

3. CONSTRUCTION OF THE INVARIANT RING

In this section we will use $Q_{50}, Q_{51}, Q_{52}, Q_{53}$ and some coefficients of the polynomial $M_4(T)$ to construct the invariant ring of five-dimensional orthogonal groups. To analysis $M_4(T)$, we must know something about $H_4(T)$. So far, all we have known about $H_4(T)$ is just its degree and leading coefficient. Hence we need more information about it. We first give some preliminary results.

Lemma 3.1. G'_3 is an irreducible polynomial in $\mathbb{F}_q[Y_0, Y_1, Y_2]$.

Proof:

$$\begin{aligned} G_3 &= F_2^q Y_3 + f(Y_0, Y_1, Y_2) \\ &= [(Y_0^{\frac{q+1}{2}} + Y_1)(Y_0^{\frac{q+1}{2}} - Y_1)]^q Y_3 + f(Y_0, Y_1, Y_2), \end{aligned}$$

for some polynomial $f(Y_0, Y_1, Y_2)$. By Lemma 2.5, $(Y_0^{\frac{q+1}{2}} + Y_1)^q$ divides G_3 . So we have

$$G_3 = (Y_0^{\frac{q+1}{2}} + Y_1)^q [(Y_0^{\frac{q+1}{2}} - Y_1)^q Y_3 + f_1(Y_0, Y_1, Y_2)].$$

That is

$$G'_3 = (Y_0^{\frac{q+1}{2}} - Y_1)^q Y_3 + f_1(Y_0, Y_1, Y_2).$$

Since G'_3 is linear in Y_3 and $Y_0^{\frac{q+1}{2}} - Y_1$ is an irreducible polynomial in $\mathbb{F}_q[Y_0, Y_1, Y_2]$, if we can show that $Y_0^{\frac{q+1}{2}} - Y_1$ does not divide $f_1(Y_0, Y_1, Y_2)$, then we are done. The coefficient of Y_2^{q+1} in G_3 is $(Y_0^q)^{\frac{q+1}{2}} + Y_1^q = (Y_0^{\frac{q+1}{2}} + Y_1)^q$. If $Y_0^{\frac{q+1}{2}} - Y_1$ divides $f_1(Y_0, Y_1, Y_2)$, then $Y_0^{\frac{q+1}{2}} - Y_1$ will divide G_3 . But it is obvious that $Y_0^{\frac{q+1}{2}} - Y_1$ does not divide $(Y_0^{\frac{q+1}{2}} + Y_1)^q$ which is the coefficient of Y_2^{q+1} in G_3 . Hence G'_3 is an irreducible polynomial in $\mathbb{F}_q[Y_0, Y_1, Y_2]$. \square

Lemma 3.2. Consider the ring homomorphism

$$\begin{aligned} \phi : \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3] &\longrightarrow \mathbb{F}_q[Q_{30}, Q_{31}, Q_{32}, Q_{33}] \\ Y_i &\mapsto Q_{3i}. \end{aligned}$$

Then $\ker\phi = \langle G'_3 \rangle$.

Proof: Since $\ker\phi$ is a prime ideal of height one in the UFD $\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3]$, it is a principal ideal generated by an irreducible polynomial.

By lemma 2.2,

$$\begin{aligned} \phi(G_3) &= 0 \\ \phi((Y_0^{\frac{q+1}{2}} + Y_1)^q G'_3) &= 0 \\ \phi((Y_0^{\frac{q+1}{2}} + Y_1)^q) \phi(G'_3) &= 0. \end{aligned}$$

Since Q_{30} and Q_{31} are algebraically independent, $\phi((Y_0^{\frac{q+1}{2}} + Y_1)^q) \neq 0$. In other words, $\phi(G'_3) = 0$. By lemma 3.1, we conclude that $\ker\phi = \langle G'_3 \rangle$. \square

Next we describe the polynomial $H_2(T)$.

- Lemma 3.3.** (a) Let R_{20} and R_{21} denote the leading coefficient and the coefficient of T^{2q} of $H_2(T)$, respectively. Then $R_{20} = G_1$ and $R_{21} = (Y_2 - Y_0^{\frac{q^2+1}{2}}) + f_{21}(Y_0, Y_1)$, for some polynomial $f_{21}(Y_0, Y_1)$.
 (b) All the coefficients of $H_2(T)$ belong to the $\mathbb{F}_q[Y_0, Y_1]$ -module generated by R_{20} and R_{21} .

