduce from [Men12, 4.5(4)] that

$$\begin{aligned}\textstyle\iint \gamma(y)\theta(x)\bullet\boldsymbol{\eta}(V,x)\,\mathrm{d}f(x)\,y\,\mathrm{d}\|\delta V\|\,x \\ = \textstyle\int \langle \theta(x), (\|V\|,m)\operatorname{ap}\mathrm{D}\,f_\gamma(x)\circ\tau(x)\rangle\,\mathrm{d}\|V\|x \\ + \textstyle\int \big(\int \gamma\,\mathrm{d}f(x)\big)\,\tau(x)\bullet \mathrm{D}\,\theta(x)\,\mathrm{d}\|V\|\,x.\end{aligned}$$

Let D be a countable dense subset of E_s such that D is a vector space over the field $\mathbf{Q}$ of rational numbers and let

$$A = \operatorname{dmn}\tau \cap \bigcap_{\gamma\in D} \operatorname{dmn}(\|V\|,m)\operatorname{ap}\mathrm{D}\,f_\gamma.$$

By [Men12, 4.5(2)], $\|V\|(U \sim A) = 0$. Note that $f_{\gamma_1+\gamma_2} = f_{\gamma_1} + f_{\gamma_2}$ for $\gamma_1,\gamma_2 \in E$. For $x \in A$, we define a bilinear form $B(x) : \mathbf{R}^n \times D \to \mathbf{R}$ over $\mathbf{Q}$ by the requirement

$$B(x)(v,\gamma) = \langle v, (\|V\|,m)\operatorname{ap}\mathrm{D}\,f_\gamma(x)\circ\tau(x)\rangle \quad \text{for } v \in \mathbf{R}^n \text{ and } \gamma \in E_s.$$

Since $\|B(x)\| \le \lim_{r\to 0+}\operatorname{Lip}(f|\mathbf{B}(x,r)) < \infty$, $B(x)$ extends uniquely to a continuous bilinear form $\mathbf{R}^n \times E_s \to \mathbf{R}$ over $\mathbf{R}$; by 3.1.10, there exists $F^s(x) \in (\mathbf{R}^n \otimes E_s)^*$ such that $\|F^s(x)\| = \|B(x)\|$ and

$$\langle v\otimes\gamma, F^s(x)\rangle = B(x)(v,\gamma) \quad \text{for } v \in \mathbf{R}^n \text{ and } \gamma \in D.$$

Since the image of the canonical map $\mathbf{R}^n \otimes D \to \mathbf{R}^n \otimes E_s$ is dense, where the tensor product $\mathbf{R}^n \otimes D$ is taken over $\mathbf{Q}$, we conclude from [Men12, 4.5(1)] and 3.2.6 that F^s is $\|V\|$ measurable. By [Men16, 3.1], the image of the canonical map $\mathscr{D}(U, \mathbf{R}^n) \otimes E_s \to \mathscr{D}(U, \mathbf{R}^n \otimes E_s)$ is dense whence we infer

$$T^s(\psi) = \textstyle\int \langle \psi(x), F^s(x) \rangle \,\mathrm{d}\|V\|\, x \quad \text{whenever } \psi \in \mathscr{D}(U, \mathbf{R}^n \otimes E_s),$$

and the assertion follows. Finally, the postscript follows from [Men16, 8.7]. □

The following lemma shows that the join of two generalized weakly differentiable Young functions is again generalized weakly differentiable provided one of them is locally Lipschitzian.

