Chapter 5 C_0-evolution System and the Conditional Stability for the Solutions of Abstract Semilinear Differential Equations

§ 5-1 Main Results

In the preceding chapter, we get sufficient conditions to ensure that the zero solution to the abstract semilinear equation (4.1) is conditionally stable and conditionally asymptotically stable. In that chapter, we considered the linear parts and forcing term function of the equation together. However, if the equation can be linearized, we may approach the desired conclusion by some perturbation theory of linear operators. We consider the asymptotic behavior of some solutions of the abstract semilinear initial value problem:

$$\begin{cases} \frac{d}{dt}u(t) = A(t)u(t) + f(t, u(t)) \\ u(0) = \xi_0 \end{cases} \tag{5.1}$$

where the family of operators $\{A(t): t \geq 0\}$ generates a non-trivial C_0-evolution system $\{U(t, s): 0 \leq s \leq t < \infty\}$ on a Banach space X, and the forcing term function $f: [0, \infty) \times X \to X$ satisfies the following conditions:

(F1) $f(t,x)$ is continuous in $t \in [0,\infty)$ for each fixed $x \in X$.

(F2) f is locally Lipschitz continuous respect to x in X with Lipschitz constant γ, that is,

$$|f(t,x) - f(t,y)| \leq \gamma |x - y| \qquad \text{for all } t \geq 0 \text{ and } |x|, |y| \leq \alpha.$$

(F3) $f(t,0) = 0$ for all $t \geq 0$.

Here we assume that there exists none trivial supplementary projections P_1, P_2 and P_3 on the Banach space X such that $P_i X = X_i$ for i=1, 2, 3. Where the dimensions of X_1 and X_2 are finite and the C_0-evolution system $\{U(t,s): 0 \leq s \leq t < \infty\}$ satisfies the following conditions:

(A1) $\{U(t,s):0\le s\le t<\infty\}$ restricted on X_1 and X_2 are total evolution system (here $U(t,s)x$ is defined by $U(t,s)x=U(s,t)^{-1}x$ for all $t<s$, $x\in X_1$ and $x\in X_2$).

(A2) $U(t,s)P_j=P_jU(t,s)$ for all $0\le s\le t<\infty$ and j=1, 2, 3.

(A3) $\int_0^t\|U(t,\tau)P_3\|d\tau+\int_t^\infty\|U(t,\tau)P_1\|d\tau\le K$ for all $0\le t<\infty$.

(A4) $\|U(t,s)P_2\|\le L_2$ for all $0\le s,\ t<\infty$.

Furthermore, the forcing term function f satisfies the condition (F4):

(F4) $\int_0^\infty|P_2f(\tau,\varphi(\tau))-P_2f(\tau,\phi(\tau))|d\tau\le\gamma_2\|\varphi-\phi\|_\infty$ for all φ, $\phi\in D$, where $D=\{\varphi\in C([0,\infty);X):\|\varphi\|_\infty\le\alpha\}$, $\alpha>0$ and $\|\cdot\|_\infty$ denotes supremum norm on $C([0,\infty);X)$.

Under the above notations and assumptions, we have following results:

Lemma 5.1 Suppose that the C_0-evolution system $\{U(t,s):0\le s\le t<\infty\}$ satisfies conditions (A1)~(A4) and the function $f:[0,\infty)\times X\to X$ satisfies conditions (F1)~(F4). Let $\xi_3\in X_3$ and the operator $G:D\to C([0,\infty);X)$ is defined by

$$
\begin{aligned}
(G\varphi)(t)=U(t,0)\xi_3&+\int_0^tU(t,\tau)P_3f(\tau,\varphi(\tau))d\tau\\
&-\int_t^\infty U(t,\tau)P_2f(\tau,\varphi(\tau))d\tau\\
&-\int_t^\infty U(t,\tau)P_1f(\tau,\varphi(\tau))d\tau
\end{aligned}
$$

for all $\varphi\in D$, then G is well-defined and

$$\|G\varphi-G\phi\|_\infty\le(\gamma K+\gamma_2L_2)\|\varphi-\phi\|_\infty\qquad\text{for any }\phi,\ \varphi\in D,$$

where γ, K, γ_2, L_2 are the constants in (A3), (A4), (F2) and (F4).

Proof. From the conditions (A1), (A3), (A4), (F3) and (F4), for any $\varepsilon>0$, $t_1>t_2\ge0$ and $\varphi\in D$, we have

$$
\begin{aligned}
&\left|G\varphi(t_1)-G\varphi(t_2)\right| \\
&=\left|\{U(t_1,0)\xi_3-U(t_2,0)\xi_3\}+\int_0^{t_1}U(t_1,\tau)P_3f(\tau,\varphi(\tau))d\tau\right. \\
&\qquad -\int_{t_1}^{\infty}U(t_1,\tau)P_2f(\tau,\varphi(\tau))d\tau-\int_{t_1}^{\infty}U(t_1,\tau)P_1f(\tau,\varphi(\tau))d\tau \\
&\qquad -\int_0^{t_2}U(t_2,\tau)P_3f(\tau,\varphi(\tau))d\tau \\
&\qquad \left.+\int_{t_2}^{\infty}U(t_2,\tau)P_2f(\tau,\varphi(\tau))d\tau+\int_{t_2}^{\infty}U(t_2,\tau)P_1f(\tau,\varphi(\tau))d\tau\right| \\
&\le\left|U(t_1,0)\xi_3-U(t_2,0)\xi_3\right|+\int_0^{t_2}\left|\{U(t_1,\tau)-U(t_2,\tau)\}P_3f(\tau,\varphi(\tau))\right|d\tau \\
&\qquad +\int_{t_2}^{t_1}\left|U(t_1,\tau)P_3f(\tau,\varphi(\tau))\right|d\tau+\int_{t_2}^{t_1}\left|U(t_2,\tau)P_2f(\tau,\varphi(\tau))\right|d\tau \\
&\qquad +\int_{t_2}^{t_1}\left|U(t_2,\tau)P_1f(\tau,\varphi(\tau))\right|d\tau \\
&\qquad +\int_{t_1}^{\infty}\left|\{U(t_1,\tau)-U(t_2,\tau)\}(P_1+P_2)f(\tau,\varphi(\tau))\right|d\tau .
\end{aligned}
$$

Since $\left|f(\tau,\varphi(\tau))\right|\le\gamma\left|\varphi(\tau)\right|\le\gamma\alpha$ on the interval $[0,\infty)$,

$$
\int_{t_1}^{\infty}\left\|U(t_1,\tau)P_1\right\|d\tau\le K \quad\text{and}\quad \int_{t_2}^{\infty}\left\|U(t_2,\tau)P_1\right\|d\tau\le K,
$$

there exists a constant $T_1>t_1$ such that

$$
\begin{aligned}
&\int_T^{\infty}\left|U(t_1,\tau)P_1f(\tau,\varphi(\tau))\right|d\tau+\int_T^{\infty}\left|U(t_2,\tau)P_1f(\tau,\varphi(\tau))\right|d\tau \\
&\le\int_T^{\infty}\left\|U(t_1,\tau)P_1\right\|\left|f(\tau,\varphi(\tau))\right|d\tau+\int_T^{\infty}\left\|U(t_2,\tau)P_1\right\|\left|f(\tau,\varphi(\tau))\right|d\tau \\
&\le\gamma\alpha\left(\int_T^{\infty}\left\|U(t_1,\tau)P_1\right\|d\tau+\int_T^{\infty}\left\|U(t_2,\tau)P_1\right\|d\tau\right) \\
&<\varepsilon
\end{aligned}
$$

for any $T\ge T_1$. From the condition (A4) and (F4), we obtain that $\left\|U(t,\tau)P_2\right\|\le L_2$ for all $0\le\tau,\ t<\infty$ and $\int_0^{\infty}\left|P_2f(\tau,\varphi(\tau))\right|d\tau\le\gamma_2\left\|\varphi\right\|_\infty\le\gamma_2\alpha<\infty$. Thus there exists a constant $T_2>t_1$ such that

$$
\begin{aligned}
&\int_T^{\infty}\left|U(t_1,\tau)P_2f(\tau,\varphi(\tau))\right|d\tau+\int_T^{\infty}\left|U(t_2,\tau)P_2f(\tau,\varphi(\tau))\right|d\tau \\
&\le\int_T^{\infty}\left\|U(t_1,\tau)P_2\right\|\left|P_2f(\tau,\varphi(\tau))\right|d\tau+\int_T^{\infty}\left\|U(t_2,\tau)P_2\right\|\left|P_2f(\tau,\varphi(\tau))\right|d\tau
\end{aligned}
$$

$$\le 2L_2\int_T^{\infty}\left|P_2 f(\tau,\varphi(\tau))\right|d\tau$$

$$< \varepsilon$$

for any $T \ge T_2$. Let $T_0 = \max\{T_1, T_2\}$, then

$$\int_{T_0}^{\infty}\left|U(t_1,\tau)P_1 f(\tau,\varphi(\tau))\right|d\tau + \int_{T_0}^{\infty}\left|U(t_2,\tau)P_1 f(\tau,\varphi(\tau))\right|d\tau$$

$$+\int_{T_0}^{\infty}\left|U(t_1,\tau)P_2 f(\tau,\varphi(\tau))\right|d\tau + \int_{T_0}^{\infty}\left|U(t_2,\tau)P_2 f(\tau,\varphi(\tau))\right|d\tau < 2\varepsilon$$

and

$$\left|G\varphi(t_1) - G\varphi(t_2)\right|$$

$$\le \left|U(t_1,0)\xi_3 - U(t_2,0)\xi_3\right| + \int_0^{t_2}\left|\{U(t_1,\tau)-U(t_2,\tau)\}P_3 f(\tau,\varphi(\tau))\right|d\tau$$

$$+\int_{t_2}^{t_1}\left|U(t_1,\tau)P_3 f(\tau,\varphi(\tau))\right|d\tau + \int_{t_2}^{t_1}\left|U(t_2,\tau)P_2 f(\tau,\varphi(\tau))\right|d\tau$$

$$+\int_{t_2}^{t_1}\left|U(t_2,\tau)P_1 f(\tau,\varphi(\tau))\right|d\tau$$

$$+\int_{t_1}^{T_0}\left|\{U(t_1,\tau)-U(t_2,\tau)\}(P_1+P_2) f(\tau,\varphi(\tau))\right|d\tau + 2\varepsilon .$$