Proof: (a) First, by Lemma 2.6, $R_{20} = G_1$. Next, we compute R_{21} .

$$\begin{aligned}
 H_2(T) &= (-P_2(T))^{\frac{q+1}{2}} + P_2'(T) \\
 &= \begin{vmatrix} Y_1 - T^{q+1} & Y_0 - T^2 \\ Y_0^q - T^{2q} & Y_1 - T^{q+1} \end{vmatrix}^{\frac{q+1}{2}} + \begin{vmatrix} Y_1 - T^{q+1} & Y_0^q - T^{2q} \\ Y_2 - T^{q^2+1} & Y_1^q - T^{q^2+q} \end{vmatrix} \\
 (-P_2(T))^{\frac{q+1}{2}} &= (Y_1^2 - 2Y_1T^{q+1} + T^{2q+2} - Y_0^{q+1} + Y_0T^{2q} + Y_0^qT^2 - T^{2q+2})^{\frac{q+1}{2}} \\
 &= (Y_1^2 - 2Y_1T^{q+1} - Y_0^{q+1} + Y_0T^{2q} + Y_0^qT^2)^{\frac{q+1}{2}} \\
 &\equiv (Y_0^{q+1} - 2Y_0^{\frac{q+1}{2}}T^{q+1} - Y_0^{q+1} + Y_0T^{2q} + Y_0^qT^2)^{\frac{q+1}{2}} \pmod{G_1} \\
 &\equiv [Y_0(-2Y_0^{\frac{q-1}{2}}T^{q+1} + T^{2q} + Y_0^{q-1}T^2)]^{\frac{q+1}{2}} \pmod{G_1} \\
 &\equiv [Y_0(T^q - Y_0^{\frac{q-1}{2}}T)]^{\frac{q+1}{2}} \pmod{G_1} \\
 &\equiv Y_0^{\frac{q+1}{2}}(T^q - Y_0^{\frac{q-1}{2}}T)^{q+1} \pmod{G_1} \\
 &\equiv Y_0^{\frac{q+1}{2}}(T^{q^2+q} - Y_0^{\frac{q-1}{2}}T^{q^2+1} - Y_0^{\frac{q^2-q}{2}}T^{2q} + Y_0^{\frac{q^2-1}{2}}T^{q+1}) \pmod{G_1} \\
 &\equiv Y_0^{\frac{q+1}{2}}T^{q^2+q} - Y_0^qT^{q^2+1} - Y_0^{\frac{q^2+1}{2}}T^{2q} + Y_0^{\frac{q^2+q}{2}}T^{q+1} \pmod{G_1}. \\
 P_2'(T) &= Y_1^{q+1} - Y_1T^{q^2+q} - Y_1^qT^{q+1} + T^{(q+1)^2} - Y_0^qY_2 + Y_0^qT^{q^2+1} + Y_2T^{2q} - T^{(q+1)^2} \\
 &= Y_1^{q+1} - Y_1T^{q^2+q} - Y_1^qT^{q+1} - Y_0^qY_2 + Y_0^qT^{q^2+1} + Y_2T^{2q}.
 \end{aligned}$$

Hence the coefficient of T^{2q} of $H_2(T)$ is $R_{21} = (Y_2 - Y_0^{\frac{q^2+1}{2}}) + G_1f_1(Y_0, Y_1)$, for some polynomial $f_1(Y_0, Y_1)$. Let $f_{21} = G_1f_1(Y_0, Y_1)$. Then $R_{21} = (Y_2 - Y_0^{\frac{q^2+1}{2}}) + f_{21}(Y_0, Y_1)$.

(b) Since

$$\begin{aligned}
 H_2(T) &= (-P_2(T))^{\frac{q+1}{2}} + P_2'(T) \\
 &\equiv (Y_0^{\frac{q+1}{2}} - Y_1)T^{q^2+q} + (Y_2 - Y_0^{\frac{q^2+1}{2}})T^{2q} + (Y_0^{\frac{q^2+q}{2}} - Y_1^q)T^{q+1} \\
 &\quad + (Y_1^{q+1} - Y_0^qY_2) \pmod{G_1} \\
 &\equiv (Y_2 - Y_0^{\frac{q^2+1}{2}})T^{2q} + (Y_0^{\frac{(q+1)^2}{2}} - Y_0^qY_2) \pmod{G_1} \\
 &= (Y_2 - Y_0^{\frac{q^2+1}{2}})(T^{2q} - Y_2) + G_1f(Y_0, Y_1),
 \end{aligned}$$

for some $f(Y_0, Y_1) \in \mathbb{F}_q[Y_0, Y_1]$. So all the coefficients of $H_2(T)$ are generated by R_{20} and R_{21} . \square

Remark 3.1. In the process of the above proof, we have $R_{20}|f_{21}$.