Lemma 4.3.12. *Suppose $V \in \mathbf{RV}_m(U)$, Z is a finite-dimensional Banach space with $\dim Z \geq 1$, $\dim Y \geq 1$, f is a generalized V weakly differentiable Young function of type Y, and g is a locally Lipschitzian $\|V\| + \|\delta V\|$ Young function of type Z. Then, $h = f \times g$ is generalized V weakly differentiable. Moreover, for $0 \leq s < \infty$ and $\|V\|$ almost all x, the function $\widetilde{H}^s$ associated with h as in 4.3.4 is characterized by*

$$\begin{aligned} \langle v \otimes \widetilde{\gamma}, \widetilde{H}^s(x) \rangle = {} & \textstyle\int \langle v \otimes \widetilde{\gamma}(y, \cdot), \widetilde{G}^s(x) \rangle \,\mathrm{d}f(x)\, y \\ & + \textstyle\int \langle v \otimes \widetilde{\gamma}(\cdot, z), \widetilde{F}^s(x) \rangle \,\mathrm{d}g(x)\, z \end{aligned}$$

whenever $v \in \mathbf{R}^n$ and $\widetilde{\gamma} \in \widetilde{E}_s(Y \times Z)$, where $\widetilde{F}^s$ and $\widetilde{G}^s$ are the functions associated with f and g as in 4.3.4 and $Y \times Z$ is endowed with the norm with value $\sup\{|y|, |z|\}$ at $(y, z) \in Y \times Z$.

Proof. Define $\widetilde{H}^s(x)$ as in the conclusion for $x \in \operatorname{dmn} \widetilde{F}^s \cap \operatorname{dmn} \widetilde{G}^s$. Observe that $\widetilde{H}^s$ is a $\|V\|$ measurable function with values in $(\mathbf{R}^n \otimes \widetilde{E}_s(Y \times Z))^*$ with respect to the $\mathbf{R}^n \otimes \widetilde{E}_s(Y \times Z)$ topology such that

$$\textstyle\int_K |\widetilde{H}^s| \,\mathrm{d}\|V\| < \infty \quad \text{whenever } K \text{ is a compact subset of } U.$$

Let $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and let $\widetilde{\gamma}(y, z) = \widetilde{\gamma}_1(y)\widetilde{\gamma}_2(z)$ for $(y, z) \in Y \times Z$, where $\widetilde{\gamma}_1 \in \widetilde{E}_s(Y)$ and $\widetilde{\gamma}_2 \in \widetilde{E}_s(Z)$. Define $g_{\widetilde{\gamma}_2}(x) = \int \widetilde{\gamma}_2 \,\mathrm{d}g(x)$ whenever $x \in U$. Approximating the locally Lipschitzian function $g_{\widetilde{\gamma}_2}$ by means of convolution and noting [Men12, 4.5(3)] and [Men16, 8.7], we compute

$$\begin{aligned} & \textstyle\iint \widetilde{\gamma}_2 \langle \theta(x) \otimes \widetilde{\gamma}_1, \widetilde{F}^s(x) \rangle \,\mathrm{d}g(x) \,\mathrm{d}\|V\|\, x \\ & \quad = \textstyle\int \big(\int \widetilde{\gamma} \,\mathrm{d}h(x) \big)\, \theta(x) \bullet \boldsymbol{\eta}(V, x) \,\mathrm{d}\|\delta V\|\, x \\ & \qquad - \textstyle\int \big(\int \widetilde{\gamma} \,\mathrm{d}h(x) \big)\, \mathrm{D}\,\theta(x) \bullet S \,\mathrm{d}V\,(x, S) \\ & \qquad - \textstyle\int \big(\int \widetilde{\gamma}_1 \,\mathrm{d}f(x) \big)\, \langle \theta(x), V\,\mathbf{D}\, g_{\widetilde{\gamma}_2}(x) \rangle \,\mathrm{d}\|V\|\, x. \end{aligned}$$

On the other hand, by 4.3.11,

$$\textstyle\iint \widetilde{\gamma}_1 \langle \theta(x) \otimes \widetilde{\gamma}_2, \widetilde{G}^s(x)\rangle \,\mathrm{d} f(x) \,\mathrm{d}\|V\|\, x \\ \qquad\qquad = \int \left(\int \widetilde{\gamma}_1 \,\mathrm{d} f(x)\right) \langle \theta(x), V\,\mathbf{D}\, g_{\widetilde{\gamma}_2}(x)\rangle \,\mathrm{d}\|V\|\, x.$$