Since the function $t \mapsto U(t,s)\xi_3$ is continuous on $0 \le s \le t < \infty$, there exists a constant $\delta_1 > 0$ such that

$$\left|U(t_1,0)\xi_3 - U(t_2,0)\xi_3\right| < \varepsilon \qquad \text{for all } \left|t_1 - t_2\right| < \delta_1 .$$

From the facts that functions $\tau \mapsto \left|U(t_1,\tau)P_3 f(\tau,\varphi(\tau))\right|$, $\tau \mapsto \left|U(t_2,\tau)P_i f(\tau,\varphi(\tau))\right|$ are continuous on the compact interval $[t_2,t_1]$ for each $i = 1,\ 2$, one obtain that they are bounded on $[t_2,t_1]$ and there exists a constant $\delta_2 > 0$ such that

$$\int_{t_2}^{t_1}\left|U(t_1,\tau)P_3 f(\tau,\varphi(\tau))\right|d\tau + \int_{t_2}^{t_1}\left|U(t_2,\tau)P_2 f(\tau,\varphi(\tau))\right|d\tau$$

$$+\int_{t_2}^{t_1}\left|U(t_2,\tau)P_1 f(\tau,\varphi(\tau))\right|d\tau < \varepsilon$$

for all $\left|t_1 - t_2\right| < \delta_2$. On the other hand, since the mappings

$$(t,\tau) \mapsto U(t,\tau)P_3 f(\tau,\varphi(\tau))$$

and

$$(t,\tau) \mapsto U(t,\tau)(P_1+P_2) f(\tau,\varphi(\tau))$$

are uniformly continuous on compact sets $\{(t,\tau): 0 \le \tau \le t \le T_0\}$ and

$\{(t,\tau):0\le t\le\tau\le T_0\}$ respectively, there exists a constant $\delta_3>0$ such that

$$\left|U(t_1,\tau)P_3f(\tau,\varphi(\tau))-U(t_2,\tau)P_3f(\tau,\varphi(\tau))\right|<\varepsilon T_0^{-1}$$

for all $|t_1-t_2|<\delta_3$, $0\le\tau\le t_2\le T_0$ and

$$\left|\{U(t_1,\tau)-U(t_2,\tau)\}(P_1+P_2)f(\tau,\varphi(\tau))\right|<\varepsilon T_0^{-1}$$

for all $|t_1-t_2|<\delta_3$, $0\le t_1\le\tau\le T_0$. Let $\delta=\min\{\delta_1,\delta_2,\delta_3\}$, then

$$\begin{aligned}|G\varphi(t_1)-G\varphi(t_2)| &\le\varepsilon+(t_2-s)\varepsilon T_0^{-1}+(T_0-t_1)\varepsilon T_0^{-1}+2\varepsilon\\ &\le 6\varepsilon \qquad \text{for all } 0\le t_2\le t_1\le t_2+\delta .\end{aligned}$$

Hence, G is well-defined and $G\varphi\in C([0,\infty);X)$ for all $\varphi\in D$. Moreover,

$$\begin{aligned}&\|G\varphi-G\phi\|_\infty\\
&\le\sup_{t\ge0}\Big\{\int_0^t\left|U(t,\tau)P_3\left(f(\tau,\varphi(\tau))-f(\tau,\phi(\tau))\right)\right|d\tau\\
&\qquad+\int_t^\infty\left|U(t,\tau)P_2\left(P_2f(\tau,\varphi(\tau))-P_2f(\tau,\phi(\tau))\right)\right|d\tau\\
&\qquad\qquad+\int_t^\infty\left|U(t,\tau)P_1\left(f(\tau,\varphi(\tau))-f(\tau,\phi(\tau))\right)\right|d\tau\Big\}\\
&\le\sup_{t\ge0}\Big\{\int_0^t\|U(t,\tau)P_3\|\gamma|\varphi(\tau)-\phi(\tau)|d\tau+\int_t^\infty\|U(t,\tau)P_1\|\gamma|\varphi(\tau)-\phi(\tau)|d\tau\Big\}\\
&\qquad+\sup_{t\ge0}\int_t^\infty\|U(t,\tau)P_2\|\left|\left(P_2f(\tau,\varphi(\tau))-P_2f(\tau,\phi(\tau))\right)\right|d\tau\\
&\le\gamma\|\varphi-\phi\|_\infty\sup_{t\ge0}\Big\{\int_0^t\|U(t,\tau)P_3\|d\tau+\int_t^\infty\|U(t,\tau)P_1\|d\tau\Big\}\\
&\qquad+L_2\sup_{t\ge0}\int_0^\infty\left|P_2f(\tau,\varphi(\tau))-P_2f(\tau,\phi(\tau))\right|d\tau\\
&\le\gamma K\|\varphi-\phi\|_\infty+\gamma_2L_2\|\varphi-\phi\|_\infty\\
&\le(\gamma K+\gamma_2L_2)\|\varphi-\phi\|_\infty\end{aligned}$$

for all ϕ, $\varphi\in D$. This lemma is proved now.

Lemma 5.2. Suppose that the C_0-evolution system $\{U(t,s):0\le s\le t<\infty\}$ satisfies conditions (A1)~(A4). Then $\lim_{t\to\infty}\|U(t,s)P_3\|=0$ for all $s\ge0$, and there is a constant $L_3>0$ such that $\|U(t,0)P_3\|\le L_3$ for all $t\ge0$.

Furthermore, if the function $f:[0,\infty)\times X \to X$ satisfies conditions (F1)~(F4) and the constants K, L_2, γ and γ_2 in (A3), (A4), (F2) and (F4) satisfy $\gamma K+\gamma_2 L_2<1$, then for any $\xi_3 \in P_3X$ with $|\xi_3|<(1-\gamma K-\gamma_2 L_2)\alpha L_3^{-1}$, the operator G is a contraction mapping from D into itself.

Proof. From the condition (A3), we obtain that $\int_0^t \|U(t,\tau)P_3\| d\tau \le K$ for all $t \ge 0$. For any fixed $s \ge 0$, let

$$\varphi(t)=\|U(t,s)P_3\|^{-1} \qquad \text{for all } t \ge s.$$

Then for any fixed $\xi \in X$ and $t \ge s \ge 0$,

$$\begin{aligned}
\left|\left(\int_s^t \varphi(\tau)d\tau\right)U(t,s)P_3\xi\right| &= \left|\int_s^t \varphi(\tau)U(t,s)P_3\xi d\tau\right| \\
&= \left|\int_s^t \varphi(\tau)U(t,\tau)P_3U(\tau,s)P_3\xi d\tau\right| \\
&\le \int_s^t \left|\varphi(\tau)U(t,\tau)P_3U(\tau,s)P_3\xi\right| d\tau \\
&\le \int_s^t \varphi(\tau)\|U(t,\tau)P_3\|\|U(\tau,s)P_3\||\xi| d\tau \\
&= |\xi|\int_s^t \varphi(\tau)\|U(\tau,s)P_3\|\|U(t,\tau)P_3\| d\tau \\
&= |\xi|\int_s^t \|U(t,\tau)P_3\| d\tau \\
&\le |\xi|\int_0^t \|U(t,\tau)P_3\| d\tau \\
&= K|\xi|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|U(t,s)P_3\|\left(\int_s^t \varphi(\tau)d\tau\right) &= \left\|\left(\int_s^t \varphi(\tau)d\tau\right)U(t,s)P_3\right\| \\
&\le K
\end{aligned}$$

for all $t \ge s \ge 0$ and

$$\varphi(t)^{-1}\int_s^t \varphi(\tau)d\tau \le K \qquad \text{for all } t \ge s \ge 0.$$

Let $\Psi(t)=\int_s^t \varphi(\tau)d\tau$ for all $t \ge s \ge 0$. Then

$$\Psi'(t)=\varphi(t)\ge \frac{1}{K}\int_s^t \varphi(\tau)d\tau=\frac{1}{K}\Psi(t),$$

and hence $\Psi'(t)\Psi(t)^{-1} \geq K^{-1}$ for all $t \geq s \geq 0$. This shows that for any fixed $t_0 > s \geq 0$,

$$\Psi(t) \geq \Psi(t_0)\exp\{K^{-1}(t-t_0)\} \qquad \text{for all } t \geq t_0.$$

and

$$\begin{aligned}\|U(t,s)P_3\| &= \varphi(t)^{-1} \\ &\leq K\Psi(t)^{-1} \\ &\leq K\Psi(t_0)^{-1}\exp\{-K^{-1}(t-t_0)\} \\ &\leq \{K\Psi(t_0)^{-1}\exp(K^{-1}t_0)\}\exp(-K^{-1}t)\end{aligned}$$

for all $t \geq t_0$. Therefore, $\lim_{t\to\infty}\|U(t,s)P_3\| = 0$ for all $s \geq 0$ and

$$\begin{aligned}\|U(t,0)P_3\| &\leq \{K\Psi(1)^{-1}\exp(K^{-1})\}\exp(-K^{-1}t) \\ &\leq K\Psi(1)^{-1}\exp(K^{-1})\end{aligned}$$

for all $t \geq 1$. With a similar proof as that for Lemma 2.1, there exists a constant $M_1 > 0$ such that $\|U(t,0)P_3\| \leq M_1$ for all $t \in [0,1]$. Let

$$L_3 = \max\{M_1, K\Psi(1)^{-1}\exp(K^{-1})\}.$$

Then $\|U(t,0)P_3\| \leq L_3$ for all $t \geq 0$.