Now we can elaborate the polynomial $H_4(T)$.

Lemma 3.4. (a) Let $H_4(T) = \sum_{k=0}^N g_k(Y_0, Y_1, Y_2, Y_3, Y_4)T^k$. Then

$$g_k(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) = 0.$$

- (b) Let R_{40} , R_{41} , and R_{42} denote the leading coefficient, the coefficient of $T^{q^3+q^2}$, and the coefficient of T^{2q^2} of $H_4(T)$, respectively. Then $R_{40} = -G'_3$, $R_{41} = -R_{20}^q Y_4 + f_{41}(Y_0, Y_1, Y_2, Y_3)$, and $R_{42} = -R_{21}^q Y_4 + f_{42}(Y_0, Y_1, Y_2, Y_3)$, for some polynomials $f_{41}(Y_0, Y_1, Y_2, Y_3, Y_4)$ and $f_{42}(Y_0, Y_1, Y_2, Y_3, Y_4)$.
- (c) All the coefficients of $H_4(T)$ belong to the $\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3]$ -module generated by R_{40} , R_{41} , and R_{42} .

Proof: (a) We first note that $A(Q_{40}, Q_{41}, Q_{42}) \neq 0$, since Q_{40} , Q_{41} , and Q_{42} are algebraically independent, where A is as in Definition 2.2. Next, by Lemma 2.2, we have

$$\begin{aligned} & H_4(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}, x_4) \\ &= \left(\frac{G_4}{A^q}\right)(Q_{40}, Q_{41}, Q_{42}, Q_{43}, Q_{44}) \\ &= 0. \end{aligned}$$

On the other hand, if we write $H_4(T) = \sum_{k=0}^N g_k(Y_0, Y_1, Y_2, Y_3, Y_4)T^k$, then

$$\begin{aligned} 0 &= H_4(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}, x_4) \\ &= \sum_{k=0}^N g_k(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34})x_4^k. \end{aligned}$$

Since $g_k(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) \in \mathbb{F}_q[x_1, x_2, x_3]$, $g_k(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) = 0$.

(b) First, by Lemma 2.6, $R_{40} = -G_3$. Next, we claim that $H_4(T) = -H_2(T)^q Y_4 + f(Y_0, Y_1, Y_2, Y_3, T)$, for some polynomial $f(Y_0, Y_1, Y_2, Y_3, T)$.

Note that

$$\begin{aligned} G_4 &= -Y_4 F_3^q + g(Y_0, Y_1, Y_2, Y_3) \\ &= -Y_4 [G_2 A]^q + g(Y_0, Y_1, Y_2, Y_3), \end{aligned}$$

for some polynomial $g(Y_0, Y_1, Y_2, Y_3)$.

By Lemma 2.5,

$$\frac{G_4}{A^q} = -Y_4 G_2^q + g_1(Y_0, Y_1, Y_2, Y_3),$$

for some polynomial $g_1(Y_0, Y_1, Y_2, Y_3)$.

Hence

$$G'_4 = -G_2^q Y_4 + g_1(Y_0, Y_1, Y_2, Y_3).$$

That is

$$H_4(T) = -H_2(T)^q Y_4 + f(Y_0, Y_1, Y_2, Y_3, T),$$

for some polynomial $f(Y_0, Y_1, Y_2, Y_3)$. So we can write

$$H_4(T) = -(R_{20}T^{q^2+q} + R_{21}T^{2q} + \sum g_j(Y_0, Y_1)T^j)^q Y_4 + f(Y_0, Y_1, Y_2, Y_3, T),$$

where summation runs over all j such that $j < q^2 + q$ and $j \neq 2q$.

By Lemma 3.3, $g_j = h_{0j}R_{20} + h_{1j}R_{21}$, where $h_{0j}, h_{1j} \in \mathbb{F}_q[Y_0, Y_1], \forall j$.