Therefore, the function $\widetilde{H}^s$ satisfies

$$\textstyle\int \langle \widetilde{\psi}(x), \widetilde{H}^s(x)\rangle \,\mathrm{d}\|V\|\, x = \iint \boldsymbol{\eta}(V,x) \bullet \widetilde{\psi}(x) \,\mathrm{d} h(x) \,\mathrm{d}\|\delta V\|\, x \\ \qquad - \iint S \bullet \mathrm{D}\, \widetilde{\psi}(x) \,\mathrm{d} h(x) \,\mathrm{d} V\,(x,S)$$

where $\widetilde{\psi} \in \mathscr{D}(U, \mathbf{R}^n \otimes \widetilde{E}_s(Y \times Z))$ with $\widetilde{\psi}(x) = \theta(x) \otimes \widetilde{\gamma}$ for $x \in U$. Using mollification, we can show that the image of the canonical monomorphism $\widetilde{E}_s(Y) \otimes \widetilde{E}_s(Z) \to \widetilde{E}_s(Y \times Z)$ is dense, see [Fed69, 1.1.3, 4.1.2, 4.1.3]. By [Men16, 3.1], the previous equation of $\widetilde{H}^s$ holds for arbitrary $\widetilde{\psi} \in \mathscr{D}(U, \mathbf{R}^n \otimes \widetilde{E}_s(Y \times Z))$ and the assertion follows from 4.3.4. □

Remark 4.3.13. The condition that g is locally Lipschitzian can not be omitted. To construct a counterexample, let R_j, V_j, V, f, g, h be as in [Men16, 8.25] and let v_j be the vector in $\mathbf{R}^2$ such that $|v_j| = 1$ and $R_j = \{r v_j : r \in \mathbf{R}, r \geq 0\}$. By 4.3.5, $\boldsymbol{\delta} \circ f$ and $\boldsymbol{\delta} \circ g$ are generalized V weakly differentiable Young functions of type $\mathbf{R}$. Let $0 \leq s < \infty$ and let T^s be the distribution associated with $\boldsymbol{\delta} \circ h$. Then, we compute

$$\begin{aligned} T^s(\psi) &= \textstyle\iint S \bullet \mathrm{D}\,\psi(x) \,\mathrm{d}\boldsymbol{\delta}_{h(x)} \,\mathrm{d} V\,(x,S) \\ &= \textstyle\int v_1 \bullet \psi(0) \,\mathrm{d}\boldsymbol{\delta}_{(1,1)} + \int v_3 \bullet \psi(0) \,\mathrm{d}\boldsymbol{\delta}_{(1,0)} \\ &\quad + \textstyle\int v_4 \bullet \psi(0) \,\mathrm{d}\boldsymbol{\delta}_{(0,1)} + \int v_5 \bullet \psi(0) \,\mathrm{d}\boldsymbol{\delta}_{(1,0)} \end{aligned}$$

whenever $\psi \in \mathscr{D}(\mathbf{R}^2, \mathbf{R}^2 \otimes E_s(\mathbf{R} \times \mathbf{R}))$. Hence, $\|T^s\|(\mathbf{R}^2 \sim \{0\}) = 0$. Choose $\gamma \in E_s(\mathbf{R} \times \mathbf{R})$ and $\theta \in \mathscr{D}(\mathbf{R}^2, \mathbf{R}^2)$ such that $\operatorname{spt}\gamma \subset \mathbf{B}((1,1), 2^{-1})$, $\gamma(1,1) \neq 0$ and $\theta(0) \bullet v_1 \neq 0$, then $T^s_{(x)}(\theta(x) \otimes \gamma) \neq 0$. Therefore, $\|T^s\|$ is not absolutely continuous with respect to $\|V\|$, hence $\boldsymbol{\delta} \circ h = (\boldsymbol{\delta} \circ f) \times (\boldsymbol{\delta} \circ g)$ is not generalized V weakly differentiable.

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