If $\gamma K + \gamma_2 L_2 < 1$ and $\xi_3 \in P_3X$ with $|\xi_3| < (1-\gamma K - \gamma_2 L_2)\alpha L_3^{-1}$, then for any $\varphi \in D$,

$$\begin{aligned}&\|C\varphi\|_\infty \\ &= \sup_{t\geq 0}\Big|U(t,0)\xi_3 + \int_0^t U(t,\tau)P_3 f(\tau,\varphi(\tau))d\tau \\ &\qquad -\int_t^\infty U(t,\tau)P_2 f(\tau,\varphi(\tau))d\tau - \int_t^\infty U(t,\tau)P_1 f(\tau,\varphi(\tau))d\tau\Big| \\ &\leq \sup_{t\geq 0}|U(t,0)P_3\xi_3| + \sup_{t\geq 0}\int_t^\infty \|U(t,\tau)P_2\|\,|P_2 f(\tau,\varphi(\tau))|\,d\tau \\ &\qquad + \sup_{t\geq s}\Big(\int_0^t \|U(t,\tau)P_3\|\,|f(\tau,\varphi(\tau))|\,d\tau \\ &\qquad\qquad + \int_t^\infty \|U(t,\tau)P_1\|\,|f(\tau,\varphi(\tau))|\,d\tau\Big)\end{aligned}$$

$$\leq \sup_{t\geq 0}\|U(t,0)P_3\||\xi_3| + \sup_{t\geq 0}\int_t^\infty \|U(t,\tau)P_2\||P_2 f(\tau,\varphi(\tau))|d\tau$$

$$+\left(\int_0^t \|U(t,\tau)P_3\|\gamma|\varphi(\tau)|d\tau + \int_t^\infty \|U(t,\tau)P_1\|\gamma|\varphi(\tau)|d\tau\right)$$

$$\leq L_3|\xi_3| + L_2\int_0^\infty |P_2 f(\tau,\varphi(\tau))|d\tau + \gamma K\|\varphi\|_\infty$$

$$\leq L_3|\xi_3| + (\gamma_2 L_2 + \gamma K)\|\varphi\|_\infty$$

$$\leq L_3\frac{(1-\gamma_2 L_2 - \gamma K)\alpha}{L_3} + (\gamma_2 L_2 + \gamma K)\alpha$$

$$=\alpha .$$

Hence, $G\varphi \in D$ for all $\varphi \in D$ and $G(D)\subset D$.

Moreover, from Lemma 5.1,

$$\|G\varphi - G\phi\|_\infty \leq (\gamma K + \gamma_2 L_2)\|\varphi - \phi\|_\infty \qquad \text{for any } \phi,\ \varphi \in D.$$

Hence, $G: D \to D$ is a contraction mapping on D with a contraction constant $\gamma K + \gamma_2 L_2$. The assertion of this lemma is established now.

Theorem 5.3. Suppose that the C_0-evolution system $\{U(t,s): 0\leq s\leq t<\infty\}$ satisfies conditions (A1)~(A4) and the function $f:[0,\infty)\times X \to X$ satisfies conditions (F1)~(F4). If the constants K, L_2, γ and γ_2 in (A3), (A4), (F2) and (F4) satisfy $\gamma K + \gamma_2 L_2 < 1$, then for any $\xi_3 \in X_3$ with $|\xi_3| < (1-\gamma K - \gamma_2 L_2)\alpha L_3^{-1}$, there exists $\xi_0 \in X$ such that $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (5.1) is bounded on $[0,\infty)$. Furthermore, $\lim_{t\to\infty}|u(t)| = 0$.

Proof. From Lemma 5.2, $G: D\to D$ is a contraction mapping on D with a contraction constant $\gamma K + \gamma_2 L_2$. Then there exists $u \in D$ such that $Gu = u$. Hence, $u(t)$ is bounded on $[0,\infty)$, and

$$u(t) = U(t,0)\xi_3 + \int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau$$

$$-\int_t^\infty U(t,\tau)P_2 f(\tau,u(\tau))d\tau - \int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau .$$

Thus

$$u(0)=\xi_3-\int_0^\infty U(0,\tau)P_2 f(\tau,u(\tau))d\tau-\int_0^\infty U(0,\tau)P_1 f(\tau,u(\tau))d\tau .$$

Let $\xi_0=u(0)\in X$. Following from the facts $P_3U(t,s)=U(t,s)P_3$ for all $t\geq s\geq 0$ and $P_jP_3=0$ for each $j=1, 2$, we have $P_3\xi_0=P_3u(0)=\xi_3$. On the other hand,

$$\begin{aligned}
&u(t)\\
&=U(t,0)\xi_3+\int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau-\int_t^\infty U(t,\tau)P_2 f(\tau,u(\tau))d\tau\\
&\qquad -\int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau\\
&=U(t,0)\xi_0+U(t,0)\int_0^\infty U(0,\tau)P_2 f(\tau,u(\tau))d\tau\\
&\qquad +U(t,s)\int_0^\infty U(0,\tau)P_1 f(\tau,u(\tau))d\tau+\int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau\\
&\qquad -\int_t^\infty U(t,\tau)P_2 f(\tau,u(\tau))d\tau-\int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau\\
&=U(t,0)\xi_0+\int_0^t U(t,\tau)(P_1+P_2+P_3)f(\tau,u(\tau))d\tau\\
&=U(t,0)\xi_0+\int_0^t U(t,\tau)f(\tau,u(\tau))d\tau \qquad \text{for any } t\geq 0.
\end{aligned}$$

This shows that $u(t)$ is a bounded mild solution to the abstract semilinear initial value problem (5.1) with initial value ξ_0 on $[0,\infty)$ which satisfies $P_3\xi_0=\xi_3$. From Theorem 2.14, the solution $u(t)$ is unique on $[0,\infty)$.

Since $|u(t)|\leq\alpha$ for all $t\geq 0$, there exists a constant $\mu\in[0,\infty)$ such that $\mu=\overline{\lim_{t\to\infty}}|u(t)|$. If $\mu>0$, then there is a constant $\theta\in(0,1)$ and $t_1\geq 0$ such that $\theta>\gamma K+\gamma_2 L_2$ and $|u(t)|\leq\theta^{-1}\mu$ for all $t\geq t_1$. From Lemma 5.2, one may have $\lim_{t\to\infty}\|U(t,0)P_3\|=\lim_{t\to\infty}\|U(t,t_1)P_3\|=0$. For any $t\geq t_1\geq 0$ with t large enough,

$$\begin{aligned}
&|u(t)|\\
&=\Big|U(t,0)\xi_3+\int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau\\
&\qquad -\int_t^\infty U(t,\tau)P_2 f(\tau,u(\tau))d\tau-\int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau\Big|
\end{aligned}$$

$$\le \|U(t,0)P_3\||\xi_3| + \|U(t,t_1)P_3\| \int_0^{t_1} |U(t_1,\tau)P_3 f(\tau,u(\tau))| d\tau$$

$$+\int_{t_1}^{t} |U(t,\tau)P_3 f(\tau,u(\tau))| d\tau + \int_t^{\infty} |U(t,\tau)P_1 f(\tau,u(\tau))| d\tau$$

$$+\int_t^{\infty} |U(t,\tau)P_2 f(\tau,u(\tau))| d\tau$$

$$\le \|U(t,0)P_3\||\xi_3| + \|U(t,t_1)P_3\| \int_0^{t_1} |U(t_1,\tau)P_3 f(\tau,u(\tau))| d\tau$$

$$+\int_{t_1}^{t} \|U(t,\tau)P_3\|\gamma|u(\tau)| d\tau + \int_t^{\infty} \|U(t,\tau)P_1\|\gamma|u(\tau)| d\tau$$

$$+\int_t^{\infty} \|U(t,\tau)P_2\||P_2 f(\tau,u(\tau))| d\tau$$

$$\le \|U(t,0)P_3\||\xi_3| + \|U(t,t_1)P_3\| \int_0^{t_1} |U(t_1,\tau)P_3 f(\tau,u(\tau))| d\tau$$

$$+\left(\int_{t_1}^{t} \|U(t,\tau)P_3\| d\tau + \int_t^{\infty} \|U(t,\tau)P_1\| d\tau\right)\gamma\theta^{-1}\mu$$

$$+L_2\int_t^{\infty} |P_2 f(\tau,u(\tau))| d\tau$$

$$\le \|U(t,0)P_3\||\xi_3| + \|U(t,t_1)P_3\| \int_0^{t_1} |U(t_1,\tau)P_3 f(\tau,u(\tau))| d\tau + K\gamma\theta^{-1}\mu$$

$$+L_2\int_t^{\infty} |P_2 f(\tau,u(\tau))| d\tau.$$

Thus $\mu = \overline{\lim_{t\to\infty}}|u(t)| \le (\gamma K + \gamma_2 L_2)\theta^{-1}\mu < \mu$. This is impossible, and hence $\mu = 0$. This shows that $\lim_{t\to\infty}|u(t)| = 0$, and this theorem is completely proved now.

With the same processes as in the proofs of Lemma 5.1 and Lemma 5.2, one may easily obtain following Lemma 5.4 and Lemma 5.5.

Lemma 5.4. Suppose the C_0-evolution system $\{U(t,s): 0 \le s \le t < \infty\}$ satisfies conditions (A1)~(A4) and the function $f:[0,\infty)\times X \to X$ satisfies conditions (F1)~(F4). For any fixed $\xi_2 \in X_2$, $\xi_3 \in X_3$, let the operator $B: D \to C([0,\infty);X)$ be defined by

$$(B\varphi)(t) = U(t,0)\xi_2 + U(t,0)\xi_3 + \int_0^t U(t,\tau)P_2 f(\tau,\varphi(\tau))d\tau$$

$$+\int_0^t U(t,\tau)P_3 f(\tau,\varphi(\tau))d\tau - \int_t^{\infty} U(t,\tau)P_1 f(\tau,\varphi(\tau))d\tau$$

for all $\varphi \in D$, then B is well-defined and

$$\|B\varphi - B\phi\|_\infty \le (\gamma K + \gamma_2 L_2)\|\varphi - \phi\|_\infty \quad \text{for any } \phi,\ \varphi \in D.$$

Where γ, K, γ_2, L_2 are the constants in (A3), (A4), (F2) and (F4).

Lemma 5.5. Suppose that the C_0-evolution system $\{U(t,s): 0 \le s \le t < \infty\}$ satisfies conditions (A1)~(A4) and the function $f:[0,\infty)\times X \to X$ satisfies conditions (F1)~(F4). If the constants K, L_2, γ and γ_2 in (A3), (A4), (F2) and (F4) satisfy $\gamma K + \gamma_2 L_2 < 1$, then for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with both $|\xi_2|$ and $|\xi_3|$ strictly less than $(1-\gamma K - \gamma_2 L_2)\alpha(L_2+L_3)^{-1}$, where L_3 is as in Lemma 5.2, the operator B is a contraction mapping from D into itself.