That is

$$H_4(T) = -R_{20}^q Y_4 T^{q^3+q^2} - R_{21}^q Y_4 T^{2q^2} - \sum (h_{0j}R_{20} + h_{1j}R_{21})^q Y_4 T^{qj} + f(Y_0, Y_1, Y_2, Y_3, T),$$

Hence the coefficient of $T^{q^3+q^2}$, and the coefficient of T^{2q^2} are

$$R_{41} := -R_{20}^q Y_4 + f_{41}(Y_0, Y_1, Y_2, Y_3),$$

and

$$R_{42} := -R_{21}^q Y_4 + f_{42}(Y_0, Y_1, Y_2, Y_3),$$

respectively, for some polynomials $f_{41}(Y_0, Y_1, Y_2, Y_3, Y_4)$, and $f_{42}(Y_0, Y_1, Y_2, Y_3, Y_4)$.

(c) By Lemma 2.6, we can write

$$\begin{aligned} H_4(T) &= -R_{20}^q Y_4 T^{q^3+q^2} - R_{21}^q Y_4 T^{2q^2} - \sum (h_{0j}R_{20} + h_{1j}R_{21})^q Y_4 T^{qj} \\ &\quad - G'_3 T^{q^4+q^2} + f_1(Y_0, Y_1, Y_2, Y_3, T) \\ &= (R_{41} - f_{41}) T^{q^3+q^2} + (R_{42} - f_{42}) T^{2q^2} - \sum (h_{0j}R_{20} + h_{1j}R_{21})^q Y_4 T^{qj} \\ &\quad - G'_3 T^{q^4+q^2} + f_1(Y_0, Y_1, Y_2, Y_3, T) \\ &= h_0 R_{40} + h_1 R_{41} + h_2 R_{42} + f_2(Y_0, Y_1, Y_2, Y_3, T), \end{aligned}$$

for some polynomials $h_0, h_1, h_2, f_1, f_2 \in \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T]$.

By (a)

$$\begin{aligned} H_4(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}, x_4) &= h_0 \cdot c + h_1 \cdot 0 + h_2 \cdot 0 + f_2(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}, x_4) \\ G'_4(Q_{40}, Q_{41}, Q_{42}, Q_{43}, Q_{44}) &= f''(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}, x_4) \\ 0 &= \sum_{k=0}^N f_k(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) x_4^k. \end{aligned}$$

Hence $f_k(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) = 0$. That is $f_k \in \langle G'_3 \rangle = \langle R_{40} \rangle$. \square

The following Lemma is an exercise in [Atiyah, Exercise 5.12]. Here we give a direct proof.

Lemma 3.5. *Let G be a finite group of automorphisms of a ring R , and let R^G denote the subring of G -invariants, that is of all $x \in R$ such that $\sigma(x) = x$ for all $\sigma \in G$. Then R is integral over R^G .*

Proof: For all $x \in R$, consider the polynomial $f(t) = \prod_{\sigma \in G} (t - \sigma(x))$ which satisfies $f(x) = 0$. Since G is finite, all coefficients of $f(t)$ belong to R^G . We complete the proof. \square

Lemma 3.6. $M_4(T) \in \mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}(T)$.

Proof: We first claim that $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$ is the integral closure of $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}$ in $\mathbb{F}_q(x_1, x_2, x_3, x_4, x_5)$. Suppose $a \in \mathbb{F}_q(x_1, x_2, x_3, x_4, x_5)$ is integral over $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}$, then a is integral over $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$. Thus $a \in \mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$, since $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$ is integral closed. The other inclusion comes from Lemma 3.5.

Since x_5 is integral over $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}$, all of the conjugates of x_5 are integral over $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}$. Because the coefficients of $M(T)$ are symmetric polynomials in the conjugates of x_5 , they are integral over $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}$. This implies that

the coefficients of $M(T)$ lie in $\mathbb{F}_q(x_1, x_2, x_3, x_4, x_5)$, because $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$ is the integral closure of $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]^{O_5(\mathbb{F}_q)}$ in $\mathbb{F}_q(x_1, x_2, x_3, x_4, x_5)$. \square

Now we begin to prove the main results.

Theorem 3.1.

$$\begin{aligned} & \mathbb{F}_q(x_1, x_2, x_3, x_4, x_5)^{O_5(\mathbb{F}_q)} \\ = & \mathbb{F}_q[Q_{50}, Q_{51}, Q_{52}, Q_{53}, \frac{R_{41}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}, \frac{R_{42}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}]. \end{aligned}$$

Proof: For simplicity, we write $R_5 = \mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$, and

$$R_5^* = \mathbb{F}_q[Q_{50}, Q_{51}, Q_{52}, Q_{53}, \frac{R_{41}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}, \frac{R_{42}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}].$$

We prove the theorem by the following programs.