Theorem 5.6. Suppose that the C_0-evolution system $\{U(t,s): 0 \le s \le t < \infty\}$ satisfies conditions (A1)~(A4) and the function $f:[0,\infty)\times X \to X$ satisfies conditions (F1)~(F4). If the constants K, L_2, γ and γ_2 in (A3), (A4), (F2) and (F4) satisfy $\gamma K + \gamma_2 L_2 < 1$. Then for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with $|\xi_2|$, $|\xi_3| < (1-\gamma K - \gamma_2 L_2)\alpha(L_2+L_3)^{-1}$, there exists $\xi_0 \in X$ such that $P_3\xi_0 = \xi_3$, $P_2\xi_0 = \xi_2$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (5.1) is bounded on $[0,\infty)$. More precisely,

$$\|u\|_\infty \le \frac{L_2}{1-\gamma K - \gamma_2 L_2}|\xi_2| + \frac{L_3}{1-\gamma K - \gamma_2 L_2}|\xi_3|.$$

Proof. From Lemma 5.5, $B: D \to D$ is a contraction mapping on D with a contraction constant $\gamma K + \gamma_2 L_2$. Hence there exists $u \in D$ such that $Bu = u$, $u(t)$ is bounded on $[0,\infty)$,

$$u(t) = U(t,0)(\xi_2+\xi_3) + \int_0^t U(t,\tau)P_2 f(\tau,u(\tau))d\tau$$
$$+\int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau - \int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau$$

and

$$u(0) = \xi_2 + \xi_3 - \int_0^\infty U(s,\tau)P_1 f(\tau,u(\tau))d\tau.$$

Let $\xi_0 = u(0) \in X$. Since $P_j U(t,s) = U(t,s) P_j$ and $P_j P_i = 0$ for $i, j \in \{1,2,3\}$ and $i \neq j$, this implies $P_2\xi_0 = P_2 u(0) = \xi_2$, $P_3\xi_0 = P_3 u(0) = \xi_3$. On the other hand,

$$\begin{aligned} u(t) &= U(t,0)(\xi_2+\xi_3) + \int_0^t U(t,\tau)P_2 f(\tau,u(\tau))d\tau \\ &\quad + \int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau - \int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau \\ &= U(t,0)\xi_0 + U(t,0)\int_0^\infty U(s,\tau)P_1 f(\tau,u(\tau))d\tau \\ &\quad + \int_0^t U(t,\tau)P_2 f(\tau,u(\tau))d\tau + \int_0^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau \\ &\quad - \int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau \\ &= U(t,0)\xi_0 + \int_0^t U(t,\tau)(P_1+P_2+P_3) f(\tau,u(\tau))d\tau \\ &= U(t,0)\xi_0 + \int_0^t U(t,\tau) f(\tau,u(\tau))d\tau \qquad \text{for any } t \geq 0. \end{aligned}$$

Thus $u(t)$ is a bounded mild solution to the abstract semilinear initial value problem (5.1) on $[0,\infty)$ with initial value ξ_0 which satisfies $P_2\xi_0 = \xi_2$, $P_3\xi_0 = \xi_3$. The uniqueness of the solution can be obtained as in the proof of Theorem 2.14 immediately. Furthermore,

$$\begin{aligned} \|u\|_\infty &= \sup_{t\geq 0}\Big| U(t,0)(\xi_2+\xi_3) + \int_0^t U(t,\tau)P_2 f(\tau,u(\tau))d\tau \\ &\quad + \int_s^t U(t,\tau)P_3 f(\tau,u(\tau))d\tau - \int_t^\infty U(t,\tau)P_1 f(\tau,u(\tau))d\tau \Big| \\ &\leq \sup_{t\geq 0}\|U(t,0)P_2\||\xi_2| + \sup_{t\geq 0}\|U(t,0)P_3\||\xi_3| \\ &\quad + \sup_{t\geq 0}\int_0^t \|U(t,\tau)P_2\||P_2 f(\tau,u(\tau))|d\tau \\ &\quad + \sup_{t\geq 0}\Big(\int_0^t \|U(t,\tau)P_3\||f(\tau,u(\tau))|d\tau \\ &\qquad + \int_t^\infty \|U(t,\tau)P_1\||f(\tau,u(\tau))|d\tau\Big) \\ &\leq \sup_{t\geq 0}\|U(t,0)P_2\||\xi_2| + \sup_{t\geq 0}\|U(t,0)P_3\||\xi_3| \\ &\quad + \sup_{t\geq 0}\int_0^t \|U(t,\tau)P_2\||P_2 f(\tau,u(\tau))|d\tau \end{aligned}$$

$$+\sup_{t\ge 0}\left(\int_0^t \|U(t,\tau)P_3\|\gamma|u(\tau)|d\tau\right.$$
$$\left.+\int_t^\infty \|U(t,\tau)P_1\|\gamma|u(\tau)|d\tau\right)$$
$$\le L_2|\xi_2|+L_3|\xi_3|+L_2\int_0^\infty |P_2 f(\tau,u(\tau))|d\tau+\gamma K\|u\|_\infty$$
$$\le L_2|\xi_2|+L_3|\xi_3|+(\gamma_2 L_2+\gamma K)\|u\|_\infty .$$

Thus $(1-\gamma K-\gamma_2 L_2)\|u\|_\infty \le L_2|\xi_2|+L_3|\xi_3|$ and

$$\|u\|_\infty \le \frac{L_2}{1-\gamma K-\gamma_2 L_2}|\xi_2|+\frac{L_3}{1-\gamma K-\gamma_2 L_2}|\xi_3| .$$

The proof of this theorem is completed now.

§ 5-2 Applications

Example 5.1 We will consider the semilinear differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x)=\Delta u(t,x)+\beta u(t,x)+f(t,x,u) & \text{on } (0,\infty)\times\Omega \\ u(t,x)=0 & \text{on } [0,\infty)\times\partial\Omega \\ u(0,x)=\xi_0(x) & \text{on } \Omega \end{cases} \qquad (5.2)$$

where $\Omega\subset R^n$ is a bounded domain with smooth boundary, $\beta>0$ is a constant, the function f satisfy conditions (F1)~(F4), and $\xi_0(\cdot)$ is in $L^2(\Omega)$. Let X be the Hilbert space $L^2(\Omega)$, and let the operator $A:D(A)\to X$ be defined by

$$A\varphi=\Delta\varphi+\beta\varphi \qquad \text{for all } \varphi\in D(A),$$

where $D(A)=\left\{\varphi\in C^1(\overline{\Omega})\cap C^2(\Omega):\varphi(x)=0 \text{ on } \partial\Omega\right\}$. Then the semilinear differential equation (5.2) can be replaced by the semilinear initial value problem:

$$\begin{cases} \dfrac{d}{dt}u(t)=Au(t)+f(t,u) & \text{on } (0,\infty) \\ u(0)=\xi_0\in X \end{cases} \qquad (5.3)$$

for all $u(\cdot)\in D(A)$. From [13, P.205], it can be shown that there exists a sequence of eigenfunctions $\{\varphi_n : n \in N\}$ corresponding to the sequence of eigenvalues $\{\lambda_n : n \in N\}$ for A and $\{\varphi_n : n \in N\}$ is an orthonormal basis for the Hilbert space X. This implies that

$$\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle \varphi_k \qquad \text{for all } \varphi \in X,$$

and the C_0-semigroup $\{S(t) : t \geq 0\}$ generated by A on X is given by

$$(S(t)\varphi)(x) = \sum_{k=1}^{\infty} \exp(\lambda_k t) \langle \varphi, \varphi_k \rangle \varphi_k(x) \qquad \text{for all } \varphi \in X,\ x \in \Omega.$$

Suppose that $\beta > 0$ be a constant such that the eigenvalues of A satisfies

$$\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \cdots \geq \operatorname{Re}\lambda_n > 0,$$

$$\operatorname{Re}\lambda_{n+1} = \operatorname{Re}\lambda_{n+2} = \cdots = \operatorname{Re}\lambda_m = 0$$

and

$$0 > \operatorname{Re}\lambda_{m+1} \geq \operatorname{Re}\lambda_{m+2} \geq \cdots\cdots.$$

We may define linear operators P_1, P_2 and P_3 on X by

$$P_1\varphi = \sum_{k=1}^{n} \langle \varphi, \varphi_k \rangle \varphi_k \qquad \text{for all } \varphi \in X,$$

$$P_2\varphi = \sum_{k=n+1}^{m} \langle \varphi, \varphi_k \rangle \varphi_k \qquad \text{for all } \varphi \in X,$$

and

$$P_3\varphi = \sum_{k=m+1}^{\infty} \langle \varphi, \varphi_k \rangle \varphi_k \qquad \text{for all } \varphi \in X.$$

Then operators P_1, P_2 and P_3 are projections on the Hilbert space X. Let X_i be the range of a projection P_i for each i=1, 2, 3. Thus dimensions of X_1 and X_2 are n and $m-n$, respectively. Let

$$U(t,s) = S(t-s) \qquad \text{for all } t \geq s \geq 0,$$

then $\{U(t,s) : 0 \leq s \leq t < \infty\}$ is a C_0-evolution system with the infinitesimal generator $A(t) \equiv A$. Since

$$\begin{aligned} S(t)\varphi &= S(t)\sum_{k=1}^{n} \langle \varphi, \varphi_k \rangle \varphi_k \\ &= \sum_{k=1}^{n} \langle \varphi, \varphi_k \rangle S(t)\varphi_k \\ &= \sum_{k=1}^{n} \langle \varphi, \varphi_k \rangle \left(\sum_{j=1}^{\infty} \exp(\lambda_j t) \langle \varphi_k, \varphi_j \rangle \varphi_j \right) \\ &= \sum_{k=1}^{n} \exp(\lambda_k t) \langle \varphi, \varphi_k \rangle \varphi_k \in X_1 \qquad \text{for any } t \geq 0 \text{ and } \varphi \in X_1, \end{aligned}$$

and similarly, $S(t)\varphi=\sum_{k=n+1}^{m}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k$ for all $t\ge 0$ and $\varphi\in X_2$. This implies that $\{S(t)|_{X_1}:t\in R\}$ and $\{S(t)|_{X_2}:t\in R\}$ are C_0-groups on X_1 and X_2, respectively. Therefore, the condition (A1) holds. On the other hand, since

$$\begin{aligned}P_1S(t)\varphi&=P_1\sum_{k=1}^{\infty}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k\\&=\sum_{k=1}^{\infty}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle P_1\varphi_k\\&=\sum_{k=1}^{\infty}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\left(\sum_{j=1}^{n}\langle\varphi_k,\varphi_j\rangle\varphi_j\right)\\&=\sum_{k=1}^{n}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k\\&=\sum_{k=1}^{n}\langle\varphi,\varphi_k\rangle S(t)\varphi_k\\&=S(t)P_1\varphi\qquad\text{for all } t\ge 0 \text{ and } \varphi\in X,\end{aligned}$$

and

$$\begin{aligned}P_2S(t)\varphi&=P_2\sum_{k=1}^{\infty}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k\\&=\sum_{k=1}^{\infty}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle P_2\varphi_k\\&=\sum_{k=1}^{\infty}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\left(\sum_{j=n+1}^{m}\langle\varphi_k,\varphi_j\rangle\varphi_j\right)\\&=\sum_{k=n+1}^{m}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k\\&=\sum_{k=n+1}^{m}\langle\varphi,\varphi_k\rangle S(t)\varphi_k\\&=S(t)P_2\varphi\qquad\text{for all } t\ge 0 \text{ and } \varphi\in X.\end{aligned}$$

This shows that $P_1S(t)=S(t)P_1$ and $P_2S(t)=S(t)P_2$ on X for all $t\ge 0$. Thus,

$$P_3S(t)=(I-P_1-P_2)S(t)=S(t)-P_1S(t)-P_2S(t)=S(t)P_3$$

on X for all $t\ge 0$. Hence, the condition (A2) holds.

For all $t\in R$ and $\varphi\in X$,

$$\begin{aligned}|S(t)P_1\varphi|&=\left|\sum_{k=1}^{n}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k\right|\\&\le\sum_{k=1}^{n}|\exp(\lambda_k t)||\langle\varphi,\varphi_k\rangle||\varphi_k|\\&\le\sum_{k=1}^{n}|\exp(\lambda_k t)||\varphi||\varphi_k|^2\\&\le\left(\sum_{k=1}^{n}\exp(t\mathrm{Re}\lambda_k)\right)|\varphi|,\end{aligned}$$

and hence

$$\|U(t,\tau)P_1\| = \|S(t-\tau)P_1\| \le \sum_{k=1}^{n} \exp((t-\tau)\mathrm{Re}\lambda_k)$$

for all $t \ge 0$, $\tau \ge 0$. Let $\omega = 2^{-1}\min\{\mathrm{Re}\lambda_n, -\mathrm{Re}\lambda_{m+1}\}$, then $\mathrm{Re}\lambda_k > \omega > 0$ and

$$(t-\tau)\mathrm{Re}\lambda_k < (t-\tau)\omega \le 0 \qquad \text{for all } t \le \tau < \infty,\ k = 1, 2, 3, \cdots, n.$$

Hence $\|U(t,\tau)P_1\| \le n\exp(\omega(t-\tau))$ for all $t \le \tau < \infty$. We define a function $V : [0,\infty) \times X \to R$ by

$$V(t,\varphi) = \left|\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi\right| \qquad \text{for all } t \in [0,\infty),\ \varphi \in X.$$

Then for all $t \in [0,\infty)$ and $\varphi \in X$,

$$\begin{aligned}
&\frac{d}{dt}V(t,\varphi) \\
&= \frac{d}{dt}\left|\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi\right| \\
&= \frac{d}{dt}\left\langle \exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi, \exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi\right\rangle^{\frac{1}{2}} \\
&= \frac{\mathrm{Re}\left\langle \frac{d}{dt}\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi, \exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi\right\rangle}{V(t,\varphi)}.
\end{aligned}$$

The fact that

$$\begin{aligned}
&\frac{d}{dt}\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi \\
&= -(\omega+\lambda_{m+1})\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi + \exp(-(\omega+\lambda_{m+1})t)S(t)AP_3\varphi \\
&= -\omega\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi - \lambda_{m+1}\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi \\
&\qquad + \exp(-(\omega+\lambda_{m+1})t)S(t)\sum_{k=m+1}^{\infty}\langle\varphi,\varphi_k\rangle A\varphi_k \\
&= -\omega\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi \\
&\qquad - \lambda_{m+1}\exp(-(\omega+\lambda_{m+1})t)S(t)\sum_{k=m+1}^{\infty}\langle\varphi,\varphi_k\rangle\varphi_k \\
&\qquad + \exp(-(\omega+\lambda_{m+1})t)S(t)\sum_{k=m+1}^{\infty}\langle\varphi,\varphi_k\rangle\lambda_k\varphi_k \\
&= -\omega\exp(-(\omega+\lambda_{m+1})t)S(t)P_3\varphi \\
&\qquad + \exp(-(\omega+\lambda_{m+1})t)S(t)\sum_{k=m+1}^{\infty}\langle\varphi,\varphi_k\rangle(\lambda_k-\lambda_{m+1})\varphi_k
\end{aligned}$$

implies that

$$\left\langle \frac{d}{dt}\exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi, \exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi \right\rangle$$

$$=\left\langle -\omega\exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi, \exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi \right\rangle$$

$$+\Big\langle \exp\left(-(\omega+\lambda_{m+1})t\right)S(t)\sum_{k=m+1}^{\infty}\langle\varphi,\varphi_k\rangle(\lambda_k-\lambda_{m+1})\varphi_k,$$

$$\exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi \Big\rangle.$$

$$=-\omega V(t,\varphi)^2$$

$$+\left|\exp\left(-(\omega+\lambda_{m+1})t\right)\right|^2\left\langle \sum_{k=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\langle\varphi,\varphi_k\rangle S(t)\varphi_k, S(t)P_3\varphi \right\rangle.$$

Moreover, from the equalities

$$\left\langle \sum_{k=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\langle\varphi,\varphi_k\rangle S(t)\varphi_k, S(t)P_3\varphi \right\rangle$$

$$=\left\langle \sum_{k=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k, \sum_{j=m+1}^{\infty}\exp(\lambda_j t)\langle\varphi,\varphi_j\rangle\varphi_j \right\rangle$$

$$=\sum_{k=m+1}^{\infty}\sum_{j=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\exp(\lambda_k t)\overline{\exp(\lambda_j t)}\langle\varphi,\varphi_k\rangle\overline{\langle\varphi,\varphi_j\rangle}\langle\varphi_k,\varphi_j\rangle$$

$$=\sum_{k=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\left|\exp(\lambda_k t)\right|^2\left|\langle\varphi,\varphi_k\rangle\right|^2\left|\varphi_k\right|^2$$

$$=\sum_{k=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\left|\exp(\lambda_k t)\right|^2\left|\langle\varphi,\varphi_k\rangle\right|^2,$$

one obtain that

$$\frac{d}{dt}V(t,\varphi)$$

$$=\frac{\operatorname{Re}\left\langle \frac{d}{dt}\exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi, \exp\left(-(\omega+\lambda_{m+1})t\right)S(t)P_3\varphi \right\rangle}{V(t,\varphi)}$$

$$=-\omega V(t,\varphi)+\Big\{V(t,\varphi)^{-1}\left|\exp\left(-(\omega+\lambda_{m+1})t\right)\right|^2$$

$$\times\operatorname{Re}\sum_{k=m+1}^{\infty}(\lambda_k-\lambda_{m+1})\left|\exp(\lambda_k t)\right|^2\left|\langle\varphi,\varphi_k\rangle\right|^2\Big\}$$

$$=-\omega V(t,\varphi)+\Big\{V(t,\varphi)^{-1}\left|\exp\left(-(\omega+\lambda_{m+1})t\right)\right|^2$$

$$\times\sum_{k=m+1}^{\infty}(\operatorname{Re}\lambda_k-\operatorname{Re}\lambda_{m+1})\left|\exp(\lambda_k t)\right|^2\left|\langle\varphi,\varphi_k\rangle\right|^2\Big\}$$

$$\le-\omega V(t,\varphi).$$

This shows that

$$V(t,\varphi) \le \exp(-\omega t) V(0,\varphi)$$
$$= \exp(-\omega t)|P_3\varphi|$$
$$\le \exp(-\omega t)|\varphi|$$

and hence

$$\left|\exp(-(\omega+\lambda_{m+1})t) S(t) P_3\varphi\right| \le \exp(-\omega t)|\varphi|$$

for all $t \ge 0$, $\varphi \in D(A)$. Since $D(A)$ is dense in X for any $\varphi \in X$, there is a sequence $\{\varphi_j : j \in N\}$ in $D(A)$ such that $\lim_{j\to\infty} \varphi_j = \varphi$. This implies that

$$\left|\exp(-(\omega+\lambda_{m+1})t) S(t) P_3\varphi\right| = \lim_{j\to\infty}\left|\exp(-(\omega+\lambda_{m+1})t) S(t) P_3\varphi_j\right|$$
$$\le \lim_{j\to\infty} \exp(-\omega t)|\varphi_j|$$
$$= \exp(-\omega t)|\varphi| \qquad \text{for all } t \ge 0,\ \varphi \in X.$$

Hence

$$\left|\exp(-(\omega+\lambda_{m+1})t)\right| \left\|S(t) P_3\right\| = \left\|\exp(-(\omega+\lambda_{m+1})t) S(t) P_3\right\|$$
$$\le \exp(-\omega t) \qquad \text{for all } t \ge 0.$$

This shows that

$$\exp(-\omega t)\exp(-\operatorname{Re}\lambda_{m+1} t)\left\|S(t) P_3\right\| \le \exp(-\omega t) \qquad \text{for all } t \ge 0$$

and

$$\left\|U(t,\tau) P_3\right\| = \left\|S(t-\tau) P_3\right\|$$
$$\le \exp((t-\tau)\operatorname{Re}\lambda_{m+1})$$
$$\le \exp(-\omega(t-\tau)) \qquad \text{for all } t \ge \tau \ge 0.$$

Therefore, for all $0 \le t < \infty$,

$$\int_0^t \left\|U(t,\tau) P_3\right\| d\tau + \int_t^\infty \left\|U(t,\tau) P_1\right\| d\tau$$
$$\le \int_0^t \exp(-\omega(t-\tau)) d\tau + \int_t^\infty n \exp(\omega(t-\tau)) d\tau$$
$$= 1 - \omega^{-1}\exp(-\omega t) + n\omega^{-1}$$
$$\le 1 + n\omega^{-1}.$$

This implies that the condition (A3) holds with $K = 1 + n\omega^{-1}$ and $L_3 = 1$. For all $t \in R$ and $\varphi \in X$,

$$\begin{aligned}|S(t)P_2\varphi| &= \left|\sum_{k=n+1}^{m}\exp(\lambda_k t)\langle\varphi,\varphi_k\rangle\varphi_k\right| \\ &\le \sum_{k=n+1}^{m}|\exp(\lambda_k t)||\langle\varphi,\varphi_k\rangle||\varphi_k| \\ &\le \sum_{k=n+1}^{m}\exp(\operatorname{Re}\lambda_k t)|\varphi||\varphi_k|^2 \\ &\le (m-n)|\varphi|.\end{aligned}$$

Thus $\|U(t,s)P_2\| = \|S(t-s)P_2\| \le m-n$ for all $0 \le s,\ t < \infty$, and hence the condition (A4) holds with $L_2 = m-n$.