Step1: $R_5^* \subset R_5^{O_5(\mathbb{F}_q)}$.

Step1 is immediate by lemma 3.4 and Lemma 3.6.

Step2: $R_5^*[1/R_{20}(Q_{50}, Q_{51})]$ is a UFD.

Let $K(Y_0, Y_1, Y_2, Y_3) = R_{21}^q R_{41} - R_{20}^q R_{42}$. By Lemma 3.4, we get $K(Q_{30}, Q_{31}, Q_{32}, Q_{33}) = 0$. Hence $K = -G'_3 g$ for some polynomial $g(Y_0, Y_1, Y_2, Y_3)$, by Lemma 3.2.

That is

$$R_{21}^q R_{41} - R_{20}^q R_{42} = -G'_3 g = R_{40} g$$

Hence

$$R_{21}^q \left(\frac{R_{41}}{R_{40}} \right) - R_{20}^q \left(\frac{R_{42}}{R_{40}} \right) = g.$$

So we can write

$$\frac{R_{42}}{R_{40}} = \left(\frac{1}{R_{20}^q} \right) \left[R_{21}^q \left(\frac{R_{41}}{R_{40}} \right) - g \right].$$

This yields that

$$\begin{aligned} & R_5^* \left[\frac{1}{R_{20}(Q_{50}, Q_{51})} \right] \\ = & \mathbb{F}_q[Q_{50}, Q_{51}, Q_{52}, Q_{53}, \frac{R_{41}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}, \frac{R_{42}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}, \\ & \frac{1}{R_{20}(Q_{50}, Q_{51})}] \\ = & \mathbb{F}_q[Q_{50}, Q_{51}, Q_{52}, Q_{53}, \frac{R_{41}(Q_{50}, Q_{51}, Q_{52}, Q_{53}, Q_{54})}{R_{40}(Q_{50}, Q_{51}, Q_{52}, Q_{53})}] \left[\frac{1}{R_{20}(Q_{50}, Q_{51})} \right]. \end{aligned}$$

Since the last ring is a localization of a polynomial ring, it is a UFD.

Step3: $R_{20}(Q_{50}, Q_{51})$ is a prime in R_5^* .

Since $Q_{50}, Q_{51}, Q_{52}, Q_{53}$, and Q_{54} are algebraically independent, thus as a ring, we have

$$R_5^* \cong \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, \frac{R_{41}(Y_0, Y_1, Y_2, Y_3, Y_4)}{R_{40}(Y_0, Y_1, Y_2, Y_3)}, \frac{R_{42}(Y_0, Y_1, Y_2, Y_3, Y_4)}{R_{40}(Y_0, Y_1, Y_2, Y_3)}].$$

Consider the ring homomorphism

$$\begin{aligned} \varphi : \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2] &\longrightarrow \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, \frac{R_{41}(Y_0, Y_1, Y_2, Y_3, Y_4)}{R_{40}(Y_0, Y_1, Y_2, Y_3)}, \frac{R_{42}(Y_0, Y_1, Y_2, Y_3, Y_4)}{R_{40}(Y_0, Y_1, Y_2, Y_3)}] \\ Y_i &\mapsto Y_i, \\ T_i &\mapsto \frac{R_{4i}}{R_{40}}. \end{aligned}$$

Then $\ker\langle\varphi\rangle$ is a prime ideal of height one. Let $L = R_{21}^q T_1 - R_{20}^q T_2 - g$. In the process of Step2, we know $\varphi(L) = 0$. Note that L is linear in T_1 , and R_{20} and R_{21} are both irreducible polynomials in $\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]$. Since R_{21} does not divide R_{20} , L is an irreducible polynomial in $\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]$. That is $\langle L \rangle = \ker\varphi$. Thus $R_5^* \cong \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]/\langle L \rangle$.