Suppose that the forcing term function $f(t,\varphi)$ satisfies conditions (F1)~(F4) with constants γ, γ_2, α. If $\omega^{-1}(n+\omega)\gamma + (m-n)\gamma_2 < 1$, then from Theorem 5.3, for any $\xi_3 \in X_3$ with

$$|\xi_3| < \left(1-\omega^{-1}(n+\omega)\gamma-(m-n)\gamma_2\right)\alpha,$$

there exists a $\xi_0 \in X$ such that $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution $u(t)$ to the semilinear initial value problem (5.3) satisfies $\lim_{t\to\infty}|u(t)| = 0$. Furthermore, from Theorem 5.6, for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with

$$|\xi_2|,\ |\xi_3| < \omega^{-1}(m-n+1)^{-1}\{\omega-(n+\omega)\gamma-\omega(m-n)\gamma_2\}\alpha,$$

there exists a $\xi_0 \in X$ such that $P_2\xi_0 = \xi_2$, $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution $u(t)$ to the semilinear initial value problem (5.3) satisfies

$$\|u\|_\infty \le (1-\gamma K-\gamma_2 L_2)^{-1}\{(m-n)|\xi_2|+|\xi_3|\}.$$

Example 5.2 Let $s \ge 0$, $\omega_1 > 0$, $\omega_3 > 0$, ξ_1, ξ_2 and ξ_3 be given real constants. Consider the ordinary differential system:

$$\begin{cases}\dfrac{d}{dt}u_1(t) = \omega_1 u_1(t) + a_{11}(t)u_1^2(t) + a_{12}(t)u_2^2(t) + a_{13}(t)u_3^2(t) \\ \dfrac{d}{dt}u_2(t) = a_{21}(t)u_1^2(t) + a_{22}(t)u_2^2(t) + a_{23}(t)u_3^2(t) \qquad \text{for } t > s \\ \dfrac{d}{dt}u_3(t) = -\omega_3 u_3(t) + a_{31}(t)u_1^2(t) + a_{32}(t)u_2^2(t) + a_{33}(t)u_3^2(t) \\ u_1(s) = \xi_1,\ u_2(s) = \xi_2 \text{ and } u_3(s) = \xi_3\end{cases} \tag{5.4}$$

where $a_{ij} \in C([s,\infty);R)$ satisfies $\|a_{ij}\|_\infty \le M$ and $\int_s^\infty |a_{2j}(\tau)|d\tau \le L$, for some constants M and L for each $i, j = 1, 2, 3$. Let X be the Banach space R^3 with the Euclidean norm, and let projections P_1, P_2, $P_3 : X \to X$ be given by

$$P_1 x = (x_1, 0, 0) \qquad \text{for all } x = (x_1, x_2, x_3) \in X,$$

$$P_2 x = (0, x_2, 0) \qquad \text{for all } x = (x_1, x_2, x_3) \in X,$$

and

$$P_3 x = (0, 0, x_3) \qquad \text{for all } x = (x_1, x_2, x_3) \in X.$$

Suppose the operator $A : X \to X$ and the function $f : [s, \infty) \times X \to X$ are defined by

$$Ax = (\omega_1 x_1, 0, -\omega_3 x_3) \qquad \text{for all } x = (x_1, x_2, x_3) \in X,$$

and

$$f(t,x)^T = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix}$$

for all $x = (x_1, x_2, x_3) \in X$. Thus the differential equations (5.4) can be replaced by the initial value problem :

$$\begin{cases} \dfrac{d}{dt} u(t) = Au(t) + f(t, u(t)) & \text{for all } t > s \\ u(s) = \xi_s \in X \end{cases} \tag{5.5}$$

Moreover, since A is a bounded operator, the C_0-semigroup $\{S(t) : t \geq 0\}$ generated by A is a C_0-group, and it can be represented as

$$S(t)x = (\exp(\omega_1 t) x_1, x_2, \exp(-\omega_3 t) x_3) \qquad \text{for all } x = (x_1, x_2, x_3) \in X.$$

Let $U(t,s) = S(t-s)$ for all $t \geq 0$ and $s \geq 0$, then $\{U(t,s) : 0 \leq s \leq t < \infty\}$ is a C_0-evolution system with the infinitesimal generator $A(t) \equiv A$.

Let $X_j = P_j X$, for each $j = 1, 2, 3$, then $\dim X_j = 1$ for each $j = 1, 2, 3$. Since

$$S(t) P_1 x = (\exp(\omega_1 t) x_1, 0, 0) \qquad \text{for all } x = (x_1, x_2, x_3) \in X,$$

$$S(t) P_2 x = (0, x_2, 0) \qquad \text{for all } x = (x_1, x_2, x_3) \in X,$$

and

$$S(t)P_3x=(0,0,\exp(-\omega_3 t)x_3) \qquad \text{for all } x=(x_1,x_2,x_3)\in X.$$

This show that the conditions (A1) and (A2) obviously hold. Since

$$\begin{aligned}\|S(t)P_1\| &= \sup_{|x|=1}|\exp(\omega_1 t)x_1| \\ &= \exp(\omega_1 t)\sup_{|x|=1}|x_1| \\ &\le \exp(\omega_1 t)\sup_{|x|=1}|x| \\ &= \exp(\omega_1 t) \qquad \text{for all } t\in R,\end{aligned}$$

$$\|S(t)P_2\| = \sup_{|x|=1}|x_2| \le \sup_{|x|=1}|x| = 1 \qquad \text{for all } t\in R,$$

and

$$\begin{aligned}\|S(t)P_3\| &= \sup_{|x|=1}|\exp(-\omega_3 t)x_3| \\ &= \exp(-\omega_3 t)\sup_{|x|=1}|x_3| \\ &\le \exp(-\omega_3 t)\sup_{|x|=1}|x| \\ &= \exp(-\omega_3 t) \qquad \text{for all } t\in R.\end{aligned}$$

This implies that

$$\|U(t,\tau)P_1\| = \|S(t-\tau)P_1\| \le \exp(\omega_1(t-\tau)),$$

$$\|U(t,\tau)P_1\| = \|S(t-\tau)P_1\| \le 1$$

and

$$\|U(t,\tau)P_3\| = \|S(t-\tau)P_3\| \le \exp(-\omega_3(t-\tau))$$

for all $t\ge 0$ and $\tau \ge 0$. Thus

$$\begin{aligned}&\int_s^t \|U(t,\tau)P_3\|d\tau + \int_t^\infty \|U(t,\tau)P_1\|d\tau \\ &\le \int_s^t \exp\{-\omega_3(t-\tau)\}d\tau + \int_t^\infty \exp\{\omega_1(t-\tau)\}d\tau \\ &= \omega_3^{-1} - \omega_3^{-1}\exp\{-\omega_3(t-s)\} + \omega_1^{-1} \\ &\le (\omega_1\omega_3)^{-1}(\omega_1+\omega_3) \qquad \text{for all } t\ge s.\end{aligned}$$

Therefore, the conditions (A3) and (A4) hold with the constants $K=(\omega_1\omega_3)^{-1}(\omega_1+\omega_3)$, $L_2=1$ and $L_3=1$.

From the definition of the function f and the assumptions of a_{ij} for each $i, j = 1, 2, 3$, the function f is continuous in t and $f(t,0) \equiv 0$. Thus conditions (F1) and (F3) hold. Moreover, for all $t \geq s$ and x, $y \in X$ with $|x|$, $|y| \leq \alpha$, we obtain that

$$
\begin{aligned}
&|f(t,x) - f(t,y)| \\
&= \Big|\big(a_{11}(t)(x_1^2 - y_1^2) + a_{12}(t)(x_2^2 - y_2^2) + a_{13}(t)(x_3^2 - y_3^2), \\
&\qquad a_{21}(t)(x_1^2 - y_1^2) + a_{22}(t)(x_2^2 - y_2^2) + a_{23}(t)(x_3^2 - y_3^2), \\
&\qquad\qquad a_{31}(t)(x_1^2 - y_1^2) + a_{32}(t)(x_2^2 - y_2^2) + a_{33}(t)(x_3^2 - y_3^2)\big)\Big| \\
&\leq \sum_{i=1}^{3} \left| a_{i1}(t)(x_1^2 - y_1^2) + a_{i2}(t)(x_2^2 - y_2^2) + a_{i3}(t)(x_3^2 - y_3^2) \right| \\
&\leq \sum_{i=1}^{3} \sum_{j=1}^{3} |a_{ij}(t)| \left| x_j^2 - y_j^2 \right| \\
&\leq M \sum_{i=1}^{3} \sum_{j=1}^{3} (|x_j| + |y_j|)|x_j - y_j| \\
&\leq M \sum_{i=1}^{3} \sum_{j=1}^{3} (|x| + |y|)|x - y| \\
&\leq 18\alpha M |x - y|.
\end{aligned}
$$

Thus the condition (F2) holds with $\gamma = 18\alpha M$. If φ, $\phi \in C([s,\infty); X)$ satisfy $\|\varphi\|_\infty$, $\|\phi\|_\infty \leq \alpha$, then

$$
\begin{aligned}
&\int_s^\infty |P_2 f(\tau, \varphi(\tau)) - P_2 f(\tau, \phi(\tau))| d\tau \\
&= \int_s^\infty \left| \sum_{j=1}^{3} a_{2j}(\tau)\left(\varphi_j(\tau)^2 - \phi_j(\tau)^2\right) \right| d\tau \\
&\leq \int_s^\infty \sum_{j=1}^{3} |a_{2j}(\tau)| \left| \varphi_j(\tau)^2 - \phi_j(\tau)^2 \right| d\tau \\
&= \sum_{j=1}^{3} \int_s^\infty |a_{2j}(\tau)| |\varphi_j(\tau) + \phi_j(\tau)| |\varphi_j(\tau) - \phi_j(\tau)| d\tau \\
&\leq \sum_{j=1}^{3} \int_s^\infty |a_{2j}(\tau)| (|\varphi_j(\tau)| + |\phi_j(\tau)|) |\varphi_j(\tau) - \phi_j(\tau)| d\tau \\
&\leq \sum_{j=1}^{3} \int_s^\infty |a_{2j}(\tau)| (|\varphi(\tau)| + |\phi(\tau)|) |\varphi(\tau) - \phi(\tau)| d\tau \\
&\leq \sum_{j=1}^{3} \int_s^\infty |a_{2j}(\tau)| (\|\varphi\|_\infty + \|\phi\|_\infty) \|\varphi - \phi\|_\infty \, d\tau
\end{aligned}
$$

$$\leq 2\alpha\|\varphi-\phi\|_\infty \sum_{j=1}^{3}\int_s^\infty |a_{2j}(\tau)|d\tau$$

$$\leq 6L\alpha\|\varphi-\phi\|_\infty .$$

Thus the condition (F4) holds with $\gamma_2 = 6L\alpha$.

If the constant $\alpha > 0$ satisfies

$$\alpha < 6^{-1}\left(L\omega_1\omega_3 + 3M\omega_1 + 3M\omega_3\right)^{-1}\omega_1\omega_3,$$

then $\gamma K + \gamma_2 L_2 < 1$. From Theorem 5.3, for any $\xi_3 \in X_3$ with

$$|\xi_3| < (\omega_1\omega_3)^{-1}\alpha\{(1-6L\alpha)\omega_1\omega_3 - 18M\alpha(\omega_1+\omega_3)\},$$

there exists $\xi_s \in X$ such that $P_3\xi_s = \xi_3$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (5.5) satisfies $\lim_{t\to\infty}|u(t)| = 0$. This implies that $\lim_{t\to\infty} u(t) = 0$, where $u(t)$ is the mild solution to the semilinear differential equations (5.4). From Theorem 5.6, for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with

$$|\xi_2|,\ |\xi_3| < (2\omega_1\omega_3)^{-1}\alpha\{(1-6L\alpha)\omega_1\omega_3 - 18M\alpha(\omega_1+\omega_3)\},$$

there exists $\xi_s \in X$ such that $P_2\xi_s = \xi_2$, $P_3\xi_s = \xi_3$ and the corresponding unique mild solution $u(t)$ to the semilinear initial value problem (5.5) satisfies

$$\|u\|_\infty \leq (1-\gamma K - \gamma_2 L_2)^{-1}(|\xi_2| + |\xi_3|).$$

Example 5.3 We will consider the semilinear differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + \beta(t)u(t,x) + f(t,u) & \text{on } (0,\infty)\times(0,\pi) \\ u(t,0) = u(t,\pi) = 0 & \text{on } [0,\infty) \\ u(0,x) = \xi_0(x) & \text{on } (0,\pi) \end{cases} \tag{5.6}$$

where β is continuous function on $[0,\infty)$ satisfying the following conditions:

($\beta 1$) $n^2 = \inf_{t\geq 0}\beta(t) \leq \sup_{t\geq 0}\beta(t) < (n+1)^2$ for some $n \in N$,

($\beta 2$) β is a constant on $[T,\infty)$ for some $T \geq 0$,

($\beta 3$) $\int_0^\infty (\beta(\tau) - n^2)d\tau$ is finite.

Let $X = L^2[0,\pi]$ and the operator $A(t): D \to X$ be defined by

$$(A(t)\varphi)(x)=\frac{\partial^2}{\partial x^2}\varphi(x)+\beta(t)\varphi(x) \qquad \text{for all } \varphi\in D,\ t\in[0,\infty),$$

where $D=\{\varphi\in C^2(0,\pi)\cap C^1[0,\pi]:\varphi(0)=\varphi(\pi)=0\}$. Then the differential equations (5.6) can be replaced by the semilinear initial value problem:

$$\begin{cases}\dfrac{d}{dt}u(t)=A(t)u(t)+f(t,u(t)) & \text{on } (0,\infty)\\ u(0)=\xi_0\in X\end{cases} \tag{5.7}$$

where $u(\cdot)\in X$. From the definition of $A(t)$, $\varphi_k(x)=\sqrt{2\pi^{-1}}\sin(kx)$ is an eigenfunction of $A(t)$ corresponding to the eigenvalue $\lambda_k(t)=\beta(t)-k^2$ of $A(t)$ for each fixed $t\ge 0$ and for all $k\in N$. On the other hand, the sequence of functions $\{\varphi_m:m\in N\}$ forms an orthonormal basis for the Hilbert space X (cf. [13, P.231]) and each φ in X can be represented as $\varphi=\sum_{k=1}^{\infty}\langle\varphi,\varphi_k\rangle\varphi_k$ (cf. [35, P.137~P.139]). Moreover, the operator $A(t)$ generates a C_0-evolution system $\{U(t,s):0\le s\le t<\infty\}$ on the Hilbert space X which satisfies

$$U(t,s)\varphi=\sum_{k=1}^{\infty}\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right)\langle\varphi,\varphi_k\rangle\varphi_k \tag{5.8}$$

for all $\varphi\in X$ and $0\le s\le t<\infty$. Since $\lambda_k(t)=\beta(t)-k^2$ and

$$\lambda_1(t)>\lambda_2(t)>\cdots>\lambda_k(t)>\cdots \qquad \text{for all } t\in[0,\infty).$$

This implies that for all $t\in[0,\infty)$,

$$\inf_{t\ge0}\lambda_1(t)>\cdots>\inf_{t\ge0}\lambda_n(t)=0>\inf_{t\ge0}\lambda_{n+1}(t)>\inf_{t\ge0}\lambda_{n+2}(t)>\cdots$$

and

$$\sup_{t\ge0}\lambda_1(t)>\cdots>\sup_{t\ge0}\lambda_n(t)>0>\sup_{t\ge0}\lambda_{n+1}(t)>\sup_{t\ge0}\lambda_{n+2}(t)>\cdots.$$

Let operators P_1, P_2 and P_3 be projections on X which are defined by

$$P_1\varphi=\sum_{k=1}^{n-1}\langle\varphi,\varphi_k\rangle\varphi_k \qquad \text{for all } \varphi\in X,$$

$$P_2\varphi=\langle\varphi,\varphi_n\rangle\varphi_n \qquad \text{for all } \varphi\in X,$$

and

$$P_3\varphi=\sum_{k=n+1}^{\infty}\langle\varphi,\varphi_k\rangle\varphi_k \qquad \text{for all } \varphi\in X.$$

Let X_i be the range of a projection P_i for each i=1, 2, 3. Then X_1 and X_2

are finite dimension spaces. For any $0 \le s,\ t < \infty$ and $\varphi \in X_1$, we have

$$\begin{aligned} U(t,s)\varphi &= U(t,s)\sum_{k=1}^{n-1}\langle \varphi, \varphi_k\rangle \varphi_k \\ &= \sum_{k=1}^{n-1}\langle \varphi, \varphi_k\rangle U(t,s)\varphi_k \\ &= \sum_{k=1}^{n-1}\langle \varphi, \varphi_k\rangle \left(\sum_{j=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-j^2)d\tau\right)\langle \varphi_k, \varphi_j\rangle \varphi_j\right) \\ &= \sum_{k=1}^{n-1}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle \varphi_k \\ &\in X_1. \end{aligned}$$

Similarly, for all $0 \le s,\ t < \infty$ and $\varphi \in X_2$,

$$U(t,s)\varphi = \exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)\langle \varphi, \varphi_n\rangle \varphi_n \in X_2.$$

Thus $\{U(t,s)|_{X_1} : 0 \le s,\ t < \infty\}$ and $\{U(t,s)|_{X_2} : 0 \le s,\ t < \infty\}$ are total C_0-evolution systems on X_1 and X_2, respectively. Hence the condition (A1) holds. Since

$$\begin{aligned} P_1 U(t,s)\varphi &= P_1\sum_{k=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle \varphi_k \\ &= \sum_{k=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle P_1\varphi_k \\ &= \sum_{k=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle \left(\sum_{j=1}^{n-1}\langle \varphi_k, \varphi_j\rangle \varphi_j\right) \\ &= \sum_{k=1}^{n-1}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle \varphi_k \\ &= \sum_{k=1}^{n-1}\langle \varphi, \varphi_k\rangle U(t,s)\varphi_k \\ &= U(t,s)\sum_{k=1}^{n-1}\langle \varphi, \varphi_k\rangle \varphi_k \\ &= U(t,s)P_1\varphi \qquad \text{for all } 0 \le s,\ t < \infty \text{ and } \varphi \in X, \end{aligned}$$

and

$$\begin{aligned} P_2 U(t,s)\varphi &= P_2\sum_{k=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle \varphi_k \\ &= \sum_{k=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle P_2\varphi_k \\ &= \sum_{k=1}^{\infty}\exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle \varphi, \varphi_k\rangle (\langle \varphi_k, \varphi_n\rangle \varphi_n) \\ &= \exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)\langle \varphi, \varphi_n\rangle \varphi_n \end{aligned}$$

$$=\langle\varphi,\varphi_n\rangle U(t,s)\varphi_n$$
$$=U(t,s)\langle\varphi,\varphi_n\rangle\varphi_n$$
$$=U(t,s)P_2\varphi \qquad \text{for all } 0\le s,\ t<\infty \text{ and } \varphi\in X.$$

This implies that

$$P_1U(t,s)=U(t,s)P_1 \text{ and } P_2U(t,s)=U(t,s)P_2$$

on X for all $0\le s,\ t<\infty$. Hence,

$$P_3U(t,s)=(I-P_1-P_2)U(t,s)$$
$$=U(t,s)-P_1U(t,s)-P_2U(t,s)$$
$$=U(t,s)P_3$$

on X for all $t\ge s\ge 0$. Therefore, the condition (A2) holds. Since

$$\left|U(t,s)P_1\varphi\right|=\left|\sum_{k=1}^{n-1}\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right)\langle\varphi,\varphi_k\rangle\varphi_k\right|$$
$$\le\sum_{k=1}^{n-1}\left|\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right)\langle\varphi,\varphi_k\rangle\varphi_k\right|$$
$$\le\sum_{k=1}^{n-1}\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right)\left|\langle\varphi,\varphi_k\rangle\right|\left|\varphi_k\right|$$
$$\le\sum_{k=1}^{n-1}\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right)\left|\varphi\right|\left|\varphi_k\right|^2$$
$$\le\sum_{k=1}^{n-1}\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right)\left|\varphi\right|$$

for all $0\le s,\ t<\infty$ and $\varphi\in X$. This means that

$$\left\|U(t,s)P_1\right\|\le\sum_{k=1}^{n-1}\exp\left(\int_s^t\left(\beta(\tau)-k^2\right)d\tau\right) \qquad \text{for all } 0\le s,\ t<\infty.$$

If we set $\omega=2^{-1}\inf_{t\ge0}\lambda_{n-1}(t)$, then

$$\lambda_1(t)>\lambda_2(t)>\cdots>\lambda_{n-1}(t)>\omega>0 \qquad \text{for all } t\ge 0,$$

and $(t-s)\lambda_k(t)\le(t-s)\omega\le 0$ for all $0\le t\le s<\infty$, $k=1,2,\cdots,n-1$. This derives

$$\left\|U(t,s)P_1\right\|\le(n-1)\exp\left(\omega(t-s)\right) \qquad \text{for all } t\le s$$

and $\int_t^\infty\left\|U(t,\tau)P_1\right\|d\tau\le(n-1)\omega^{-1}$ is finite for all $t\ge 0$. As long as we can show that there is constants $L_3>0$ and $\eta>0$ such that

$$\|U(t,s)P_3\| \le L_3 \exp(-\eta(t-s)) \qquad \text{for all } 0 \le s \le t < \infty.$$

Then for all $t \ge 0$,

$$\int_0^t \|U(t,\tau)P_3\| d\tau \le L_3 \eta^{-1} \exp(-\eta)$$

is finite and this shows the condition (A3) to be true with the constant K which equals to $L_3\eta^{-1}\exp(-\eta)+(n-1)\omega^{-1}$.