If we can show $\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]/\langle R_{20}, L \rangle$ is an integral domain, then R_{20} is a prime in R_5^* . Note that

$$\begin{aligned} g &= R_{21}^q \left(\frac{R_{41}}{R_{40}} \right) - R_{20}^q \left(\frac{R_{42}}{R_{40}} \right) \\ &\equiv R_{21}^q \left(\frac{R_{41}}{R_{40}} \right) \pmod{R_{20}}. \end{aligned}$$

Let

$$\tilde{L} = R_{21}^q T_1 - R_{21}^q \left(\frac{R_{41}}{R_{40}} \right).$$

Then

$$\begin{aligned} L &= R_{21}^q T_1 - R_{20}^q T_2 - g \\ &\equiv R_{21}^q T_1 - R_{21}^q \left(\frac{R_{41}}{R_{40}} \right) \pmod{R_{20}} \\ &\equiv \tilde{L}. \end{aligned}$$

Since

$$\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]/\langle R_{20}, L \rangle \cong D/\tilde{L}, \text{ where } D := \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]/\langle R_{20} \rangle.$$

and since $R_{21} \equiv Y_2 - Y_0^{\frac{q^2+1}{2}}$ is an irreducible polynomial in D by Remark 3.1,

$$\begin{aligned} &\mathbb{F}_q[Y_0, Y_1, Y_2, Y_3, T_1, T_2]/\langle R_{20}, L \rangle \text{ is an integral domain} \\ \Leftrightarrow &\tilde{L} \text{ is prime in } D. \\ \Leftrightarrow &R_{21} \text{ does not divide } R_{21}^q \left(\frac{R_{41}}{R_{40}} \right) \text{ in } D. \end{aligned}$$

We first compute F_3 and F_3' modulo R_{20} to obtain R_{40} in D .

$$\begin{aligned}
F_3 &\equiv \begin{vmatrix} Y_0 & Y_0^{\frac{q+1}{2}} & Y_2 \\ Y_0^{\frac{q+1}{2}} & Y_0^q & Y_0^{\frac{q^2+q}{2}} \\ Y_2 & Y_0^{\frac{q^2+q}{2}} & Y_0^{q^2} \end{vmatrix} \\
&\equiv Y_0^{q^2+q+1} + 2Y_0^{\frac{q^2+2q+1}{2}}Y_2 - Y_0^qY_2 - Y_0^{q^2+q+1} - Y_0^{q^2+q+1} \\
&\equiv -Y_0^q(Y_2^2 - 2Y_0^{\frac{q^2+1}{2}} + Y_0^{q^2+1}) \\
&\equiv -Y_0^q(Y_2 - Y_0^{\frac{q^2+1}{2}})^2 \\
&\equiv -Y_0^qR_{21}^2.
\end{aligned}$$

$$\begin{aligned}
F'_3 &\equiv \begin{vmatrix} Y_0^{\frac{q+1}{2}} & Y_0^q & Y_0^{\frac{q^2+q}{2}} \\ Y_2 & Y_0^{\frac{q^2+q}{2}} & Y_0^{q^2} \\ Y_3 & Y_2^q & Y_0^{\frac{q^3+q^2}{2}} \end{vmatrix} \\
&\equiv Y_0^{\frac{q^3+2q^2+2q+1}{2}} + Y_0^{q^2+q}Y_3 + Y_0^{\frac{q^2+1}{2}}Y_2^{q+1} - Y_0^{q^2+q}Y_3 - Y_0^{\frac{q^3+q^2+2q}{2}}Y_2 - Y_0^{\frac{2q^2+q+1}{2}}Y_2^q \\
&\equiv Y_0^{\frac{q^2+1}{2}}Y_2(Y_2 - Y_0^{\frac{q^2+1}{2}})^q - Y_0^{\frac{2q^2+q+1}{2}}(Y_2 - Y_0^{\frac{q^2+1}{2}}) \\
&\equiv (Y_2 - Y_0^{\frac{q^2+1}{2}})^q Y_0^{\frac{q^2+1}{2}}(Y_2 - Y_0^{\frac{q^2+1}{2}}) \\
&\equiv Y_0^{\frac{q^2+1}{2}}R_{21}^{q+1}.
\end{aligned}$$

We get

$$R_{40} = G'_3 = \frac{(-F_3)^{\frac{q+1}{2}} + F'_3}{(Y_0^{\frac{q+1}{2}} + Y_1)^q} \equiv \frac{Y_0^{\frac{q^2+q}{2}}R_{21}^{q+1} + Y_0^{\frac{q^2+1}{2}}R_{21}^{q+1}}{Y_0^{\frac{q+1}{2}} + Y_0^{\frac{q+1}{2}}} \equiv R_{21}^{q+1} \pmod{R_{20}}.$$

Thus \tilde{L} is an irreducible in $D \Leftrightarrow R_{21}^2$ does not divide R_{41} in D .