On the other hand, for all $t \ge s \ge 0$ and $\varphi \in X$,

$$\begin{aligned}
|U(t,s)P_3\varphi| &= \left|\sum_{k=n+1}^{\infty} \exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)\langle\varphi,\varphi_k\rangle\,\varphi_k\right| \\
&\le \sum_{k=n+1}^{\infty} \exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)|\langle\varphi,\varphi_k\rangle||\varphi_k| \\
&\le \sum_{k=n+1}^{\infty} \exp\left(\int_s^t (\beta(\tau)-k^2)d\tau\right)|\varphi||\varphi_k|^2 \\
&\le |\varphi|\sum_{k=1}^{\infty} \exp\left(\int_s^t \left(\beta(\tau)-(n+k)^2\right)d\tau\right). \\
&= |\varphi|\sum_{k=1}^{\infty} \exp\left(\int_s^t \beta(\tau)d\tau-(n+k)^2(t-s)\right)
\end{aligned}$$

Thus

$$\|U(t,s)P_3\| \le \sum_{k=1}^{\infty} \exp\left(\int_s^t \beta(\tau)d\tau-(n+k)^2(t-s)\right) \quad \text{for all } t \ge s \ge 0.$$

Since there is a constant $0 \le \varepsilon < 1$ such that $\sup_{t\ge 0}\beta(t) \le (n+\varepsilon)^2 < (n+1)^2$,

$$\begin{aligned}
\|U(t,s)P_3\| &\le \sum_{k=1}^{\infty} \exp\left(-(n+k)^2(t-s)\right)\exp\left(\int_s^t \beta(\tau)d\tau\right) \\
&\le \sum_{k=1}^{\infty} \exp\left(-((n+\varepsilon)+(k-\varepsilon))^2(t-s)\right)\exp\left(\int_s^t (n+\varepsilon)^2 d\tau\right) \\
&= \sum_{k=1}^{\infty} \frac{\exp\left((n+\varepsilon)^2(t-s)\right)}{\exp\left(\left((n+\varepsilon)^2+2(n+\varepsilon)(k-\varepsilon)+(k-\varepsilon)^2\right)(t-s)\right)} \\
&= \sum_{k=1}^{\infty} \exp\left(-\left(2(n+\varepsilon)(k-\varepsilon)+(k-\varepsilon)^2\right)(t-s)\right) \\
&= \sum_{k=1}^{\infty} \left(\exp\left(-2(n+\varepsilon)(k-\varepsilon)(t-s)\right)\exp\left(-(k-\varepsilon)^2(t-s)\right)\right) \\
&\le \exp\left(-(1-\varepsilon)^2(t-s)\right)\sum_{k=1}^{\infty}\left(\exp\left(-2(n+\varepsilon)(k-1)(t-s)\right)\right) \\
&\le \exp\left(-(1-\varepsilon)^2(t-s)\right)\sum_{k=1}^{\infty}\left(\exp\left(-2n(t-s)\right)\right)^{(k-1)} \qquad (5.9)
\end{aligned}$$

for all $t > s \geq 0$. The constant $\left[\exp(2n(t-s))-1\right]^{-1}\exp(2n(t-s))$ is dependent on t, s and

$$\lim_{s\to t}\frac{\exp(2n(t-s))}{\exp(2n(t-s))-1}=\infty$$

So, we can not directly estimate $\|U(t,s)P_3\|$ from (5.9). To overcome this difficulty, we need to consider the parameters of the C_0-evolution system $U(t,s)$ in the following cases:

(1) The first parameter t is in the interval $[0,T+1]$ and the second parameter s satisfies $0 \leq s \leq t \leq T+1$.

(2) The first parameter t is in the interval $(T+1,\infty)$ and the second parameter s satisfies $T < t-1 \leq s \leq t < \infty$.

(3) The first parameter t is in the interval $(T+1,\infty)$ and the second parameter s satisfies $0 \leq s < t-1$

Case (1): By using the same technique as used in the proof of Lemma 2.1, one may have $M_1 = \{\|U(t,s)P_3\| : 0 \leq s \leq t \leq T+1\}$ which is a finite constant. So, we obtain the estimation

$$\|U(t,s)P_3\| \leq M_1\exp\left((1-\varepsilon)^2(T+1)\right)\exp\left(-(1-\varepsilon)^2(t-s)\right)$$

for all $0 \leq s \leq t \leq T+1$.

Case (2): From the assumption $(\beta 2)$ of the function β and (5.8), it is easy to see that $U(t,s)=U(t-s+T,T)$ for all $T < t-1 \leq s \leq t < \infty$. Therefore,

$$M_2 = \{\|U(t,s)P_3\| : t-1 \leq s \leq t\} = \{\|U(T+h,T)P_3\| : 0 \leq h \leq 1\}$$

is finite for all $T+1 < t < \infty$ and hence

$$\|U(t,s)P_3\| \leq M_2\exp\left((1-\varepsilon)^2\right)\exp\left(-(1-\varepsilon)^2(t-s)\right)$$

for all $T < t-1 \leq s \leq t$.

Case (3): From (5.9), one obtain that

$$\begin{aligned}\|U(t,s)P_3\| &\leq \exp\left(-(1-\varepsilon)^2(t-s)\right)\sum_{k=1}^{\infty}\left(\exp(-2n(t-s))\right)^{(k-1)} \\ &= \exp\left(-(1-\varepsilon)^2(t-s)\right)\left\{1+\sum_{k=1}^{\infty}\left(\exp(-2n(t-s))\right)^{k}\right\}\end{aligned}$$

$$= \exp\left(-(1-\varepsilon)^2 (t-s)\right)\left\{1+\frac{\exp(-2n(t-s))}{1-\exp(-2n(t-s))}\right\}$$

$$= \exp\left(-(1-\varepsilon)^2 (t-s)\right)\left\{1+\left(\exp(2n(t-s))-1\right)^{-1}\right\}$$

$$\leq \exp\left(-(1-\varepsilon)^2 (t-s)\right)\left\{1+\left(\exp(2n)-1\right)^{-1}\right\}$$

$$\leq \exp(2n)\left(\exp(2n)-1\right)^{-1} \exp\left(-(1-\varepsilon)^2 (t-s)\right).$$

Finally, let η and L_3 are the constants $(1-\varepsilon)^2$ and

$$\max\left\{M_1 \exp\left((1-\varepsilon)^2 (T+1)\right), M_2 \exp\left((1-\varepsilon)^2\right), \frac{\exp(2n)}{\exp(2n)-1}\right\}$$

respectively, then we may get the estimation of $\|U(t,s)P_3\|$ as

$$\|U(t,s)P_3\| \leq L_3 \exp(-\eta(t-s)) \qquad \text{for all } 0 \leq s \leq t < \infty.$$

This shows the condition (A3) to be true with the constant

$$K = L_3 (1-\varepsilon)^{-2} \exp\left(-(1-\varepsilon)^2\right) + (n-1)\omega^{-1}.$$

On the other hand, since for all $0 \leq s,\ t < \infty$ and $\varphi \in X$,

$$|U(t,s)P_2\varphi| = \left|\exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)\langle\varphi,\varphi_n\rangle\,\varphi_n\right|$$

$$\leq \exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)\left|\langle\varphi,\varphi_n\rangle\right||\varphi_n|$$

$$\leq \exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)|\varphi||\varphi_n|^2$$

$$\leq \exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)|\varphi|,$$

$\|U(t,s)P_2\| \leq \exp\left(\int_s^t (\beta(\tau)-n^2)d\tau\right)$ for all $0 \leq s,\ t < \infty$. From the assumption $(\beta 3)$ of the function β,

$$\|U(t,s)P_2\| \leq \max\left\{\exp\left(\int_0^\infty (\beta(\tau)-n^2)d\tau\right),\ \exp\left(\int_0^\infty (n^2-\beta(\tau))d\tau\right)\right\}$$

for all $0 \leq s,\ t < \infty$. This implies the condition (A4) holds, where the constant L_2 is given by

$$L_2 = \max\left\{\exp\left(\int_0^\infty (\beta(\tau)-n^2)d\tau\right),\ \exp\left(\int_0^\infty (n^2-\beta(\tau))d\tau\right)\right\}.$$

Suppose that the function $f(t,\varphi)$ satisfies conditions (F1)~(F4). If $K\gamma + L_2\gamma_2 < 1$, then followed from Theorem 5.3, for any $\xi_3 \in X_3$ which satisfies

$$|\xi_3| < (1 - K\gamma + L_2\gamma_2)\alpha L_3^{-1},$$

there exists $\xi_0 \in X$ such that $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution . $u(t)$ to the semilinear initial value problem (5.7) satisfies $\lim_{t\to\infty}|u(t)| = 0$.

On the other hand, according to Theorem 5.6, for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with both $|\xi_2|$ and $|\xi_3| < (L_2 + L_3)^{-1}(1 - K\gamma - L_2\gamma_2)\alpha$, there exists $\xi_0 \in X$ such that $P_2\xi_0 = \xi_2$, $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution $u(t)$ to the semilinear initial value problem (5.7) satisfies

$$\|u\|_\infty \le (1 - \gamma K - \gamma_2 L_2)^{-1}(L_2|\xi_2| + L_3|\xi_3|).$$