Let

$$N_1 = \begin{vmatrix} Y_1 & Y_0^q & Y_1^q \\ Y_2 & Y_1^q & Y_0^{q^2} \\ T_1 & Y_3^q & Y_2^{q^2} \end{vmatrix}, \quad N_2 = \begin{vmatrix} Y_0 & Y_1 & Y_3 \\ Y_1 & Y_0^q & Y_2^q \\ Y_3 & Y_2^q & Y_0^{q^3} \end{vmatrix}, \quad N_3 = \begin{vmatrix} Y_0 & Y_1 & Y_2 \\ Y_1 & Y_0^q & Y_1^q \\ Y_3 & Y_2^q & Y_1^{q^2} \end{vmatrix}.$$

Then

$$\begin{aligned}
N_1(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) &= \begin{vmatrix} x_1^q & -x_2^q & x_3^q \\ x_1^{q^2} & -x_2^{q^2} & x_3^{q^2} \\ x_1^{q^4} & -x_2^{q^4} & x_3^{q^4} \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} \\ x_2 & x_2^q & x_2^{q^2} \\ x_3 & x_3^q & x_3^{q^2} \end{vmatrix}; \\
N_2(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) &= \begin{vmatrix} x_1 & -x_2 & x_3 \\ x_1^q & -x_2^q & x_3^q \\ x_1^{q^3} & -x_2^{q^3} & x_3^{q^3} \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1^q & x_1^{q^3} \\ x_2 & x_2^q & x_2^{q^3} \\ x_3 & x_3^q & x_3^{q^3} \end{vmatrix}; \\
N_3(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) &= \begin{vmatrix} x_1 & -x_2 & x_3 \\ x_1^q & -x_2^q & x_3^q \\ x_1^{q^3} & -x_2^{q^3} & x_3^{q^3} \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} \\ x_2 & x_2^q & x_2^{q^2} \\ x_3 & x_3^q & x_3^{q^2} \end{vmatrix}.
\end{aligned}$$

Let $S = N_1 - (-N_2)^{\frac{q-1}{2}} N_3$. Then $S(Q_{30}, Q_{31}, Q_{32}, Q_{33}, Q_{34}) = 0$. Since

$$\begin{aligned}
S &= N_1 - (-N_2)^{\frac{q-1}{2}} N_3 \\
&= (Y_0^{q^2+q} - Y_1^{2q})Y_4 + h(Y_0, Y_1, Y_2, Y_3, Y_4) \\
&= (Y_0^{\frac{q+1}{2}} - Y_1)^q (Y_0^{\frac{q+1}{2}} + Y_1)^q Y_4 + h(Y_0, Y_1, Y_2, Y_3, Y_4),
\end{aligned}$$

for some polynomial $h(Y_0, Y_1, Y_2, Y_3, Y_4)$, so $(Y_0^{\frac{q+1}{2}} + Y_1)^q R_{41} + S \in \mathbb{F}_q[Y_0, Y_1, Y_2, Y_3]$. By Lemma 3.2, we have $(Y_0^{\frac{q+1}{2}} + Y_1)^q R_{41} + S = G'_3 f(Y_1, Y_2, Y_3)$, for some polynomial $f(Y_1, Y_2, Y_3)$. That is

$$R_{41} = \frac{G'_3 f - S}{(Y_0^{\frac{q+1}{2}} + Y_1)^q} = \frac{R_{40} f - [N_1 - (-N_2)^{\frac{q-1}{2}} N_3]}{(Y_0^{\frac{q+1}{2}} + Y_1)^q} \text{ in } D.$$

All we have to do is to compute N_1 and N_3 in D .

$$\begin{aligned}
N_1 &= \begin{vmatrix} Y_0^{\frac{q+1}{2}} & Y_0^q & Y_0^{\frac{q^2+q}{2}} \\ Y_2 & Y_0^{\frac{q^2+q}{2}} & Y_0^{q^2} \\ T_1 & Y_3^q & Y_2^{q^2} \end{vmatrix} = Y_0^{\frac{q+1}{2}} (Y_0^{\frac{q^2+q}{2}} Y_2^{q^2} - Y_0^{q^2} Y_3^q) - Y_2 (Y_0^q Y_2^{q^2} - Y_0^{\frac{q^2+q}{2}} Y_3^q) \\
&= Y_0^q Y_2^{q^2} (Y_0^{\frac{q^2+1}{2}} - Y_2) - Y_0^{\frac{q^2+q}{2}} Y_3^q (Y_0^{\frac{q^2+1}{2}} - Y_2) \\
&= R_{21} (Y_0^q Y_2^{q^2} - Y_0^{\frac{q^2+q}{2}} Y_3^q);
\end{aligned}$$

$$\begin{aligned}
N_3 &= \begin{vmatrix} Y_0 & Y_0^{\frac{q+1}{2}} & Y_2 \\ Y_0^{\frac{q+1}{2}} & Y_0^q & Y_0^{\frac{q^2+q}{2}} \\ Y_3 & Y_2^q & Y_0^{\frac{q^3+q^2}{2}} \end{vmatrix} = Y_2 (Y_0^{\frac{q+1}{2}} Y_2^q - Y_0^q Y_3) - Y_0^{\frac{q^2+q}{2}} (Y_0 Y_2^q - Y_0^{\frac{q+1}{2}} Y_3) \\
&= Y_0^{\frac{q+1}{2}} Y_2^q (Y_2 - Y_0^{\frac{q^2+1}{2}}) + Y_0^q Y_3 (Y_0^{\frac{q^2+1}{2}} - Y_2) \\
&= R_{21} (Y_0^{\frac{q+1}{2}} Y_2^q - Y_0^q Y_3);
\end{aligned}$$

$$\begin{aligned}
R_{41} &= \frac{G'_3 f - S}{(Y_0^{\frac{q+1}{2}} + Y_1)^q} \\
&= \frac{R_{40} f - [N_1 - (-N_2)^{\frac{q-1}{2}} N_3]}{(Y_0^{\frac{q+1}{2}} + Y_1)^q} \\
&= \frac{R_{21}^{q+1} f + R_{21} (Y_0^q Y_2^{q^2} - Y_0^{\frac{q^2+q}{2}} Y_3^q) + (-N_2)^{\frac{q-1}{2}} R_{21} (Y_0^{\frac{q+1}{2}} Y_2^q - Y_0^q Y_3)}{(Y_0^{\frac{q+1}{2}} + Y_1)^q}.
\end{aligned}$$

The coefficient of Y_3^q in $(Y_0^q Y_2^{q^2} - Y_0^{\frac{q^2+q}{2}} Y_3^q) + (-N_2)^{\frac{q-1}{2}} (Y_0^{\frac{q+1}{2}} Y_2^q - Y_0^q Y_3)$ is

$$\begin{aligned}
& -Y_0^{\frac{q^2+q}{2}} + (Y_0^q)^{\frac{q-1}{2}} (-Y_0^q) \\
&= -Y_0^{\frac{q^2+q}{2}} + (Y_0^q)^{\frac{q-1}{2}} (-Y_0^q) \\
&= -2Y_0^{\frac{q^2+q}{2}}.
\end{aligned}$$

Since R_{21} does not divide $-2Y_0^{\frac{q^2+q}{2}}$, R_{21} does not divide R_{41} in D . Thus $R_{20}(Q_{50}, Q_{51})$ is a prime in R_5^* .

Step4: R_5^* is a UFD. (Hence R_5^* is integrally closed.)

Since $R_{20}(Q_{50}, Q_{51})$ is a prime in R_5^* . By Nagata's lemma(Theorem 1.6), R_5^* is a UFD.

Step5: R_5 is integral over R_5^* .

Consider the series of subfields

$$\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5] \supseteq R_5^*[x_5] \supseteq R_4^*[x_5].$$

By Remark1.1, $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$ is integral over $R_4^*[x_5]$. So $\mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$ is integral over $R_5^*[x_5]$. On the other hand, by Lemma 2.7 and Lemma 3.6, $R_5^*[x_5]$ is integral over R_5^* . That is R_5 is integral over R_5^* .

R_5 is integral over R_5^* , so $R_5^{O_5(\mathbb{F}_q)}$ is integral over R_5^* . It is easy to see that R_5^* and $R_5^{O_5(\mathbb{F}_q)}$ have the same quotient field. Since R_5^* is integrally closed, we have $R_5^* = R_5^{O_5(\mathbb{F}_q)}$. \square

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